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Characterizing Hilbert space topology

by

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Abstract. It is shown that a complete separable $X \in$ ANR is homeomorphic to an open subset of a Hilbert space iff any map $\bigoplus_{n\in N} I^n \to X$ is strongly approximable by maps sending $\{I^n\}_{n\in N}$ to discrete families. Corresponding characterizations of non-separable Hilbert manifolds are also included which imply, in particular, that any Fréchet space is homeomorphic to a Hilbert space.

This paper establishes characterizations of manifolds modelled on infinite-dimensional Hilbert spaces, analogous to the following for Q-manifolds, proved by the author in [37]:

(0) *A locally compact* ANR, X, *is a manifold modelled on the Hilbert cube* Q *iff any two maps* $f, g\colon Q \to X$ *can be arbitrarily closely approximated by maps with disjoint images.*

That result was obtained by using Edward's theorem stating that $X\times Q$ is a Q-manifold, for any locally compact $X \in$ ANR, and by using Bing's shrinking criterion to show that, under the assumption of (0), $X \cong X\times Q$(¹). The approach of [37] was adopted by Mogilski, who proved in his Ph. D. thesis that if $X\times Q$ is locally homeomorphic to the Hilbert space l_2, then so is X and, more generally, that among ANR's the CE-images of l_2-manifolds are l_2-manifolds. (See [28].) However, the compactness of the Q-axis was essential for the examination of the projection $X\times Q \to X$ in both [37] and [28] and, even though an analogue of Edward's theorem is known for complete ANR's (with Q replaced by a Hilbert space), the lack of local compactness of the Hilbert spaces makes the characterizations of Hilbert manifolds more complicated and less accessible than that of Q-manifolds.

Here we apply a simple variation of Bing's shrinking criterion, stated in § 1, which is valid for non-proper maps, to examine the projection $X\times H \to H$ where $X\times H$ is a manifold modelled on a Hilbert space H of infinite dimension. In this way we show in § 2 that $X \cong H$ iff any map $H \to X$ can strongly be approximated by embeddings sending H to a Z-set in X. This preliminary characterization is improved

(¹) $\cong$ means homeomorphism.

in § 3 where we give other necessary and sufficient conditions on X to be homeomorphic, or locally homeomorphic, to H. The conditions of § 3 are formulated in terms of properties of mappings of finite dimensional metric simplicial complexes into X. In particular, we have (N denotes the set of integers):

(i) *Let X be a complete separable* AR. *Then $X \cong l_2$ iff, given a map $f\colon N\times Q\to X$ and a cover $\mathscr{U}$ of X by open sets, there is a map $g\colon N\times Q\to X$ with $\{g(\{n\}\times Q)\}_{n\in N}$ discrete in X and $\{g(y), f(y)\}_{y\in N\times Q}$ refining $\mathscr{U}$.*

(ii) *Let X be a complete* AR *and let A be a discrete space of cardinality equal to the weight of X. Then, X is homeomorphic to the Hilbert space $l_2(A)$ iff $X\cong X\times l_2$ and, for some metric inducing the topology of X, any map $A\times Q\to X$ is the ϱ-uniform limit of maps sending $\{\{a\}\times Q\}_{a\in A}$ to σ-discrete families.*

We give the following applications of the characterizations of § 3:

In § 6 we prove that two Fréchet spaces are homeomorphic iff they have the same weight. This settles the problem of the topological classification of Fréchet spaces, dating from Fréchet and Banach, which in the separable case was completed by combined results of Kadec and Anderson.

In § 5 we show that the product $\prod_{i\in N} X_i$ of complete AR's is homeomorphic to a Hilbert space provided all the X_i's are non-compact and have the same weight.

In § 4 we extend the before-mentioned result of Mogilski by showing that Hilbert manifolds of arbitrary weight are respected by a class of maps which includes proper CE-maps and proper retractions as well.

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Notation. By I we denote the segment $[0, 1]$, by I^k the k-cube and by Q the Hilbert cube I^∞. The set of integers is denoted by N and the real line by R. We write p_X for the projection $X\times Y\to X$.

Embeddings are assumed to be closed and homeomorphisms surjective. If K is an abstract simplicial complex then $K^{(n)}$ denotes its n-skeleton and $|K|$ its geometric realization equipped with the metric topology induced by the Hilbert space in which $|K|$ is naturally embedded, see [10], [22]. Given families $\mathscr{A}$, $\mathscr{B}$ of sets in a space Y we write $\mathscr{A}\prec\mathscr{B}$ if $\mathscr{A}$ refines $\mathscr{B}$ and, if $f\colon X\to Y$ and $g\colon Y\to Z$ are some maps, we let $f^{-1}(\mathscr{A}) = \{f^{-1}(A)\}_{A\in\mathscr{A}}$ and $g(\mathscr{A}) = \{g(A)\}_{A\in\mathscr{A}}$. All the undefined notions have the meaning of [10] or [18].

§ **1. Certain properties of function spaces.** $C(Y, Z)$ denotes the set of all maps (i.e. continuous functions) from Y to Z and $\operatorname{cov}(Z)$ the family of open coverings of Z. If ϱ is a bounded metric for Z we denote by $\hat\varrho$ the sup-metric on $C(Y, Z)$ induced by ϱ. We write, for $\mathscr{V}\in\operatorname{cov}(Z)$, $\alpha\in C(Z, (0,\infty))$ and $f\in C(Y, Z)$,

$$B(f, \mathscr{V}) = \{g\in C(Y, Z)\colon \{f(y), g(y)\}\prec\mathscr{V} \text{ for each } y\in Y\},$$

$$B_\varrho(f, \alpha) = \{g\in C(Y, Z)\colon \varrho(f(y), g(y))\leqslant \alpha f(y) \text{ for all } y\in Y\}.$$

Maps in $B(f, \mathscr{V})$ are said to be $\mathscr{V}$-close to f. We topologize $C(Y, Z)$ by the limitation topology τ in which each $f\in C(Y, Z)$ has $\{B(f, \mathscr{V})\colon \mathscr{V}\in\operatorname{cov}(Z)\}$ as basis of neighbourhoods. If Z is metrizable then τ coincides with the topology of uniform convergence with respect to all metrics for Z; if ϱ is a fixed metric for Z then the sets $B_\varrho(f, \alpha)$, $\alpha\in C(Z, (0,\infty))$, form a basis of (closed) τ-neighbourhoods of f.

If not stated otherwise, spaces X, Y, Z are assumed to be metrizable and all function spaces are considered in the limitation topology. $C(Y, Z)$ is in general not metrizable. We shall use the following properties of $C(Y, Z)$:

1.1. LEMMA (see [37], [33]). *Let Z be complete-metrizable, F a subspace of $C(Y, Z)$ and U_n, $n\in N$, open subsets of $C(Y, Z)$. If $U_n\cap F$ is dense in F for each n then maps in F are approximable by elements (i.e. are in the closure) of $\bigcap U_n\cap F_\varrho$, where F_ϱ denotes the ϱ-closure of F and ϱ is any metric for Z. In particular, countable intersections of dense G_δ-sets in $C(Y, Z)$ are dense in $C(Y, Z)$.*

1.2. LEMMA. *If $\mathscr{A}$ is a family of subsets of Y then $\{f\in C(Y, Z)\colon f(\mathscr{A})$ is locally finite in $Z\}$ and $\{f\in C(Y, Z)\colon f(\mathscr{A})$ is discrete in $Z\}$ are open subsets of $C(Y, Z)$.*

1.3. LEMMA. *If $Z\in$ ANR and A is a closed subset of Y then the restriction $f\mapsto f|A$ is an open map from $C(Y, Z)$ to $C(A, Z)$.*

The proofs of 1.2 and of 1.3 are omitted ([37], § 1, contains a proof of 1.3 for the case where Y is compact).

Given $\mathscr{U}\in\operatorname{cov}(Y)$, a map $f\colon Y\to Z$ is said to be a *$\mathscr{U}$-map* if there exist $\mathscr{V}\in\operatorname{cov}(Z)$ with $f^{-1}(\mathscr{V})\prec\mathscr{U}$.

1.4. LEMMA. (a) *For each $\mathscr{U}\in\operatorname{cov}(Y)$ the set of all $\mathscr{U}$-maps $f\colon Y\to Z$ is open in $C(Y, Z)$.*

(b) *Let ϱ be a complete metric for Y and $\mathscr{U}_n\in\operatorname{cov}(Y)$ be such that* $\operatorname{diam}_\varrho U<1/n$ *for $U\in\mathscr{U}_n$ and $n\in N$. Then, $f\colon Y\to Z$ is an embedding iff it is a $\mathscr{U}_n$-map for all $n\in N$. In particular, the set of embeddings $Y\to Z$ is a G_δ-set in $C(Y, Z)$.*

(c) *If $f|A\colon A\to Z$ is a $\mathscr{U}$-map, where $f\in C(Y, Z)$ and A is closed in Y, then there is a neighbourhood P of A in Y such that $f|P$ is a $\mathscr{U}$-map.*

Proof of (c). Choose a locally finite $\mathscr{V}_0\in\operatorname{cov}(Z)$ with $f^{-1}(\overline{V}_0)\cap A\prec\mathscr{U}$ for all $V_0\in\mathscr{V}_0$, and let $\mathscr{V}\in\operatorname{cov}(Z)$ be a star-refinement of $\mathscr{V}_0$. Using the paracompactness of Y construct a $\mathscr{W}\in\operatorname{cov}(Y)$ such that $\operatorname{st}(f^{-1}(\overline{V}_0)\cap A, \mathscr{W})\prec\mathscr{U}$ for all $V_0\in\mathscr{V}_0$. We put $P = \operatorname{st}(A, \mathscr{W})$. Given $V\in\mathscr{V}$. If $x\in f^{-1}(V)\cap P$ then $\{x, a\}\prec\mathscr{W}$ for some $a\in A$. We have $\{f(x), f(a)\}\prec\mathscr{V}$ and $f(x)\in V$ whence $f(a)\in\operatorname{st}(V, \mathscr{V})$. Take $V_0\in\mathscr{V}_0$ containing $\operatorname{st}(V, \mathscr{V})$; then $a\in f^{-1}(V_0)\cap A$ and

$$x\in\operatorname{st}(f^{-1}(V_0)\cap A, \mathscr{W}) \stackrel{\mathrm{df}}{=} L_V.$$

Hence $f^{-1}(V)\cap P\subset L_V\prec\mathscr{U}$, for all $V\in\mathscr{V}$.

The proof of (a) is omitted. To see (b) notice that (y_n) is ϱ-Cauchy whenever $(f(y_n))$ converges and f is a $\mathscr{U}_n$-map for all n.

Below, 1.1 and 1.4 are used to characterize maps of complete metric spaces which are approximable by homeomorphisms. A map $\pi\colon Y\to Z$ is said to be a *near-*

homeomorphism if, given $\mathcal{V} \in \text{cov}(Z)$, there is a homeomorphism $f\colon Y \to Z$ $\mathcal{V}$-close to π. The set of all near-homeomorphisms $Y \to Y$ will be denoted by $\text{NH}(Y)$.

1.5. THEOREM (Bing's shrinking criterion). *Let Y and Z be complete-metrizable spaces. A map $\pi\colon Y \to Z$ is a near-homeomorphism iff $\pi(Y)$ is dense in Z and the following condition is satisfied*:

(bi) *given $\mathcal{U} \in \text{cov}(Y)$ and $\mathcal{V} \in \text{cov}(Z)$ there are $\mathcal{W} \in \text{cov}(Z)$ and $f \in \text{NH}(Y)$ with $\pi f \in B(\pi, \mathcal{V})$ and $f\pi^{-1}(\mathcal{W}) \prec \mathcal{U}$.*

Proof. Let $\mathcal{U}_n \in \text{cov}(Y)$, $n \in N$, be such that $\text{mesh}(\mathcal{U}_n) \to 0$ with respect to some metric of Y, and let U_n be the set of all $\mathcal{U}_n$-maps $Y \to Z$. We fix a metric ϱ for Z and write $F = \{\pi h\colon h \text{ is a homeomorphism of } Y\}$. Given $\pi h \in F$ and $\mathcal{U} \in \text{cov}(Y)$ there are, by (bi), $\mathcal{W} \in \text{cov}(Z)$ and a homeomorphism f of Y with $f\pi^{-1}(\mathcal{W}) \prec h(\mathcal{U})$ and πf^{-1} as close to π as we wish. Then $\pi f^{-1}h$ is a $\mathcal{U}$-map in F which closely approximates πh. By 1.4 and 1.1 it thus follows that the set of embeddings $Y \to Z$ belonging to F_ϱ contains F, hence π, in its closure. As any map in F_ϱ is a uniform limit of maps whose images are dense in Y, the maps in F_ϱ have dense images and embeddings in F_ϱ are homeomorphisms. This completes the proof.

1.6. Remark. If π is a closed map then condition (bi) is equivalent to the following more familiar

(bi)′ given $\mathcal{U} \in \text{cov}(Y)$ and $\mathcal{V} \in \text{cov}(Z)$ there is an $f \in \text{NH}(Y)$ with $\pi f \in B(\pi, \mathcal{V})$ and $\{f\pi^{-1}(z)\}_{z\in Z} \prec \mathcal{U}$.

(Proof. If $\{f\pi^{-1}(z)\}_{z\in Z} \prec \mathcal{U}$ then $\{\pi^{-1}(z)\}_{z\in Z} \prec f^{-1}(\mathcal{U})$, whence $\pi^{-1}(\mathcal{W}) \prec f^{-1}(\mathcal{U})$ for $\mathcal{W} = \{Z \setminus \pi(Y \setminus U)\colon U \in f^{-1}(\mathcal{U})\} \in \text{cov}(Z)$. Thus (bi)′ $\Rightarrow$ (bi), provided π is closed.)

Condition (bi)′ is the one which Bing showed in [11] to characterize near-homeomorphisms between compacta. Later, Bing characterization has been extended in [26] and [17], [25] to show that (bi)′ distinguishes near-homeomorphisms among proper surjections of locally compact metric spaces and of complete metric spaces, respectively. (Actually, in all these papers a parametric version of (bi)′ was considered to give a necessary and sufficient condition on π to be conjugate to the 1-level of a small pseudo-isotopy of X.)

§ 2. **Characterizing Hilbert manifolds I.** We recall that a closed set K is said to be a *Z-set in X*, written $K \in \mathcal{Z}(X)$, iff the set $\{f \in C(Q, X)\colon f(Q) \cap K = \emptyset\}$ is dense in $C(Q, X)$. Embeddings whose images are Z-sets are called *Z-embeddings*. In this section we prove.

2.1. PROPOSITION. *A complete connected* ANR, X, *is a manifold modelled on the infinite-dimensional Hilbert space H of the same weight as X iff the following condition is satisfied*

(*) *for any complete-metrizable space Y with $wY \leqslant wX$ the set of Z-embeddings $Y \to X$ is dense in $C(Y, X)$.*

Proof. Assume that X is a manifold modelled on H and fix Y and $u \in C(Y, X)$. If $w\colon X \times H \times H \to X$ is any homeomorphism and $v\colon Y \to H$ is any embedding then $w(u \times v \times 0)$ is a Z-embedding of Y into X; therefore the necessity of (*) follows from the fact that $p_X\colon X \times H \times H \to X$ is a near-homeomorphism ([5], [29]) and Y embedds into H ([10], p. 193).

The proof of the sufficiency part involves the main idea of Edwards (see [13]) and author's [34] result on products of ANR's and Hilbert spaces.

Given a closed subset K of a space Y let $(Y \times H)_K$ denote $(Y \setminus K) \times H \cup K$, equipped with the topology generated by open subsets of $(Y \setminus K) \times H$ and by sets $U \cap K \cup (U \setminus K) \times H$, where U is open in Y. Define $(H \times Y)_K$ similarly. By $\pi_K\colon Y \times H \to (Y \times H)_K$ we denote the map which is the identity on $(Y \setminus K) \times H$ and p_Y on $K \times H$.

2.2. LEMMA. *If $K \in \mathcal{Z}(X)$ then $\pi_K\colon X \times H \to (X \times H)_K$ is a near-homeomorphism; moreover $K \in \mathcal{Z}((X \times H)_K)$.*

Proof. A homeomorphism $g\colon X \times H \to (X \times H)_K$ with $g(x, 0) = x$ for $x \in K$ is defined in the proof of Proposition 5.1 of [36]; using the formulas there it is easy to see that g may be taken to approximate π_K as close as we wish. Since $K \times \{0\} \in \mathcal{Z}(X \times H)$ we have $K = g(K \times \{0\}) \in \mathcal{Z}((X \times H)_K)$.

Let J denote $[0, \infty]$ and $C = (H \times J)_{\{0\}}$, the metric cone over H.

2.3. LEMMA. *Given $\mathcal{V}_0 \in \text{cov}(X)$ there are $K = \overline{K} \subset X$ and $g \in \text{NH}(X \times C)$ such that*

(iii) $p_X g \in B(p_X, \mathcal{V}_0)$, $g(K \times C) = X \times H \times \{\infty\}$ *and*

(iv) *the sets $g(P \times C)$, where P is open in X, form a basis of neighbourhoods of points of $X \times H \times \{\infty\}$ in $X \times C$.*

Proof. Let $\mathcal{W} \in \text{cov}(X)$ satisfy $\text{st}^5(\mathcal{W}) \prec \mathcal{V}_0$; since $X \in \text{ANR}$ there is by (*) a Z-embedding $u\colon X \times H \times \{\infty\} \to X$ $\mathcal{W}$-homotopic to p_X. With $K = \text{im}(u)$ there is by 2.2 a homeomorphism $f\colon X \times C \to (X \times C)_K$ such that $p_X f$ is $\mathcal{W}$-homotopic to p_X; then $f^{-1}\pi_K \in \text{NH}(X \times C)$ and $f^{-1}(K) \in \mathcal{Z}(X \times C)$. Since $X \times C$ is a Hilbert manifold [34], by the unknotting theorem for Z-sets there is a homeomorphism h of $X \times C$ with $p_X h$ $\text{st}^3(\mathcal{W})$-close to p_X and

$$h | X \times H \times \{\infty\} = f^{-1}u\colon X \times H \times \{\infty\} \to f^{-1}(K)$$

(see [4], [12], [33]). We let $g = h^{-1}f^{-1}\pi_K$.

Proof of 2.1. We shall apply 1.5 to show that $X \times C \to X$ is a near-homeomorphism. This will conclude the proof since $X \times C \cong X \times H \times J$ by 2.2 and $X \times H \times J$ is an H-manifold by [34].

Given $\mathcal{V} \in \text{cov}(X)$ and $\mathcal{U} \in \text{cov}(X \times C)$. Take $\alpha\colon X \to (0, \infty)$ and $\mathcal{V}_0 \in \text{cov}(X)$ such that $\text{st}(\mathcal{V}_0) \prec \mathcal{V}$ and

(i) $\{(x, h, t) \in X \times (H \times J)_{\{0\}}\colon t \leqslant \alpha(x), x \in V\}$ refines $\mathcal{V}$, for each $V \in \mathcal{V}_0$.

Then, take $\mathscr{U}_0 \in \mathrm{cov}(X\times H)$ and $\gamma\colon H \to (0, \infty)$ such that

(ii) $\{(x, h, t) \in X\times H\times (0, \infty)\colon (x, h)\in U$ and $t-\alpha(x, h)\in [i\gamma(x, h), (i+3)\gamma(x, h)]\}$ refines $\mathscr{U}$, for each $U\in \mathscr{U}_0$ and $i\in N$.

(Construction. By compactness of $[0, \infty]$ there is a locally finite $\mathscr{U}_0\in \mathrm{cov}(X\times H)$ and positive reals δ_U, $U\in\mathscr{U}_0$, such that $\inf\alpha|U>0$ and $U\times[t, t+\delta_U]\prec\mathscr{U}$ for each $U\in\mathscr{U}_0$ and $t\geqslant \inf\alpha|U$. By the well known theorem on separation of semicontinuous functions by continuous ones ([18], p. 236) there is a $\gamma\colon X\times H\to(0, \infty)$ whose graph misses the closed subset $\bigcup\{U\times[\delta_U, \infty]\colon U\in\mathscr{U}_0\}$ of $X\times H\times J$.)

Let $g\in \mathrm{NH}(X\times C)$ satisfy (iii) and (iv). Then, there are an open neighbourhood P_C of K in X and $\mathscr{P}\in\mathrm{cov}(P_0)$ such that

(v) $p_{X\times H}\,g(P\times H\times J)\prec\mathscr{U}_0$ for $P\in\mathscr{P}$.

Let $\beta_1\colon X\times H\to (0, \infty)$ be a map with $S_1 = \{(x, h, t)\colon t\geqslant\beta_1(x, h)\}\subset g(P_0\times H\times J)$. Using (i), (ii) and the separation theorem again we get an open neighbourhood P_1 of K in X and a map $\beta_2\colon X\times H\to(0, \infty)$, $\beta_2\geqslant 2\beta_1$, with $\mathrm{cl}(P_1)\subset P_0$, $g(P_1\times H\times J)\subset S_1$ and $S_2 = \{(x, h, t)\colon t\geqslant\beta_2(x, h)\}\subset g(P_1\times H\times J)$. Inductively, we construct maps $\beta_i\colon X\times H\to(0, \infty)$ and open neighbourhoods P_i of K in X such that $\mathrm{cl}(P_i)\subset P_{i-1}$, $\beta_i\geqslant 2\beta_{i-1}$ and

(vi) $\{(x, h, t)\colon t\geqslant\beta_{i+1}(x, h)\}\subset g(P_i\times H\times J)\subset\{(x, h, t)\colon t\geqslant\beta_i(x, h)\}$ for $i = 1, 2, \ldots$

Let μ be a homeomorphism of $X\times(H\times J)_{(0)}$ preserving the sets $\{x\}\times\{h\}\times J$ and carrying the graph of β_i onto that of $\alpha+(i-1)\gamma$, for each $i\in N$. We put $f=\mu g$; then $p_X f\in B(p_X, \mathrm{st}(\mathscr{V}_0))$ and it remains to show that, given $x\in X$, there is a neighbourhood G of X such that $f(G\times C)\prec\mathscr{U}$. We consider 3 cases

1^0 $x\notin P_0$. Take $V\in\mathscr{V}_0$ containing x and put $G = V\setminus \mathrm{cl}P_1$. Then $p_X f(G\times C)\prec\mathscr{V}$ and $f(G)\subset\{(x, h, t)\in X\times(H\times J)_{(0)}\colon t\leqslant\alpha(x, h)\}$. Hence $f(G\times C)\prec\mathscr{U}$, by (i).

2^0 For some i, $x\in P_i\setminus P_{i+1}$. Take $P\in\mathscr{P}$ containing x and put $G = P\cap P_i\setminus\mathrm{cl}P_{i+2}$. Then $p_{X\times H}f(G\times C)\prec\mathscr{U}_0$ and

$$f(G\times C)\subset\{(x, h, t)\colon t-\alpha(x, h)\in[i\gamma(x, h), (i+3)\gamma(x, h)]\},$$

by (v) and (vi). Hence $f(G\times C)\prec\mathscr{U}$, by (i).

3^0 $x\in K$. The existence of the required neighbourhood G of x follows from (iii).

§ 3. Characterizing Hilbert manifolds II. Throughout this section we assume that X is a connected complete-metrizable ANR and A is a discrete space of cardinality wX. Our purpose is to establish the following characterizations:

3.1. THEOREM. *X is a Hilbert manifold iff the following two conditions are satisfied:*

(∗1) *for each $n\in N$ the set of maps $A\times I^n\to X$ sending $\{\{a\}\times I^n\}_{a\in A}$ to a discrete family in X is dense in $C(A\times I^n, X)$;*

(∗2) *for any sequence (K_n) of finite-dimensional simplicial complexes having not more than wX vertices the set*

(1) $$\{f\in C(\bigoplus_{n\in N}|K_n|, X)\colon \{f(|K_n|)\}_{n\in N} \text{ is locally finite in } X\}$$

is dense in $C(\bigoplus_{n\in N}|K_n|, X)$.

3.2. COROLLARY. *X is an l_2-manifold iff it is separable and*

$$\{f\in C(\bigoplus_{n\in N} I^n, X)\colon \{f(I^n)\}_{n\in N} \text{ is discrete in } X\}$$

is dense in $C(\bigoplus_{n\in N} I^n, X)$.

3.3. COROLLARY. *X is a Hilbert manifold iff $X\times l_2\cong X$ and there are metrics ϱ_n of X such that, for all $n\in N$, the set*

$$\{f\in C(A\times I^n, X)\colon \{f(\{a\}\times I^n)\}_{a\in A} \text{ is } \sigma\text{-discrete in } X\}$$

is ϱ_n-dense in $C(A\times I^n, X)$.

The necessity of all of the conditions mentioned follows from 2.1.

The proofs of the sufficiency parts involve the following properties of metric simplicial complexes (they are presumably known but the author could not find them in the literature):

3.4. LEMMA. *Let $Y\in$ ANR and $\mathscr{U}\in\mathrm{cov}(Y)$. Then, there is a locally finite-dimensional simplicial complex K with not more than wY vertices and maps $v\colon Y\to|K|$, $w\colon |K|\to Y$ such that $wv\in B(\mathrm{id}_Y, \mathscr{U})$ and, writing $\mathscr{U}_0$ for the cover of $|K|$ by open stars of the vertices of K, we have $v^{-1}(\mathscr{U}_0)\prec\mathscr{U}$. Moreover, K may be taken to be finite-dimensional if Y is, and locally finite if Y is separable.*

Proof. Assume first that Y is an open subset of a normed space $(E, \|\ \|)$. Passing to a refinement we may assume that $\mathscr{U}$ consists of convex subsets of E. Let $\mathscr{W}$ be a star-refinement of $\mathscr{U}$ such that the nerve K of $\mathscr{W}$ is locally finite-dimensional; the existence of $\mathscr{W}$ follows from a result of Dowker [16], p. 209. If Y is separable (finite-dimensional) then $\mathscr{W}$ may be taken star-finite (of finite order). Define $v\colon Y\to|K|$ and $w\colon |K|\to E$ by

$$v(x) = (\lambda_W(x))_{W\in\mathscr{W}} \quad\text{and}\quad w((t_W)) = \sum_{W\in\mathscr{W}} t_W x_W$$

where (λ_W) is a locally finite partition of unity on Y with $\lambda_W(Y\setminus W) = \{0\}$ for all $W\in\mathscr{W}$, and $\{x_W\colon W\in\mathscr{W}\}$ is a system of points of Y with $x_W\in W$, for all $W\in\mathscr{W}$. It is easy to see that $w(|K|)\subset X$ and v and w satisfy the required conditions.

The general case now follows by considering Y as a retract of an open subset of a normed linear space (see [10], p. 68).

3.5. LEMMA. *Let K be an n-dimensional simplicial complex, $n<\infty$, and let P be an open neighbourhood of $|K^{(n-1)}|$ in $|K|$. Then, the convex hulls C_σ of $|\sigma|\setminus P$, $\sigma\in K^{(n)}$, form a discrete family in $|K|$.*

Proof. Otherwise, there is an $x \in \bar{C} \setminus C$, where $C = \bigcup \{C_\sigma\colon \sigma \in K^{(n)}\}$. Then, $x \in |\tau|$ for some $\tau \in K^{(n-1)}$. By simple Hilbert-space geometry, $\operatorname{dist}(|\sigma| \setminus P, |\tau|) = \operatorname{dist}(C_\sigma, |\tau|)$ for all $\sigma \in \operatorname{st}(\tau, K)$. Since $|\tau| \subset P$, we get $\operatorname{dist}(C \cap |\operatorname{st}(\tau, K)|, |\tau|) > 0$, contrary to the fact that $x \in \bar{C} \cap |\tau|$.

The reduction of 3.1 to 2.1 is divided into three lemmas:

3.6. LEMMA. *Assume that X satisfies (*1) and, in addition, $X \times Q \cong X$. If K is a finite-dimensional simplicial complex with $\leqslant wX$ vertices and $\mathcal{U}$ is the cover of $|K|$ by open stars of the vertices of K, then the set of $\mathcal{U}$-maps $|K| \to X$ is dense in $C(|K|, X)$.*

Proof. If $\dim K = 0$ then the assertion follows from (*1). Suppose that $\dim K = n$ and the lemma is proved for complexes of dimension $n-1$; we shall show that, given $f\colon |K| \to X$ and a neighbourhood G of f in $C(|K|, X)$, there is a $\mathcal{U}$-map $g \in G$.

By 1.3 and the inductive assumption we may assume that f is a $\mathcal{U}$-map on $|K^{(n-1)}|$. Let P be a neighbourhood of $|K^{(n-1)}|$ in $|K|$ with $f|P$ a $\mathcal{U}$-map, and let $\mathcal{V} \in \operatorname{cov}(X)$ be such that

(2) if $g\colon |K| \to X$ is $\operatorname{st}(\mathcal{V})$-close to f then $g \in G$ and $g|P$ is a $\mathcal{U}$-map.

By 3.5 there is a cover $\{C_\sigma\colon \sigma \in K^n\}$ of $|K| \setminus P$, discrete in $|K|$, such that $C_\sigma \cong I^n$ and $C_\sigma \subset \operatorname{int}|\sigma|$ for each $\sigma \in K^{(n)}$. By (*1) and 1.2 there is a map $g_0\colon |K| \to X$ sending $\{C_\sigma\colon \sigma \in K^{(n)}\}$ to a discrete family and $\mathcal{V}$-close to f. Let $\{D_\sigma\colon \sigma \in K^{(n)}\}$ be a discrete family of open subsets of $|K|$ such that $C_\sigma \subset D_\sigma \subset \operatorname{int}|\sigma|$ for $\sigma \in K^{(n)}$. Take $\lambda\colon |K| \to [0, 1]$ with $\lambda(y) = 1$ if y is in one of the C_σ's and $\lambda(y) = 0$ if y is in none of the D_σ's, and define $\bar{g}\colon |K| \to X \times [0, 1]$ by $\bar{g}(y) = (g_0(y), \lambda(y))$ for $y \in |K|$. Then, $\bar{g}$ is a $\mathcal{U}$-map: if $\mathcal{W} \in \operatorname{cov}(X)$ is such that, for each $W \in \mathcal{W}$, $g_0^{-1}(W) \cap P \prec \mathcal{U}$ and W intersects at most one member of $\{g_0(D_\sigma)\colon \sigma \in K^{(n)}\}$, then $\bar{g}^{-1}(W \times [s, t]) \prec \mathcal{U}$ for all $W \in \mathcal{W}$ and $s, t \in [0, 1]$ with $|s-t| < 1$.

Since $X \cong X \times Q$ there is a homeomorphism $h\colon X \times [0, 1] \to X$ $\mathcal{V}$-close to p_X (see [4], [29]). We let $g = h\bar{g}$.

3.7. LEMMA. *Let K be a simplicial complex with $\leqslant wX$ vertices and $\mathcal{U}$ the cover of $|K|$ by the stars of the vertices of K. If X satisfies (*1) then there is a dense G_δ-set F in $C(|K|, X \times Q)$ such that $\{g^{-1}(x)\}_{x \in X} \prec \mathcal{U}$ for all $g \in \{p_X f\colon f \in F\}$.*

Proof. Given integers m, n, consider for each $\sigma \in K^{(m)} \setminus K^{(m-1)}$ the image of $|\sigma|$ under the $(1-1/n)$-homothery with respect to the barycenter of $|\sigma|$, and denote by $\mathcal{A}_{m,n}$ the discrete family of so obtained subsets of $|K|$. By (*1) and the compactness of Q, the sets

$$F_{m,n} = \{f \in C(|K|, X \times Q)\colon p_X f(\mathcal{A}_{m,n}) \text{ is discrete in } X\}$$

are open and dense in $C(|K|, X \times Q)$. We let $F = \bigcap F_{m,n}$.

3.8. LEMMA. *Assume X satisfies (*1) and (*2) and, in addition, $X \cong X \times Q$. Then, for any complete metrizable space Y with $wY \leqslant wX$, the set of all embeddings $Y \to X$ is a dense G_δ in $C(Y, X)$.*

Proof. By 1.1 and 1.4 it suffices to show that

(3) given $\mathcal{U} \in \operatorname{cov}(Y)$, $\mathcal{V} \in \operatorname{cov}(X)$ and $f\colon Y \to X$, there is a $\mathcal{U}$-map $g\colon Y \to X$ $\mathcal{V}$-close to f.

Since $X \in \mathrm{ANR}$ there is an extension $\tilde{f}\colon \tilde{Y} \to X$ of f such that $\tilde{Y} \in \mathrm{ANR}$. By 3.4 there are a locally finite-dimensional simplicial complex K and maps $v\colon \tilde{Y} \to |K|$, $w\colon |K| \to \tilde{Y}$ with $\tilde{f}wv$ as close to f as we wish and $v^{-1}(\mathcal{U}_0) \prec \mathcal{U}$, where $\mathcal{U}_0$ is the cover of $|K|$ by open stars of the vertices of K. Then, $g_0 v$ is a $\mathcal{U}$-map for any $\mathcal{U}_0$-map $g_0\colon |K| \to X$; this observation reduces the proof of (3) to the case where

(4) $Y = |K|$, for some l.f.d. simplicial complex K, and $\mathcal{U}$ is the cover of $|K|$ by open stars of its vertices.

Let $\emptyset = W_0 \subset W_1 \subset \ldots$ be open finite-dimensional subsets of $|K|$ which cover $|K|$; without loss of generality we assume that $\overline{W}_i \subset W_{i+1}$ for all i (if not, replace W_i by $\{y \in |K|\colon \max_{n \leqslant i} \lambda_n(y) > 1/i\}$, where $\lambda_n\colon |K| \to [0, 1]$ satisfy $\lambda_n^{-1}(0) = |K| \setminus W_n$, $n \in N$). The family $\{\overline{W}_{2i+1} \setminus W_{2i}\}_{i \geqslant 0}$ is discrete in $|K|$; let S_0 be the union of its members. There is a discrete family $\{D_i\}_{i \geqslant 0}$ of open finite-dimensional subsets of $|K|$ such that $\overline{W}_{2i+1} \setminus W_{2i} \subset D_i$ for all i. Then $D_i \in \mathrm{ANR}$ for all i (see [22]) whence, by 3.4, (*2) and 1.3, there is a map $h\colon |K| \to X$ with $h \in B(f, \mathcal{V})$ and $\{h(\overline{W}_{2i+1} \setminus W_{2i})\}_{i \geqslant 0}$ locally finite in X. By 3.7 we may require in addition that the sets $h(\overline{W}_{2i+1} \setminus W_{2i})$ are pair-wise disjoint and hence form a discrete collection. There are $\mathcal{U}$-maps $h_n\colon |K^{(n)}| \to X$ with $h_n \in B(h| |K^{(n)}|, \mathcal{V}_1)$ for all $n \geqslant 0$, where $\mathcal{V}_1$ is such that any map in $B(h|S_0, \mathcal{V}_1)$ sends $(\overline{W}_{2i+1} \setminus W_{2i})_{i \geqslant 0}$ to a discrete family and extends to a map $|K| \to X$ $\mathcal{V}$-close to f (see 3.6, 1.2 and 1.3). With $n(i) = \dim W_{2i+2}$ for $i \geqslant 0$, we extend the map $S_0 \to X$ given by

$$x \mapsto h_{n(i)}(x) \quad \text{for} \quad x \in \overline{W}_{2i+1} \setminus W_{2i} \text{ and } i \geqslant 0$$

to a map $g_0\colon |K| \to X$ $\mathcal{V}$-close to g. Clearly, $g_0|S_0$ is a $\mathcal{U}$-map

Similarly, we construct $g_1\colon |K| \to X$ which is a $\mathcal{U}$-map on $S_1 = \bigcup_{i \geqslant 1} \overline{W}_{2i} \setminus W_{2i-1}$ and is so close to g_0 that $g_1 \in B(g, \mathcal{V})$ and $g_1|S_0$ is a $\mathcal{U}$-map. Let P be a neighbourhood of S_0 with $g_1|P$ a $\mathcal{U}$-map, and let $\lambda\colon |K| \to I$ satisfy $\lambda^{-1}(1) \subset |K| \setminus P$ and $\lambda^{-1}(0) \supset S_0$. Define $\bar{g}\colon |K| \to X \times I$ by $\bar{g}(y) = (g_1(y), \lambda(y))$; then $\bar{g}$ is a $\mathcal{U}$-map on $S_0 \cup S_1 = |K|$ (Proof omitted). We let $g = w\bar{g}$ where $w\colon X \times I \to X$ is a homeomorphism sufficiently close to p_X.

Proof of 3.1. Let K be a simplicial complex with $\leqslant wX$ vertices and $\mathcal{U}$ the cover of $|K|$ by the stars of the vertices of K. By 3.7, 3.8 and 1.1, there is a dense G_δ-set F_0 in $C(|K|, X \times Q)$ consisting of embeddings and such that $\{g^{-1}(x)\}_{x \in X} \prec \mathcal{U}$ for $g \in \{p_X f\colon f \in F_0\}$. The maps $p_X f$, $f \in F_0$, being closed it follows that they are $\mathcal{U}$-maps (cf. Remark 1.6); hence the $\mathcal{U}$-maps are dense in $C(|K|, X)$. As in the proof of 3.8, this implies that the embeddings are dense in $C(S, X)$, for any complete metric space S of weight $\leqslant wX$.

Applying this with $S = Y \oplus A \times Q$ we infer that, for any complete metric space Y with $wY \leqslant wX$, the sets

$$F_n = \{f \in C(Y, X)\colon f \text{ is an embedding and there is an } 1/n\text{-net } \{g_a\}_{a \in A} \text{ in } C(Q, X) \text{ with } \{f(Y)\} \cup \{g_a(Q)\}_{a \in A} \text{ discrete in } X\}$$

are dense in $C(Y, \mathcal{U})$. Since the F_n's are of type G_δ it follows that $\bigcap F_n$ is a dense subset of $C(Y, X)$ consisting of Z-embeddings. By 2.1, X is a Hilbert manifold.

Proof of 3.2. If X is separable then the complex occurring in the proof of 3.8 may be taken to be star-finite and the sets W_i in that proof to be compact polyhedra. Therefore in order that X be an l_2-manifold it suffices that the set (1) be dense in $C(\bigoplus_{n \in N} |K_n|, X)$, for any sequence (K_n) of finite complexes. Considering $\bigoplus_{n \in N} |K_n|$ as a subset of $\bigoplus_{n \in N} I^n$, a map $g\colon \bigoplus_{n \in N} |K_n| \to X$ extends to a map $\bigoplus_{n \in N} I^n \to X$ provided each $g(|K_n|)$ is contractible in X. Therefore, under condition of 3.2, it follows from the local contractibility of X that each $x \in X$ has an l_2-manifold neighbourhood.

The proof of 3.3 involves the following

Lemma. *Let Y_a, $a \in A$, be metrizable spaces and let $f\colon \bigoplus_{a \in A} Y_a \to X$ be a map such that $\{f(Y_a)\}_{a \in A}$ is σ-discrete in X. If $X \times l_2 \cong X$ then f is approximable by maps $g\colon \bigoplus_{a \in A} Y_a \to X$ with $\{g(Y_a)\}_{a \in A}$ discrete in X.*

Proof. Let $\{A_n\}_{n \in N}$ be a decomposition of A such that $\{f(Y_a)\}_{a \in A_n}$ is discrete in X, for all n. We let $g = vu$, where $v\colon X \times (0, \infty) \to X$ is a homeomorphism close to p_X (see [4], [29]) and

$$u(y) = (f(y), n) \in X \times (0, \infty) \quad \text{for} \quad n \in N \text{ and } y \in \bigcup \{Y_a\colon a \in A_n\}.$$

Proof of 3.3. It follows from the lemma that X satisfies (∗2) and that to verify (∗1) it suffices to check if, given $f\colon A \times I^n \to X$ and $\alpha\colon X \to (0, \infty)$, there is a $g \in B_{\varrho_n}(f, 2\alpha)$ with $\{g(\{a\} \times I^n)\}_{a \in A}$ σ-discrete in X.

To this end let

$$A_i = \{a \in A\colon \inf_{q \in I^n} \alpha(a, q) \in [1/i, 1/i-1)\}, \quad i \in N.$$

By assumption, there are $g_i\colon A_i \times I^n \to X$, $i \in N$, with $\varrho_n(g_i, f|A_i \times I^n) < 1/i$ and $\{g_i(\{a\} \times I^n)\}_{a \in A}$ σ-discrete in X. We let $g(a, q) = g_i(a, q)$ for $i \in N$ and $(a, q) \in A_i \times I^n$.

§ 4. Spaces finely dominated by Hilbert manifolds.

We say that $p\colon M \to X$ is a *$\mathcal{U}$-domination*, where $\mathcal{U} \in \operatorname{cov}(X)$, if p is proper and there is a map $q\colon X \to M$ with pq $\mathcal{U}$-homotopic to id_X.

4.1. Theorem. *Let X be a complete ANR. If, for every $\mathcal{U} \in \operatorname{cov}(X)$, X is $\mathcal{U}$-dominated by a Hilbert manifold then X is a Hilbert manifold itself.*

4.2. Remark. Let $p\colon M \to X$ be a proper map of ANR's. If p is either a retraction or a CE-map then it is a $\mathcal{U}$-domination for each $\mathcal{U} \in \operatorname{cov}(X)$. (The latter case follows from infinite-dimensional versions of Lacher's theorem, see [23] and [33].)

Proof of 4.1. We check that X satisfies (∗1). Given $f\colon A \times I^n \to X$ and $\mathcal{U} \in \operatorname{cov}(X)$. Let $p\colon M \to X$ and $q\colon X \to M$, where M is a Hilbert manifold, be as in the definition of a $\mathcal{U}$-domination. If $f_0\colon A \times I^n \to M$ is a sufficiently close approximation to qf with $\{f_0(a \times I^n)\}_{a \in A}$ discrete in M, then $g = pf_0$ $\mathcal{U}$-approximates f; moreover $\{g(a \times I^n)\}_{a \in A}$ consists of compacta and is locally finite (we use the properness of p). Therefore the relation

$$a \sim b \text{ iff there are } a_1, \ldots, a_n \in A \text{ with } a = a_1,\ a_n = b \text{ and } g(a_i \times I^n) \cap g(a_{i+1} \times I^n) \neq \emptyset \text{ for } i < n$$

decomposes A into countable sets A_λ, $\lambda \in \Lambda$, such that $\{g(A_\lambda \times I^n)\}_{\lambda \in \Lambda}$ is discrete in X. Moreover, it follows from assumptions that

$$\{h \in C(I^n \times \{1, 2\}, X)\colon h(I^n \times 1) \cap h(I^n \times 2) = \emptyset\}$$

is dense in $C(I^n \times \{1, 2\}, X)$; therefore one can apply Baire property of $C(A_\lambda \times I^n, X)$ to get $\mathcal{U}$-approximations $h_\lambda\colon A_\lambda \times I^n$ to $g|A_\lambda \times I^n$ such that still $h = \bigoplus_{\lambda \in \Lambda} h_\lambda$ has $\{h(A_\lambda \times I^n)\}_{\lambda \in \Lambda}$ discrete and $\{h(a \times I^n)\}_{a \in A_\lambda}$ locally finite for each λ, but in addition each $\{h(a \times I^n)\}_{a \in A_\lambda}$ consists of disjoint sets. Then, h $\operatorname{st}(\mathcal{U})$-approximates f and sends $\{a \times I^n\}_{a \in A}$ to a discrete family.

The verification of (∗2) is trivial.

§ 5. Infinite products which are Hilbert spaces.

5.1. Theorem. *Let $X_1, X_2, \ldots$ be complete AR's. In any of the following cases $X = \prod_i X_i$ is homeomorphic to a Hilbert space:*

(a) *$wX = \aleph_0$ and infinitely many of the X_i's are non-compact,*

(b) *$wX > \aleph_0$ and $\sup_{i > n} wX_i = wX$, for each $n \in N$.*

Remark. Let Y be a complete non-compact AR. By 5.2, the product Y^∞ is homeomorphic to a Hilbert space; in particular this is true for $Y = J(\mathfrak{M})$, the $\mathfrak{M}$-hedgehog (see [18], pp. 172, 197), as was conjectured by de Groot. That $J(\aleph_0)^\infty \cong l_2$ has already been shown by Curtis and Vo-Thanh-Liem in a recent paper [14] which covers also some other special cases of 5.2.

In notation of 5.1 equip X with the metric $\varrho((x_i), (y_i)) = \max_{i \in N} \varrho_i(x_i, y_i)$, where ϱ_i is a metric for X_i with $\varrho_i \leqslant 2^{-i}$, for each $i \in N$.

Lemma. *Let H be the Hilbert space of weight wX. If all the X_i's contain closed subsets homeomorphic to H, then $X \cong H$.*

Proof. By 3.1 and [20] it suffices to show that, given a complete metric space Y with $wY \leqslant wX$ and maps $f\colon Y \to X$, $\alpha\colon X \to (0, 1)$, there is an embedding $g\colon Y \to X$ with $g \in B_\varrho(f, \alpha)$.

By assumption, for each $i \in N$ there is an embedding $\varphi_i\colon Y \to X_i$. Since $X_i \in \mathrm{AR}$ there are maps $c_i\colon X_i \times X_i \times [0, \infty) \to X_i$, $i \in N$, such that $c_i(x_1, x_2, t) = x_1$ for $t \leqslant 1$ and $c_i(x_1, x_2, t) = x_2$ for $t \geqslant 2$. We define g by the formulas

$$p_i g(y) = c_i\big(p_i f(y), \varphi_i(y), 2^i \alpha f(y)\big) \quad \text{for} \quad y \in Y,$$

where $p_i\colon X \to X_i$ denotes the natural projection. Given $y \in Y$, if $\alpha f(y) \in [2^{-n-1}, 2^{-n}]$ then $2^i \alpha f(y) \leqslant 1$ for $i \leqslant n$, whence $p_i f(y) = p_i g(y)$ for $i \leqslant n$ and

$$\varrho\big(f(y), g(y)\big) \leqslant 2^{-n-1} \leqslant \alpha f(y).$$

Moreover, if $(g(y_n))$ converges for a given sequence (y_n) then write $x = \lim_n g(y_n)$ and consider $\varepsilon = \inf_{n \in N} \alpha f(y_n)$. If $\varepsilon = 0$ then $(f(y_{i(n)}))$ converges to x for any subsequence $(y_{i(n)})$ of (y_n) with $\lim \alpha f(y_{i(n)}) = 0$; this however yields $\alpha(x) = 0$ which is impossible. Thus $\varepsilon > 0$ and for n so large that $2^n \varepsilon \geqslant 2$ we get $p_n g(y_i) = \varphi_n(y_i)$ for all i. Since φ_n is an embedding (y_n) converges and g is an embedding.

Proof of 5.1. Let A be a discrete space of cardinality wX. Using the lemma and considering products of infinitely many of the X_i's instead of the X_i's themselves, we reduce the problem to demonstrating that X contains closed subsets homeomorphic to $l_2(A)$. In case (a) simply observe that each $X_i \times X_{i+1}$ contains a closed copy of $[0, \infty)$ (cf. [14]) and separable complete metric spaces embed into $[0, \infty)^\infty$ by means of a sequence of partitions of unity. In case (b) assume without loss of generality $wX = \lim_i wX_{2i}$ and write $\tilde{X} = \prod_i X_{2i}$. By the preceeding argument $\tilde{X} \times l_2$ is homeomorphic to a closed subset of X and by 3.3 it remains to show that, given $n \in N$ and $f\colon A \times Q \to \tilde{X}$, there is a $g\colon A \times Q \to \tilde{X}$ with $\{g(\{a\} \times Q)\}_{a \in A}$ σ-discrete in $\tilde{X}$ and $\hat{\varrho}(g, f) \leqslant 2^{-n}$. Put $Y = \prod_{i \leqslant n} X_{2i}$ and $Z = \prod_{i > n} X_{2i}$; then $wZ = wX$. Pick a point from each member of a σ-discrete basis in Z to get a set $\{z(a)\colon a \in A\}$ σ-discrete in Z and define $g\colon A \times Q \to \tilde{X}$ by $p_Y g = p_Y f$ and $p_Z g(\{a\} \times Q) = \{z(a)\}$ for $a \in A$ to complete the proof.

§ 6. The topological classification of Fréchet spaces. By a Fréchet space we mean any locally convex complete-metrizable topological vector space. The purpose of this section is to prove the following

6.1. THEOREM. *Any Fréchet space, X, is homeomorphic to a Hilbert space.*

The separable version of 6.1, to which we shall refer as to the Kadec–Anderson theorem, was obtained by combined efforts of Kadec [24], Anderson [2] and Bessaga and Pełczyński [8]. Shorter proofs of the results of Kadec and Anderson were given in [3], [9] and [10].

For non-separable spaces many special cases of 6.1 have been established, including results of Bessaga [6], Troyanski [32], Gutman [19], Terry–Toruńczyk ([31] and [35]) stating, respectively, that Banach spaces which are either reflexive or are of the form $c_0(A)$ or are weakly compactly generated are homeomorphic to a Hilbert space, as is any Fréchet space homeomorphic to its own countable product.

Other results on this subject proved till 1975 were contained in [7] and in [10], Chapter VII.

In the proof of 6.1 we use Kadec–Anderson theorem and results 6.2 and 6.3 below:

6.2. PROPOSITION. *Let X be a complete connected* ANR *such that $X \times l_2 \cong X$. If $K \in \mathscr{Z}(X)$ for any closed subset K of X with $wK < wX$, then X is a Hilbert manifold.*

Proof. Equip X with a metric ϱ and let A be a discrete space of cardinality wX. By 3.3 it suffices to show that, given $\varepsilon > 0$ and $f\colon A \times Q \to X$, there is a $g\colon A \times Q \to X$ with $\hat{\varrho}(g, f) \leqslant \varepsilon$ and $\{g(\{a\} \times Q\}_{a \in A}$ σ-discrete in X.

Let $<$ be a well-ordering of A with $\operatorname{card}\{a' \in A\colon a' < a\} < wX$, for all $a \in A$. We construct $g|\{a\} \times Q$ by induction on a so that

$$\delta(a) = \inf_{a' < a} \varrho\big(g(\{a\} \times Q), g(\{a'\} \times Q)\big) > 0,$$

for all $a \in A$. Let $g|\{a_0\} \times Q = f|\{a_0\} \times Q$ for the minimal element a_0 of $(A, <)$ and, if $g|\{a'\colon a' < a\} \times Q$ has been defined for a certain $a > a_0$, write K for the closure of $g(\{a'\colon a' < a\} \times Q)$. Clearly $wK \leqslant \aleph_0 \operatorname{card}\{a'\colon a' < a\} < wX$ whence $K \in \mathscr{Z}(X)$. To complete the induction let $g|\{a\} \times Q$ be an ε-approximation of $f|\{a\} \times Q$ whose image misses K.

Write $A_k = \{a \in A\colon \delta(a) > 1/k\}$ for $k \in N$. Evidently, $\bigcup A_k = A$ and $\{g(\{a\} \times Q)\}_{a \in A_k}$ is discrete in X for all $k \in N$; thus g has the desired properties.

6.3. LEMMA. *Let U be a connected open subset of a non-separable metrizable topological group X. Then, $K \in \mathscr{Z}(U)$ whenever K is a closed set in U with $wK < wU$; in particular, $X \times l_2$ is a Hilbert manifold provided X is a complete* ANR.

Proof. We may assume that $1_X \in U$. It follows from the homogeneity of X that $wV < wU$ for no open subset V of U (see [30], p. 497). Therefore, given $f\colon Q \to U$, there is a sequence (a_n) of points of $A = U \setminus \{xy^{-1}\colon x \in K, y \in f(Q)\}$ converging to 1_X. We write $g_n(q) = a_n f(q)$ for $n \in N$ and $q \in Q$; then $g_n(Q) \subset U$ for large n's and $g_n(Q) \cap K = \emptyset$ for all n's, showing that $K \in \mathscr{Z}(U)$. Hence if $X \in \mathrm{ANR}$ then 6.2 applies to any component of $X \times l_2$.

Proof of 6.1. Let X be an infinite-dimensional Fréchet space. Then $X \in \mathrm{AR}$ by Dugundji theorem (see [15], p. 188) and, if X_0 is any closed separable linear subspace of X of infinite dimension, then $X_0 \cong l_2$ by Kadec–Anderson theorem. A theorem of Bartle and Graves hence gives $X \cong X_1 \times l_2$, for some X_1 (see [10], p. 86). Thus $X \cong X_1 \times l_2 \times l_2 \cong X \times l_2$ and the result follows from 6.3 and Henderson's [20] theorem that Hilbert spaces are the only contractible Hilbert manifolds.

In connection with the subject of this section let us ask the following (cf. [10])

QUESTION. Let X be a complete-metrizable topological group. If $X \in \mathrm{ANR}$, is X locally homeomorphic to a Hilbert space, of finite or infinite dimension?

Appendix to § 6. A proof of Kadec–Anderson theorem. For the sake of completeness let us apply 3.2 to give a short argument for the Kadec–Anderson theorem which was used in the proof of 6.1 and states that $X \cong l_2$ for any separable infinite-dimensional Fréchet space X. We consider 3 cases:

1° X is a Banach space. Let ϱ be the metric induced by the norm $\| \ \|$ of X. By 3.2 and Dugundji theorem it suffices to show that, given $\alpha: X \to (0, \infty)$ and $f: \bigoplus_{n\in N} I^n \to X$, there is a $g \in B_\varrho(f, \alpha)$ with $\{g(I^n)\}_{n\in N}$ discrete in X.

We construct $g|I^n$ by induction on n.

Assume that, for a certain $n \geqslant 1$, g has been defined on $\bigoplus_{k<n} I^k$ so that each $g(I^k)$, $k<n$, is contained in a finite-dimensional linear subspace of X and

$(5)_n$ $\quad \|g(y)-f(y)\| \leqslant 2\alpha f(y)$ and $\|g(y)-g(y_1)\| \geqslant \frac{1}{2}\alpha f(y)$ for all y, y_1 such that $y_1 \in I^k$, $y \in I^l$ and $k<l<n$.

Let $h: I^n \to X$ be a piece-wise linear map with $\|h(y)-f(y)\| < \frac{1}{2}\inf_{z\in I^n} \alpha f(z)$ for $y \in I^n$, and let E be the linear span of $g(I^1) \cup \ldots \cup g(I^{n-1}) \cup h(I^n)$; by construction, $\dim(E) < \infty$. Take a vector $x \in X$ of norm $\leqslant \frac{3}{2}$ which is at distance 1 from E and write $g(y) = h(y) + \alpha f(y)\cdot x$ for $y \in I^n$ to complete the inductive step.

We claim that the map $g: \bigoplus_{n\in N} I^n \to X$ which satisfies $(5)_n$ for all n carries $\{I^n\}_{n\in N}$ to a discrete family in X. In fact, otherwise there exist integers $k_1 < k_2 < \ldots$ and points $y_i \in I^{k_i}$ such that $(g(y_i))$ converges. By (5), $\lim \alpha f(y_i) = 0$ and $(f(y_i))$ converges; hence $\alpha(\lim f(y_i)) = 0$ which is impossible.

2° $X = R^\infty$. This case is covered by 5.2([1]).

3° The general case. We follow an argument of [8]. By a theorem of Eidelheit either X is a Banach space, and then $X \cong l_2$ by 1°, or there is a closed linear subspace X_0 of X with $X/X_0 \cong R^\infty$; see [10], p. 184. By 2° and the Bartle–Graves theorem we then have $X \cong l_2 \times X_0$, whence $X \cong l_2$ by [8] or [34].

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([1]) However, let us notice that in the proof of 5.2(a) we are using Anderson–Mc Charen theorem [4] established originally by applying the fact that $l_2 \cong R^\infty$.

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