Plane indecomposable continua no composant of which is accessible at more than one point

by

Michel Smith (Auburn, Al.)

Abstract. A method is described for constructing plane indecomposable continua that have the property that no composant of the continuum is arcwise accessible from the compliment of the continuum at more than one point. Further the pseudo-arc is embedded in the plane with this property.

Introduction. Beverly Brechner has an example of an embedding of the pseudo-arc in the plane which she conjectures has the property that no composant of it is accessible at more than one point. Here such an embedding is constructed and a technique for producing plane continua with this property is presented. (The author wishes to acknowledge the fact that Wayne Lewis has recently announced similar results.)

DEFINITIONS and NOTATIONS. Space is assumed to be the plane, denoted by E^2, with the standard Euclidean distance d. By a disc is meant a homeomorphic copy of the square disc $[0, 1]\times[0, 1]$. If Z is a bounded set or a point and ε is a positive number, then $S(Z, \varepsilon)$ denotes the open set $\{x|\ d(x, Z)<\varepsilon\}$. If H is a set, then $\mathrm{Int}(H)$ denotes the interior of H and $\mathrm{Bd}(H)$ denotes the boundary of H. If M is a set and H is a subset of M, then H is said to be *accessible from the complement of* M if there is a point x in H and an arc α with x as one of its endpoints so that $\alpha-\{x\}\subset E^2-M$ and the set H is said to be *accessible from the complement of* M *at the point* x.

THEOREM 1. *Suppose* S *is the square disc* $[0, 1]\times[0, 1]$, I *is* $[0, 1]\times\{0\}$, K *is the Cantor set lying in* I *and* $D_1, D_2, \ldots$ *is a sequence of discs lying in* S *so that*:

(1) $D_{n+1}\subset D_n$ *and* $D_1 = S$,

(2) $K\subset \mathrm{Bd}(D_n)$ *for all positive integers* n, *and*

(3) *if* $\varepsilon>0$ *and* P *is a point of* $I-K$ *then there exists an integer* N *and an integer* $n>N$ *so that* $P\in D_N-D_n$ *and every point of* D_n *lies within* ε *of the component of* D_N-D_n *containing* P.

Then the common part M *of the discs* $D_1, D_2, \ldots$ *is an indecomposable plane continuum uncountably many composants of which are accessible from the complement* *of M, in particular no two points of $K-\{(0,0),(1,0)\}$ lie in the same composant of M.*

Proof. Suppose $M = \bigcap_{i=1}^{\infty} D_i$ and $D_1, D_2, \ldots$ satisfy the hypothesis of the theorem. Since $K \in D_i$ for each i then $K \in M$. Suppose x and y are two points of K and neither x nor y is $(0,0)$ or $(1,0)$. Let α be an arc with endpoints x and y so that $\alpha-\{x,y\}$ lies in E^2-S. Suppose further that x and y lie in the same composant of M and H is a proper subcontinuum of M containing x and y. Let z be a point of $M-H$. Now $H \cup \alpha$ separates E^2 into two open sets U and V, one of which, say U, is unbounded; and $z \in E^2-H \cup \alpha = U \cup V$.

Case I. Suppose z lies in the bounded component V of $E^2-H \cup \alpha$. There is a positive number ε so that $S(z, \varepsilon)$ is a subset of V. Let P be a point of $I-K$ which lies in U, there is such a point since neither x nor y is $(0,0)$ or $(1,0)$. There is an integer N and an integer $n>N$ so that $P \notin D_n$ and every point of D_n lies within ε of the component W of D_N-D_n containing P. Thus z lies within ε of W so there is a point Q in $W \cap S(z, \varepsilon)$. But then there is an arc β lying in W containing P and Q but $Q \in S(z, \varepsilon) \subset V$, $P \in V$, and $W \subset E^2-H \cup \alpha$ which is a contradiction.

Case II. Suppose z belongs to the unbounded open set U. Then let P be a point of $I-K$ which lies in V, and the argument used in Case I applies only now β would be constructed to be a subset of W which would lie in V.

Thus no proper subcontinuum of M contains both x and y. So no two points of $K-\{(0,0),(1,0)\}$ lie in the same composant of M. So uncountably many composants of M are accessible. The indecomposability of M follows from the fact that M is irreducible between each pair of points in $K-\{(0,0),(1,0)\}$.

THEOREM 2. *There exists a pseudo-arc M in the plane such that no composant of M is accessible from the complement of M at more than one point.*

First some definitions which will be used in the construction must be made:

Suppose D is a disc and K is a Cantor set in $\mathrm{Bd}(D)$ then the sequence $\{S_i\}_{i=1}^{\infty}$ is said to be a *defining sequence of segments in* $\mathrm{Bd}(D)$ *for* K if it is true that each element of $\{S_i\}_{i=1}^{\infty}$ is an arc minus its endpoints which lies in $\mathrm{Bd}(D)$, no two elements of $\{S_i\}_{i=1}^{\infty}$ intersect, and $K = \mathrm{Bd}(D)-\bigcup_{i=1}^{\infty} S_i$.

Suppose that α is an arc with endpoints x and x', H is a point set and H intersects α. Then the point p is the *first point of H in α in the order x to x'* means that $p \in H$ and either $p = x$ or $x \notin H$ and the component of $\alpha-\{p\}$ containing x contains no point of H. If H is closed and H intersects α then there always exists such a point p.

If G is a finite collection of open sets, then the collection C of open sets is said to *refine* G if the closure of every element of C is a subset of some element of G. The collection G is called a *chain* if G is a finite collection $g_1, g_2, \ldots, g_n$ of open sets called the *links* of G so that g_i intersects g_j if and only if $|i-j| \leqslant 1$, and $\bar{g}_i$ intersects $\bar{g}_j$ if and only if $|i-j| \leqslant 1$. If G is a chain, $\varepsilon>0$, and for each link $g \in G$ $\mathrm{diam}(g)<\varepsilon$ then G is called an *ε-chain*.

If G is a chain and C is a chain then C is said to be *crooked in* G if and only if it is true that:

1) C refines G;

2) if G' is a subchain of G having at least five links and C' is a subchain of C whose first and last links lie in the first and last links of G' respectively, then there are links c_i and c_j of C' with c_i preceding c_j such that c_i lies in the next to the last link of G' and c_j lies in the second link of G; and

3) every link of G contains the closure of some link of C. For theorems concerning crookedness the reader should consult Bing [1] and Moise [3]. The following theorem will be needed in the construction.

THEOREM A. *Suppose $C_1, C_2, \ldots$ is a sequence of chains so that C_{i+1} is crooked in C_i and for each positive integer n C_n is a $(1/n)$-chain. Then the common part $M = \bigcap_{n=1}^{\infty} C_n^*$ is a pseudo-arc.*

Suppose D is a disc and $\varepsilon>0$, by an *ε-partition* P of the disc D is meant a collection $p_1, p_2, \ldots, p_n$ of discs with diameters less than ε whose union is D so that if two elements of P intersect, then they do so along a common boundary which is an arc; so that $p_i \cap p_j = \mathrm{Bd}(p_i) \cap \mathrm{Bd}(p_j)$. A *partition* is an ε-partition for some positive number ε. By a *chain-like partition* P of a disc is meant a partition $p_1, p_2, \ldots, p_n$ so that p_i intersects p_j if and only if $|i-j| \leqslant 1$; note that if $i<j$, then $p_i \cup p_{i+1} \cup \ldots \cup p_j$ is a disc.

The chain-like partition $P = p_1, p_2, \ldots, p_n$ of the disc D is said to *induce a covering* $C = c_1, c_2, \ldots, c_n$ of D if it is true that:

1) $p_i \subset c_i$, $i = 1, 2, 3, \ldots, n$;

2) the elements of C are the interiors of discs;

3) $\bar{c}_i$ intersects $\bar{c}_j$ if and only if p_i intersects p_j; and

4) $p_i \cap \bar{c}_j$ is a disc or $\emptyset$ for all $i, j = 1, 2, \ldots, n$.

The following lemmas are needed for the proof of the theorem. Lemmas 1 thru 5 are fairly elementary and are stated without proof. Lemmas 6 and 7 define an embedding of the pseudo-arc in the plane as the common part of a sequence of discs. Lemma 8 states that the continuum obtained in Lemma 7 satisfies the conclusion of Theorem 2.

LEMMA 1. *If $\delta>0$, D is a disc, and P is a chain-like partition of D; then P induces a covering $C = c_1, c_2, \ldots, c_n$ such that for each integer $i = 1, 2, \ldots, n$* $\mathrm{diam}(c_i) < \mathrm{diam}(p_i)+\delta$.

LEMMA 2. *If $\varepsilon>0$, D is a disc, and P is a chain-like ε-partition of D, then P induces a covering C of D which is an ε-chain.*

LEMMA 3. *If $\delta>0$, D is a disc, G is a covering of D, $P = p_1, p_2, \ldots, p_n$ is a chain-like partition of D each element of which is a subset of some element of G; then P induces a covering C which is a chain which covers D and refines G so that* $\mathrm{diam}(c_i) < \mathrm{diam}(p_i)+\delta$ *for $i = 1, 2, \ldots, n$.*

LEMMA 4. *Suppose D is a disc, G is a chain which covers D, J is an arc lying in* $\mathrm{Bd}(D)$, *and* $\varepsilon>0$. *Then there exstis a disc D' such that:*

(1) *$D'\subset D$ and J does not intersect* $\mathrm{Bd}(D')$,

(2) *there is a chain-like ε-partition Q of D' which induces a covering C of D' which is an ε-chain which refines G and so that every element of G contains an element of C, and*

(3) *every point of D' lies within ε of some point of the component of $D-D'$ containing J.*

LEMMA 5. *Suppose D is a disc, $C = c_1, c_2, \ldots, c_k$ is a chain covering D, and $\varepsilon>0$. Then there exists a disc D', and a chain-like ε-partition P' of D' so that:*

(1) $D'\subset D$,

(2) *P' induces an ε-chain covering D' which refines C, and*

(3) *every point of* $\mathrm{Bd}(D)$ *lies within ε of D'.*

LEMMA 6. *Suppose D is a disc, $\varepsilon>0$, $\delta>0$, $P = p_1, p_2, \ldots, p_n$ is a chain-like partition of D, C is a covering of D induced by P, K is a Cantor set lying in* $\mathrm{Bd}(D)$, *and J is an arc in* $\mathrm{Bd}(D)-K$. *Then there exists a disc H and a chain-like ε-partition Q of H which induces an ε-chain E covering H so that:*

(1) *E is crooked in C,*

(2) $K\subset\mathrm{Bd}(H)$,

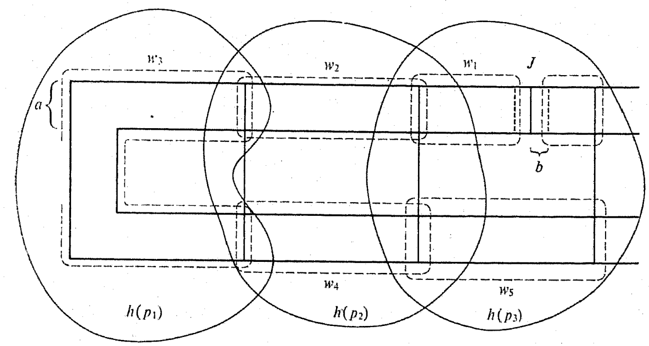
(3) *$J\subset D-H$, and*

(4) *every point of H lies within δ of the component of $(E^2-H)\cap D$ which contains J.*

Proof. There is a homeomorphism h from E^2 onto itself which maps D onto $[0, n]\times[0, 1]$ so that $h(p_i) = [i-1, i]\times[0, 1]$. There is a number γ so that if x and y are points of D and $d(x, y)<\gamma$, then $d(h^{-1}(x), h^{-1}(y))<\mathrm{Min}\{\varepsilon, \delta\}$. There is a number $a<\frac{1}{4}\gamma$ so that some vertical or horizontal segment lying in $h(J)$ has a diameter greater than a. Let A be the annulus $[0, n]\times[0, 1]-[a, n-a]\times[a, 1-a]$. There is a horizontal or vertical line segment L and a positive number b so that L lies in A and intersects $h(J)$ and both components of $\mathrm{Bd}(A)$, $\overline{S(L, b)}\cap A$ is a rectangle which does not intersect $h(K)$ and $A-S(L, b)$ is a disc. Let $B = A-S(L, b)$, so B is a disc.

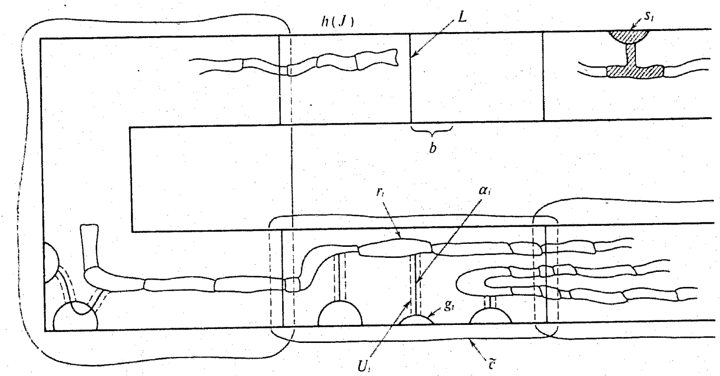
The collection $\{w|\ w$ is a component of $[i-1, i]\times[0, 1]\cap B$ for some integer $i\}$ is a partition W of B.

In particular suppose i is an integer and $h(p_i)$ does not intersect $S(L, b)$; then if i is 1 or n $h(p_i)\cap B$ is an element of W, and if i is neither 1 nor n $h(p_i)\cap B$ is the union of two elements of W. Now if $h(p_i)$ intersects $S(L, b)$, $h(p_i)$ could be the union of three elements of W. W induces a covering $\tilde{C}$ of B which refines the covering $\{h(c)|\ c\in C\}$. In particular suppose the element $\tilde{c}$ of $\tilde{C}$ is chosen to be $S(w, \delta_w)$ then δ_w can be chosen small enough so that (except for the elements of W lying in $h(p_1)$ or $h(p_n)$) $S(w, \delta_w)\cap B$ is a rectangle; in any case δ_w can be chosen small enough so that if Z is a horizontal or vertical line segment lying in A with endpoints



on the two components of $\mathrm{Bd}(A)$ then Z is a subset of every element of $\tilde{C}$ that it intersects. Let us say that $\tilde{C}$ satisfies condition Z if it satisfies this latter condition. Further δ_w can be chosen so that $\mathrm{Bd}(S(w, \delta_w))$ does not intersect $h(K)$.

There is a disc I and a chain-like $\frac{1}{4}\gamma$-partition $R = r_1, r_2, \ldots, r_m$ of I which induces the $\frac{1}{4}\gamma$-chain V which covers I so that V is crooked in $\tilde{C}$ and so that $\overline{V}^*$ intersects both components of $\mathrm{Bd}(S(L, b))\cap\mathrm{Int}(A)$ and lies in $\mathrm{Int}\,A$. Thus any horizontal or vertical line segment intersecting both components of $\mathrm{Bd}(A)$ and which does not intersect $S(L, b)$ must intersect some element of V.



There is a finite collection G of regions covering K so that:

1) the elements of G are the interiors of circles of diameter less than $\frac{1}{4}\gamma$,

2) the closures of two elements of G do not intersect,

3) the closure of each element of G intersects neither $\overline{V}^*$ nor $h(J)$, and

4) each element of G is a subset of every element of $\tilde{C}$ that it intersects.

Define a corner of B to be any element in the set:

$$\{[0, a]\times[1-a, a],\ [0, a]\times[0, a],\ [1-a, 1]\times[1-a, 1],\ [1-a, 1]\times[0, a]\}.$$

For each element g_i of G not intersecting a corner of B there is an arc α_i so that: α_i is a horizontal or vertical line segment, α_i intersects $\bar{g}_i$ at one of its endpoints and only one element r_{k_i} of R at the other endpoint, α_i is a subset of every element of $\tilde{C}$ that it intersects, and α_i does not lie in the boundary of any element of $\tilde{C}$; condition (Z) insures that this can be done (care need be taken to insure that α_i does not intersect a common boundary point of r_{k_i} and $r_{k_{i-1}}$ or of r_{k_i} and $r_{k_{i+1}}$, as there are only four such points this is easily accomplished.) Note that $\operatorname{diam}(\alpha_i)<\frac{1}{4}\gamma$. For each α_i there is a regular open set $U_i\subset S(\alpha_i,\gamma_i)\cap B$ with $\gamma_i<\frac{1}{4}\gamma$ containing α_i so that $\bar{U}_i$ intersects no element of $R\cup G$ distinct from r_{k_i} or g_i and so that $(\bar{g}_i\cap B)\cup\bar{U}_i\cup r_{k_i}$ is a disc which is a subset of every element of $\tilde{C}$ which contains r_{k_i}; further if $s_{k_i}=(\bar{g}_i\cap B)\cup\bar{U}_i\cup r_{k_i}$, then s_{k_i} intersects s_{k_j} if and only if r_{k_i} intersects r_{k_j}. If F is one of the corners of B then pick an element r_k intersecting $\operatorname{Int}(F)$ and an arc α_{r_k} connecting r_k and all of the elements of G which intersect F and an open set U_{r_k} so that $r_k\cup(\bar{U}_{r_k}\cup B)\cup\overline{\{q\mid g\in G \text{ and } g \text{ intersects } F\}^*}$ is a disc which does not intersect $s_{k'}$ for k' so that $|k'-k|>1$. Now if $S=s_1,s_2,\ldots,s_m$, then S^* is a disc, S is a chain-like partition of S^* which has the property that if $c\in C$, then $s_i\subset h(c)$ if and only if $r_i\subset h(c)$. Note that $\operatorname{diam}(U_i)<\frac{1}{2}\gamma$, $\operatorname{diam}(r_{k_i})<\frac{1}{4}\gamma$ and $\operatorname{diam}(g_i)<\frac{1}{4}\gamma$; so each element of S has diameter less than γ. Thus S is a chain-like γ-partition of the disc S^* and so induces a γ-chain $\tilde{E}=\tilde{e}_1,\tilde{e}_2,\ldots,\tilde{e}_m$ which covers S^* and is crooked in $\tilde{C}$.

Let $H=h^{-1}(S^*)$; $Q=h^{-1}(s_1)$, $h^{-1}(s_2),\ldots,h^{-1}(s_m)$; and $E=h^{-1}(\tilde{e}_1)$, $h^{-1}(\tilde{e}_2),\ldots,h^{-1}(\tilde{e}_m)$. Then:

1) E is crooked in C since $\tilde{E}$ is crooked in $\tilde{C}$;

2) $K\subset\operatorname{Bd}(H)$ because $h(K)\subset G^*\subset S^*$; and

3) $J\subset D-H$ since no element of G intersects $h(J)$ and each element of R lies in $\operatorname{Int}(A)$. Condition (4) follows from the fact that if Z is the component of $(E^2-H)\cap D$ which contains J then $h(Z)$ contains the bounded component of E^2-A and every point of B lies within $\sqrt{2}a$ of this component. Hence every point of $h(H)$ lies within $\sqrt{2}a$ of $h(Z)$ and $a<\frac{1}{4}\gamma$ so every point of H must lie within δ of Z.

From the preceding lemmas it follows by induction that.

LEMMA 7. *There exists sequences $D_1,D_2,\ldots$; $P_1,P_2,\ldots$; $K_1,K_2,\ldots$; $\{S_i^1\}_{i=1}^{\infty}$, $\{S_i^2\}_{i=1}^{\infty},\ldots$; $R_1,R_2,\ldots$; $J_1,J_2,\ldots$; and $U_1,U_2,\ldots$ so that for each positive integer n:*

(1) *D_n is a disc and $D_n\subset D_{n-1}$ for $n>1$;*

(2) *P_n is a chain-like $1/n$-partition of D_n which induces a $1/n$-chain C_n which covers D_n so that if $n>1$ C_n is crooked in C_{n-1}, and so that $d(D_n,E^2-C_n^*)<1/n$;*

(3) *K_n is a Cantor set in $\operatorname{Bd}(D_n)$, if $n>1$ K_n contains K_{n-1}, $\{S_i^n\}_{i=1}^{\infty}$ is a defining sequence of segments in $\operatorname{Bd}(D_n)$ for K_n which are ordered in non-increasing order by diameter, $\operatorname{diam}(S_i^n)<1/n$ for each positive integer i, and $\operatorname{Max}\{\operatorname{diam}(S_i^n)\}_{i=1}^{\infty}<\operatorname{Max}\{\operatorname{diam}(S_i^{n-1})\}_{i=1}^{\infty}$;*

(4) *$R_1=S_1^1$ and R_n is an element of $\bigcup_{j=1}^{n}(\{S_i^j\}_{i=1}^{\infty})-\{R_i\}_{i=1}^{n-1}$ which has diameter $\operatorname{Max}\{\operatorname{diam}(S_i^j)\mid S_i^j\neq R_k$ for positive integers i, j, and k with $j\leqslant n$ and $0<k<n\}$ (so that if $R_n=S_i^j$, then $j\leqslant n$) and J_n is a point lying in R_n;*

(5) *if $n>1$ D_n does not contain J_{n-1}, U_{n-1} is the component of $D_{k_n}-D_n$ containing J_{n-1} where k_n is the last integer so that $J_{n-1}\in\operatorname{Bd}(D_{k_n})$; and*

(6) *every point of D_n lies within $1/n$ of U_{n-1} for all $n>1$.*

It follows from Theorem A that if M is the common part of a sequence of discs $D_1,D_2,\ldots$ which satisfy the conclusion of Lemma 7, then M is a pseudo-arc. To complete the proof of Theorem 2 it is only necessary to show that M is irreducible between each pair of points accessible from the complement of M. Since not all the conditions of Lemma 7 are needed to show this the following lemma is proved.

LEMMA 8. *Suppose that there exist sequences $D_1,D_2,D_3,\ldots$; $K_1,K_2,K_3,\ldots$; $\{S_i^1\}_{i=1}^{\infty}$, $\{S_i^2\}_{i=1}^{\infty}$, $\{S_i^3\}_{i=1}^{\infty},\ldots$; $R_1,R_2,R_3,\ldots$; $J_1,J_2,J_3,\ldots$; and $U_1,U_2,U_3,\ldots$ so that for each positive integer n:*

(1) *D_n is a disc and $D_n\subset D_{n-1}$ for $n>1$;*

(2) *K_n is a Cantor set in $\operatorname{Bd}(D_n)$, K_n contains K_{n-1}, $\{S_i^n\}_{i=1}^{\infty}$ is a defining sequence of segments in $\operatorname{Bd}(D_n)$ for K_n which are ordered in non-increasing order by diameter, $\operatorname{diam}(S_i^n)<1/n$ for each positive integer i, and*

$$\operatorname{Max}\{\operatorname{diam}(S_i^n)\}_{i=1}^{\infty}<\operatorname{Max}\{\operatorname{diam}(S_i^{n-1})\}_{i=1}^{\infty};$$

(3) *$R_1=S_1^1$, R_n is an element $S_{i_n}^{j_n}$ of $\bigcup_{j=1}^{n}(\{S_i^j\}_{i=1}^{\infty})-\{R_i\}_{i=1}^{n-1}$ which has diameter $\operatorname{Max}\{\operatorname{diam}(S_i^j)\mid S_i^j\neq R_k$ for positive integers i, j, and k with $j\leqslant n$ and $0<k<n\}$;*

(4) *if $n>1$ D_n does not contain J_{n-1}, and U_{n-1} is the component of $D_{k_n}-D_n$ containing J_{n-1} where k_n is the last integer so that $J_{n-1}\in\operatorname{Bd}(D_{k_n})$, $J_n\in R_n$; and*

(5) *every point of D_n is within $1/n$ of U_{n-1} for all $n>1$.*

Then if $M=\bigcap_{n=1}^{\infty}D_n$ then M is an indecomposable plane continuum no composant of which is accessible at more than one point.

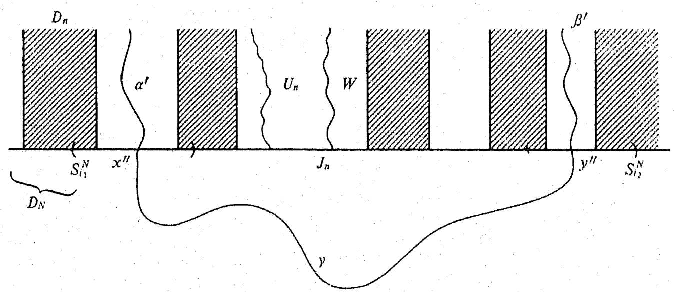
Proof. Some preliminary observations need to be made. It follows from conditions (2) and (3) that if $R_n=S_i^j$, then $j\leqslant n$. If $\varepsilon>0$ and n is a positive integer then the set $\{S_i^n\mid \operatorname{diam}(S_i^n)>\varepsilon\}$ is finite so that the sets $\{R_n\mid n=1,2,3,\ldots\}$ and $\{S_i^j\mid i,j=1,2,3,\ldots\}$ are equal.

Suppose that x and y are two points of M lying in the same composant C_p of M and that C_p is accessible from the complement of M at both x and y. Let α and β be non-intersecting arcs so that x is an endpoint of α, y is an endpoint of β, $\alpha\cap M=\{x\}$, and $\beta\cap M=\{y\}$; further choose α and β so that the endpoints x' and y' of α and β respectively which do not belong to M also do not belong to D_1. There is a proper subcontinuum H of M containing x and y. Let $z\in M-H$.

The three cases are similar, the most complicated will be presented: neither x nor y lies in any of the sets of the sequence $\{K_n\}_{n=1}^{\infty}$. Let N be a positive integer so that $1/N<d(\alpha,\beta)$. For each positive integer n the set $\{i\mid \alpha \text{ intersects } S_i^n\}$ is finite,

otherwise α would intersect K_n and hence M. (In the other cases if $x \in K_n$ for some n then α can be chosen to intersect D_n at only one point.) Let $S_{i_1}^N$ be the element of $\{S_i^N|\ i = 1, 2, 3, \ldots\}$ first intersected by α in the order from x to x', and let $S_{i_2}^N$ be the element of $\{S_i^N|\ i = 1, 2, 3, \ldots\}$ first intersected by β in the order from y to y'. Since $\mathrm{diam}(S_{i_1}^N) < 1/N$ and $d(\alpha, \beta) > 1/N$ it follows that $S_{i_1}^N \neq S_{i_2}^N$, and hence $S_{i_1}^N$ and $S_{i_2}^N$ do not intersect.

Let x'' be the first point of α in the order from x to x' lying in $S_{i_1}^N$ and y'' be the first point of β in the order from y to y'' lying in $S_{i_2}^N$. There is an arc γ with end-points x'' and y'' so that $\gamma - \{x'', y''\}$ does not intersect D_N. Let α' be the subarc of α from x to x'' and let β' be the subarc of β from y to y'', so that $\alpha' \cup \gamma \cup \beta'$ is an arc from x to y. Thus $H \cup \alpha' \cup \gamma \cup \beta'$ separates E^2 into two mutually exclusive open sets U and V, both of which contain $\alpha' \cup \gamma \cup \beta'$ in their boundary, let U be the unbounded domain and let V be the bounded one. Let ε be a positive number so that $S(z, \varepsilon)$ does not intersect $H \cup \alpha' \cup \gamma \cup \beta'$, so that $S(z, \varepsilon)$ is either a subset of U or a subset of V.



Case (i). Suppose $S(z, \varepsilon) \subset U$. The set of elements of the sequence $\{S_i^N\}_{i=1}^{\infty}$ which belong to V is infinite. There is an integer N' so that $1/N' < \varepsilon$. Thus there is an integer $n > \mathrm{Max}\{N, N'\}$ so that $R_n = S_i^N$ for some integer i and $S_i^N \subset V$. So $J_n \in S_i^N$. Let W be the component of $D_N - D_n$ containing J_n. W does not intersect $\gamma - \{x'', y''\}$ since $W \subset D_N$ and $\gamma - \{x'', y''\}$ does not intersect D_N. If W intersects α then there is an arc A lying in $W \cup \alpha'$ intersecting both S_i^N and $S_{i_1}^N$ and missing M but then $A \cup (\alpha' - \{x\})$ separates D_N into two sets both of which intersect M and whose union contains M which is not possible (note that for all positive integers j both end-points of S_j^N lie in M). Similarly W does not intersect β'. So W does not intersect $H \cup \alpha' \cup \gamma \cup \beta'$, and hence must be a subset of V since W intersects S_i^N. $U_{n-1} \subset W$ since U_{n-1} is a component of $D_{k_n} - D_n$ which is a subset of $D_N - D_n$ and U_{n-1} and W intersect (from the definition of $U_{n-1} k_n \geqslant N$ since $J_{n-1} \in R_{n-1} \subset D_{n-1} \subset D_N$ so $J_n \in D_N$). So every point of D_n lies within $1/n$ of W. Some point of $S(z, \varepsilon)$ intersects W since $1/n < 1/N' < \varepsilon$. So $S(z, \varepsilon)$ intersects both U and V, which is a contradiction. A similar argument can be used in case (ii): $S(z, \varepsilon) \subset V$. Thus it has been shown that x and y lie in different composants of M, so the theorem has been proven.

Observation. There exists an indecomposable continuum M is the plane which is not hereditarily indecomposable such that no composant of M is accessible at more than one point of M. To see this let $D_1 = [0, 1] \times [0, 1]$ and let α be the arc $\{\frac{1}{2}\} \times [0, \frac{1}{2}]$. Then a sequence $D_1, D_2, D_3, \ldots$ can be constructed satisfying the hypothesis of Lemma 8 so that $\alpha \subset D_i$ for all positive integers i. Thus $\alpha \subset M = \bigcap_{i=1}^{\infty} D_i$.

References

[1] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. 15 (1948), pp. 729–742.

[2] B. Brechner, *On stable homomorphisms and embeddings of the pseudo-arc*, to appear.

[3] E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its non-degenerate sub-continua*, Trans. Amer. Math. Soc. 63 (1948), pp. 581–584.

[4] — *A note on the pseudo-arc*, Trans. Amer. Math. Soc. 64 (1949), pp. 57–58.

[5] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Pub. 13 (1962).

[6] M. H. Proffitt, *Concerning accessibility in the plane*, Comment Math. Prac. Mat. 17 (1973/74), pp. 441–443.

AUBURN UNIVERSITY
AUBURN, ALABAMA

Accepté par la Rédaction le 2. 10. 1978