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Spaces of order arcs in hyperspaces

by

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Abstract. Let X be a metric continuum and let 2^X and $C(X)$ denote respectively the space of closed subsets and the space of subcontinua of X topologized with the Hausdorff metric. An order arc in 2^X ($C(X)$) is an arc α contained in 2^X ($C(X)$) such that if $A, B \in \alpha$, then $A \subseteq B$ or $B \subseteq A$. Let $\Gamma(2^X)$ ($\Gamma(C(X))$) denote the space of order arcs in 2^X ($C(X)$) together with the singletons $\{A\}$, $A \in 2^X$ ($C(X)$), topologized with the Hausdorff metric on 2^{2^X}. In this paper we prove that if X is locally connected, then $\Gamma(2^X)$ is homeomorphic with the Hilbert cube Q and if, in addition, X contains no arc with interior, then $\Gamma(C(X))$ is homeomorphic with Q.

1. Introduction. Let X be a continuum (i.e., a compact connected metric space containing more than one point). The *hyperspaces* of X are the spaces 2^X, consisting of all nonempty closed subsets of X, and $C(X)$, consisting of the connected elements in 2^X, each with the Hausdorff metric H. Basic facts about hyperspaces may be found in [13] and [9].

An *order arc in* 2^X (resp., $C(X)$) is an arc $\alpha \subset 2^X$ (resp. $\alpha \subset C(X)$) such that if $A, B \in \alpha$, then $A \subset B$ or $B \subset A$. Order arcs in hyperspaces were first constructed in [2], as a part of the proof of the following:

1.1. THEOREM. *For any continuum X, 2^X and $C(X)$ are each arcwise connected continua.*

However, the fact that the construction in [2] yielded an arc was not noted until later in [10, Lemma 5]. Since the publication of these two papers, order arcs have been used extensively in studying hyperspaces. However, spaces of order arcs have undergone almost no investigation. In this paper we investigate the spaces

$$\Gamma(2^X) = \{\alpha \subset 2^X\colon \alpha \text{ is an order arc}\} \cup \{\{A\}\colon A \in 2^X\}$$

and

$$\Gamma(C(X)) = \{\alpha \subset C(X)\colon \alpha \text{ is an order arc}\} \cup \{\{A\}\colon A \in C(X)\}$$

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with the metric obtained from the Hausdorff metric for 2^{2^X}, which we denote by H^2 (see also [13, (1. 31.1)]). Elements of the form $\{A\}$, $A \in 2^X$ or $C(X)$, are included in order that the two spaces be compact. Such singleton elements are, of course, not arcs, but, without confusion, we will call $\Gamma(2^X)$ (resp., $\Gamma(C(X))$ the *space of order arcs in* 2^X (resp., $C(X)$). We will also use the notation $F_1(Y) = \{\{y\}: y \in Y\}$ to denote the *subspace of singletons in* 2^Y, where Y is any continuum. Note that $F_1(Y)$ is naturally isometric to Y.

Except for the compactness of the spaces, which was proved in [10], and some results in [13], stated as 2.1 through 2.3 below, no results about $\Gamma(2^X)$ and $\Gamma(C(X))$ have appeared in the literature. *Our main purpose in this paper is to study the structure of $\Gamma(2^X)$ and $\Gamma(C(X))$ when X is a Peano continuum (i.e., a locally connected continuum). We also obtain some results for X not necessarily Peanian.*

The following important results in the theory of hyperspaces motivate much of our discussion. (The symbol $\approx$ means "is homeomorphic to", Q denotes the Hilbert cube, and a free arc in X is an arc $\gamma \subset X$ such that γ without its endpoints is an open subset of X.)

1.2. THEOREM. *Let X be a continuum. Then the following are equivalent* [17]: (a) *X is locally connected*; (b) *2^X is locally connected*; (c) *2^X is an* AR *(i.e., absolute retract)*; (d) *$C(X)$ is locally connected*; (e) *$C(X)$ is an* AR.

1.3. THEOREM. *Let X be a Peano continuum. Then $2^X \approx Q$ and, furthermore, $C(X) \approx Q$ if and only if X contains no free arc* ([4] *and* [5]).

In [13, (1.27.3)] it was noted that $\Gamma(2^X)$ has the same homotopy type as 2^X (see 2.3 below), and questions were raised concerning possible analogues of 1.2 and 1.3 for spaces of order arcs. Theorem 4.3 below provides an analogue for 1.2. We also show (Theorem 5.2) that, if X is a Peano continuum, $\Gamma(2^X) \approx Q$ and, if, in addition, X contains no free arc, then $\Gamma(C(X)) \approx Q$. Example 5.3 shows that X Peanian need not imply that $\Gamma(C(X)) \approx Q$. We do not know which Peano continua X have the property that $\Gamma(C(X)) \approx Q$.

2. Preliminaries. In this section we summarize the results from the theory of hyperspaces and continua and from infinite dimensional topology which will be used later in the paper. For sets A and B, $A \setminus B$ denotes the complement of B in A. By a mapping or map we mean a continuous function.

The first tool we need is the notion of segments, which is due to Kelley [9]. A *Whitney map for* 2^X is a map $\omega\colon 2^X \to [0, +\infty)$ such that $\omega(\{x\}) = 0$ for each $x \in X$ and $\omega(A) < \omega(B)$ if $A, B \in 2^X$ and $A \subset B \neq A$. A *segment with respect to ω from* $A_0 \in 2^X$ *to* $A_1 \in 2^X$ is a map $\sigma\colon [0,1] \to 2^X$ such that $\sigma(0) = A_0$, $\sigma(1) = A_1$, $\sigma(t_1) \subset \sigma(t_2)$ if $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, and

$$\omega(\sigma(t)) = (1-t)\cdot\omega(\sigma(0)) + t\cdot\omega(\sigma(1))$$

for each $t \in [0,1]$. Let

$$S_\omega(2^X) = \{\sigma\colon [0,1] \to 2^X \mid \sigma \text{ is a segment or a constant function}\}$$

and

$$S_\omega(C(X)) = \{\sigma \in S_\omega(2^X) \mid \sigma(t) \in C(X) \text{ for each } t \in [0,1]\},$$

and let each of these be topologized by the usual "supremum metric" for continuous functions. These two spaces are called *spaces of segments*. The following theorem indicates the fundamental relationship between spaces of segments and spaces of order arcs.

2.1. THEOREM [13, (1.30)]. *For any continuum X and any given Whitney map ω for 2^X, $S_\omega(2^X) \approx \Gamma(2^X)$ and $S_\omega(C(X)) \approx \Gamma(C(X))$. Moreover, the function $f_\omega\colon S_\omega(2^X) \to \Gamma(2^X)$ defined by $f_\omega(\sigma) = \sigma([0,1])$ is a homeomorphism and $f_\omega|S_\omega(C(X))) = \Gamma(C(X))$.*

Thus spaces of order arcs and individual order arcs are parametrized in a well-behaved manner by segments. For this reason, the study of spaces of order arcs is facilitated by using segments. Throughout the paper we will refer to the homeomorphism f_ω defined in the statement of Theorem 2.1.

Properties of hyperspaces are reflected in spaces of order arcs. For example, the following retraction theorem guarantees that a hyperspace has the same homotopy type as the corresponding order arc space.

2.2. THEOREM (see [13, (1.203.3)]). *For any continuum X, $F_1(2^X)$ and $F_1(C(X))$ are strong deformation retracts of $\Gamma(2^X)$ and $\Gamma(C(X))$ respectively.*

Proof. Let f_ω be the homeomorphism defined in Theorem 2.1. Define homotopies $k\colon S_\omega(2^X) \times [0,1] \to \Gamma(2^X)$ and $h\colon \Gamma(2^X) \times [0,1] \to \Gamma(2^X)$ by $k(\sigma, s) = \sigma([0, 1-s])$ and $h(\alpha, s) = k(f_\omega^{-1}(\alpha), s)$. Then h is clearly a strong deformation retraction. Moreover, it follows from the final equality in Theorem 2.1 that the restriction of h to $\Gamma(C(X)) \times [0,1]$ is a strong deformation retraction onto $F_1(C(X))$. ■

2.3. COROLLARY (see [13, (1.27.3)]). *For any continuum X, $\Gamma(2^X)$ and $\Gamma(C(X))$ are arcwise connected continua of the same homotopy type as 2^X and $C(X)$ respectively.*

Proof. By [10, proof of Lemma 4], $\Gamma(2^X)$ and $\Gamma(C(X))$ are compact. Therefore, the corollary is an immediate consequence of Theorems 1.1 and 2.2. ■

We now state four lemmas about order arcs which will be used throughout this paper. The first three are a mixture of results and parts of proofs in [10]. The fourth one is, for the most part, 2.3 of [9]. We indicate where they appear in [13] so that their proofs may be found easily by the reader.

2.4. LEMMA (see [13, (1.4)]). *Let A be a nondegenerate subcontinuum of 2^X. Then A is an order arc if and only if $A, B \in A$ implies $A \subset B$ or $B \subset A$.*

2.5. LEMMA (see [13, (1.5) and (1.6)]). *If α is an order arc in 2^X, then $\bigcap \alpha$ and $\bigcup \alpha$ are elements of α and, in fact, are the two end points of α.*

2.6. LEMMA (see [13, (1.11)]). *If $\alpha \in \Gamma(2^X)$ such that $\bigcap \alpha \in C(X)$, then $\alpha \subset C(X)$. Hence $\Gamma(C(X)) = \{\alpha \in \Gamma(2^X)\colon \bigcap \alpha \in C(X)\}$.*

2.7. LEMMA (see [13, (1.25)]). *Let $A_0, A_1 \in 2^X$. Then the following are equivalent:*

(1) *For any Whitney map ω, there exists some $\sigma \in S_\omega(2^X)$ such that $\sigma(0) = A_0$ and $\sigma(1) = A_1$;*

(2) *There exists some $\alpha \in \Gamma(2^X)$ such that $\bigcap \alpha = A_0$ and $\bigcup \alpha = A_1$;*

(3) *$A_0 \subset A_1$ and each component of A_1 intersects A_0.*

Next we recall some notions which will be used in case X is a Peano continuum. A metric d for X is said to be *convex* provided that, given $x, y \in X$, there exists some $z \in X$ such that $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$. If d is convex, then for any two distinct points $x, y \in X$, there is some arc $\gamma \subset X$ which is isometric to the interval $[0, d(x, y)]$ and whose end points are x and y [11]. Consequently, *balls with respect to a convex metric, that is, sets of the form $\{y \in X: d(x, y) \leqslant t\}$, $x \in X$, $t \geqslant 0$, are always arcwise connected.*

If X is a Peano continuum, then there is a convex metric for X which yields the original topology ([1] or [12]). Thus, for the sake of simplicity, we will assume that the given metrics for Peano continua are convex.

For any $\varepsilon \geqslant 0$ and any $A \in 2^X$, we define

$$K_\varepsilon(A) = \bigcup_{a \in A} \{x \in X: d(x, a) \leqslant \varepsilon\}.$$

We will use the following well known result.

2.8. Lemma. *If a continuum X has a convex metric d, then the function $\varphi: 2^X \times [0, +\infty) \to 2^X$ defined by $\varphi(A, t) = K_t(A)$ is continuous.*

For an arbitrary metric d, the function φ defined above need not be continuous. For necessary and sufficient conditions for such "continuity of balls", see [14]. Also note that, if X has a convex metric, then, for any $\alpha \in \Gamma(C(X))$ and for any $t \geqslant 0$, $\{\varphi(A, t): A \in \alpha\} \in \Gamma(C(X))$.

We conclude this section by observing that the proof of Theorem 5.2 uses Theorem 2.9 below, which is a version of a recent result by Toruńczyk. The proof of this theorem utilizes a considerable amount of Q-manifold theory, but its application in this paper demands only an understanding of the following basic terminology. A closed subset A of a separable metric space Y is a *Z-set in Y* provided that, for each $\varepsilon > 0$, there exists a mapping f_ε: $Y \to Y \setminus A$ such that $d(f_\varepsilon(y), y) < \varepsilon$ for each $y \in Y$. A mapping $g: Y \to Y$ is a *Z-map* provided that $g[Y]$ is a Z-set in Y. (Basic properties of Z-sets may be found in [3].)

2.9. Theorem [16]. *Let Y be a compact* AR. *If the identity map of Y onto Y is the uniform limit of Z-maps, then $Y \approx Q$.*

3. The order arc spaces as retracts.

The results of this section concern the existence of retractions onto the spaces of order arcs. In the next section we use these results to characterize when the spaces of order arcs are in fact absolute retracts.

3.1. Lemma. *There exists a retraction $R: \Gamma(2^X) \to \Gamma(C(X))$ if and only if there exists a retraction $r: 2^X \to C(X)$.*

Proof. Suppose the retraction R is given. Let $i: 2^X \to \Gamma(2^X)$ be the natural embedding $i(A) = \{A\}$. Let $\varrho: \Gamma(C(X)) \to C(X)$ be defined by $\varrho(\alpha) = \bigcap \alpha$. By the proof of Theorem 2.2, $i\varrho$ is a retraction. It therefore follows easily that $r = \varrho Ri$: $2^X \to C(X)$ is a retraction.

Now suppose the retraction r is given. Note that, for any segment $\sigma \in S_\omega(2^X)$ and any $t \in [0, 1]$, $r \circ \sigma([0, t])$ is a continuum. Let f_ω be the homeomorphism from Theorem 2.1. Since union is continuous [9, p. 23], it is routine to verify that the function $\gamma: \Gamma(2^X) \times [0, 1] \to 2^X$ defined by

$$\gamma(\alpha, t) = \bigcup \{r \circ \sigma(\tau): \tau \in [0, t], \sigma = f_\omega^{-1}(\alpha)\}$$

is continuous.

Define R by $R(\alpha) = \{\gamma(\alpha, t): t \in [0, 1]\}$. If α is any fixed order arc in 2^X, then $R(\alpha)$ is the continuous image of $[0, 1]$ and thus a continuum. Moreover, if $t_1 < t_2$, then by definition $\gamma(\alpha, t_1) \subset \gamma(\alpha, t_2)$. Thus by Lemma 2.4 $R(\alpha)$ is an order arc. Since $\bigcap R(\alpha) = \gamma(\alpha, 0) = r(\bigcap \alpha) \in C(X)$, by Lemma 2.6 $R(\alpha) \in \Gamma(C(X))$. The continuity of R follows immediately from the continuity of γ. Since r is a retraction, it is clear that R is a retraction. ■

3.2. Corollary. *If X is a Peano continuum, then $\Gamma(C(X))$ is a retract of $\Gamma(2^X)$.*

Proof. If X is a Peano continuum, then it follows from Theorem 1.2 that $C(X)$ is a retract of 2^X. ■

Remark. $C(X)$ is not always a retract of 2^X [7], so $\Gamma(C(X))$ is not always a retract of $\Gamma(2^X)$. At the present time, there is no known example of a non-locally connected continuum X for which $C(X)$ *is* a retract of 2^X.

3.3. Lemma. *For any continuum X, $F_1(\Gamma(2^X))$ is a retract of $2^{\Gamma(2^X)}$.*

Proof. Let f_ω be the homeomorphism from $S_\omega(2^X)$ onto $\Gamma(2^X)$ defined in 2.1. For each $\mathscr{A} \in 2^{\Gamma(2^X)}$ and each $t \in [0, 1]$, let

$$\gamma(\mathscr{A}, t) = \bigcup \{[f_\omega^{-1}(\alpha)](t): \alpha \in \mathscr{A}\}.$$

By using the continuity of f_ω^{-1} and of the union function [9, p. 23], it follows easily that the formula above defines a continuous function $\gamma: 2^{\Gamma(2^X)} \times [0, 1] \to 2^X$. For each $\mathscr{A} \in 2^{\Gamma(2^X)}$, let

$$R(\mathscr{A}) = \{\gamma(\mathscr{A}, t): t \in [0, 1]\}.$$

Since γ is continuous, $R(\mathscr{A})$ is a subcontinuum of 2^X for each $\mathscr{A} \in 2^{\Gamma(2^X)}$. Also, for each $\mathscr{A} \in 2^{\Gamma(2^X)}$, $\gamma(\mathscr{A}, t_1) \subset \gamma(\mathscr{A}, t_2)$ if $t_1 \leqslant t_2$. Thus, by Lemma 2.4, we have that $R(\mathscr{A}) \in \Gamma(2^X)$ for each $\mathscr{A} \in 2^{\Gamma(2^X)}$. Also, since γ is continuous, R is continuous. Therefore, by letting $r(\mathscr{A}) = \{R(\mathscr{A})\}$ for each $\mathscr{A} \in 2^{\Gamma(2^X)}$, we see that r is the desired retraction from $2^{\Gamma(2^X)}$ onto $F_1(\Gamma(2^X))$. ■

4. Locally connected spaces of order arcs.

In 4.1 and 4.2 below we assume that X is a Peano continuum whose metric d is convex. In Section 2 we noted that, under these assumptions, the function $\varphi: 2^X \times [0, +\infty) \to 2^X$ defined by $\varphi(A, t) = K_t(A)$ is continuous. We now consider a similar function $\Phi: \Gamma(2^X) \times [0, +\infty) \to \Gamma(2^X)$, defined by

$$\Phi(\alpha, t) = \{K_t(A): A \in \alpha\}.$$

We first note that the range of Φ is indeed $\Gamma(2^X)$. The proof of this statement uses Lemma 2.4 in the same way this lemma was used to prove similar statements in 3.1 and 3.3.

4.1. Lemma. *The function Φ defined above is continuous.*

Proof. The lemma follows easily by using the homeomorphism f_ω of 2.1 and by using Lemma 2.8. ■

4.2. Lemma. *For any $\alpha \in \Gamma(2^X)$ and any $\varepsilon > 0$, the set*

$$\mathscr{B} = \{\beta \in \Gamma(2^X):\ H^2(\alpha, \beta) < \varepsilon\}$$

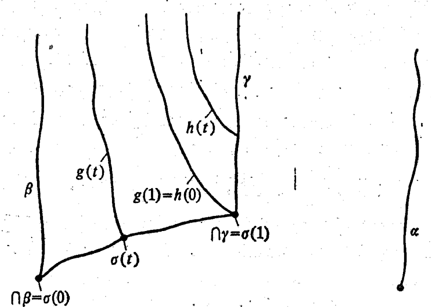
is pathwise connected.

Proof. Choose any $\beta \in \mathscr{B}$. We will prove that there is a path in $\mathscr{B}$ from α to β. Let $r = H^2(\alpha, \beta)$, so $r < \varepsilon$. Then for any $t \in [0, r]$, it is clear that

$$H^2(\{K_t(A):\ A \in \alpha\}, \alpha) \leqslant r < \varepsilon .$$

Thus we can define a path $f\colon [0, r] \to \mathscr{B}$ by $f(t) = \Phi(\alpha, t)$, where Φ is the map from Lemma 4.1. Let $\gamma = f(r) = \{K_r(A):\ A \in \alpha\}$. Since $f(0) = \alpha$, f is a path from α to γ.

We will complete the proof by showing that there is also a path in $\mathscr{B}$ from β to γ.



The picture above illustrates the strategy. We begin the construction by proving that there is a segment from $\bigcap\beta$ to $\bigcap\gamma$. It is an elementary exercise to show that $H^2(\alpha, \beta) = r$ implies that $H(\bigcap\alpha, \bigcap\beta) \leqslant r$. It is also obvious that $\bigcap\gamma = K_r(\bigcap\alpha)$. Thus we have

(a) $\bigcap\beta \subset \bigcap\gamma$.

Now let C be any component of $\bigcap\gamma$ and let c be any element of C. Since $\bigcap\gamma = K_r(\bigcap\alpha)$, there exists some $a \in \bigcap\alpha$ such that $d(a, c) \leqslant r$. Let

$$D = \{x \in X:\ d(a, x) \leqslant r\} .$$

As was noted in Section 2, since d is a convex metric, D is arcwise connected. Moreover, by definition $D \subset K_r(\bigcap\alpha) = \bigcap\gamma$, so $D \subset C$. But since $H(\bigcap\alpha, \bigcap\beta) \leqslant r$, there exists some $b \in \bigcap\beta$ such that $d(a, b) \leqslant r$ and, thus, $b \in C$. Since C was arbitrary, we have

(b) Each component of $\bigcap\gamma$ contains some point of $\bigcap\beta$.

Therefore, by Lemma 2.7, it follows from (a) and (b) that there is some segment $\sigma \in S_\omega(2^X)$ such that $\sigma(0) = \bigcap\beta$ and $\sigma(1) = \bigcap\gamma$.

Define a map $g\colon [0, 1] \to \Gamma(2^X)$ by $g(t) = \{\sigma(t) \cup B:\ B \in \beta\}$. The continuity of g is obvious and it is easy to use Lemma 2.4 to verify that, for each $t \in [0, 1]$, $g(t)$ is indeed an element of $\Gamma(2^X)$. Moreover, for any $t \in [0, 1]$ and any $A \in \alpha$, $\sigma(t) \subset \bigcap\gamma \subset K_r(A)$. Thus it follows that if $A \in \alpha$ and $B \in \beta$ such that $H(A, B) \leqslant r$, then $H(A, \sigma(t) \cup B) \leqslant r$. Therefore, it is clear that $H^2(\alpha, g(t)) < \varepsilon$, that is, $g(t) \in \mathscr{B}$, for each $t \in [0, 1]$. Finally, note that $g(0) = \beta$ and, although $g(1)$ need not be the same as γ, we have $\bigcap g(1) = \bigcap\gamma$.

Define a map $h\colon [0, 1] \to \Gamma(2^X)$ as follows: Let η be any homeomorphism of $[0, 1]$ onto γ such that $h(0) = \bigcap\gamma$ and $h(1) = \bigcup\gamma$, and let

$$h(t) = \{\eta(s):\ s \leqslant t\} \cup \{\eta(t) \cup B:\ B \in \beta\} .$$

It is again routine to prove that h is continuous and that $h(t) \in \mathscr{B}$ for each $t \in [0, 1]$. Note, moreover, that $h(0) = g(1)$. Finally, it is an easy exercise to show that $H^2(\alpha, \beta) = r$ implies that $\bigcup\beta \subset K_r(\bigcup\alpha)$. Then, since $K_r(\bigcup\alpha) = \bigcup\gamma = \eta(1)$, it follows that $\eta(1) \cup B = \eta(1)$ for each $B \in \beta$ and hence $h(1) = \gamma$.

Therefore, the maps g and h can be used to define the desired path in $\mathscr{B}$ from β to γ. ■

We now combine our previous results to conclude that the spaces of order arcs are locally connected (in fact, are AR's) precisely when X is a Peano continuum.

4.3. Theorem. *Let X be a continuum. The following are equivalent:*

(a) X is locally connected.

(b) $\Gamma(2^X)$ is locally connected.

(c) $\Gamma(2^X)$ is an AR.

(d) $\Gamma(C(X))$ is locally connected.

(e) $\Gamma(C(X))$ is an AR.

Remark. Theorem 5.2 in the next section will allow us to add to this list of equivalent conditions the following:

(f) $\Gamma(2^X)$ is a Hilbert cube.

Proof of theorem. Statement (a) implies statement (b) by Lemma 4.2. Statement (b) implies statement (c) as a consequence of Theorem 1.2 and Lemma 3.3. Statement (c) implies statement (a) as a consequence of Theorems 2.2 and 1.2. Therefore, (a), (b) and (c) are equivalent.

Together (a) and (c) imply (e) as a consequence of Corollary 3.2. Clearly, (e) implies (d). Finally, (d) implies (a), again as a consequence of Theorems 2.2 and 1.2. ■

5. $\Gamma(2^X)$ a Hilbert cube.

The following lemma is used in the proof of Theorem 5.2. The proof of the lemma is an adaptation of [6, Lemma 5.4] and [15, Lemma 4.2].

5.1. LEMMA. *Let X be a Peano continuum and let Y be a closed subspace of X having non-empty interior. Then*

$$\Gamma_Y(2^X) = \{\alpha \in \Gamma(2^X)\colon\ Y \subset \bigcap \alpha\}$$

is a Z-set in $\Gamma(2^X)$. If Y contains no free arcs in X, then

$$\Gamma_Y(C(X)) = \{\alpha \in \Gamma(C(X))\colon\ Y \subset \bigcap \alpha\}$$

is a Z-set in $\Gamma(C(X))$.

Proof. Suppose first that Y contains an arc J which is free in X. Without loss of generality we assume that X has a convex metric and that J is isometric to the interval $[-1, 1]$ with the usual metric; so, in order to simplify notation, we identify J with this interval. Let

$$M = \{A \in 2^X\colon\ 0 \notin A \text{ and } A \cap [-1, 1] \neq \emptyset\}.$$

For each $A \in M$, if $A \cap [-1, 0] \neq \emptyset$, let $a^- = \sup A \cap [-1, 0]$, and if $A \cap [0, 1] \neq \emptyset$, let $a^+ = \inf A \cap [0, 1]$. The following defines a map $f\colon M \to 2^X$.

$$f(A) = \begin{cases} \{a^+\} & \text{if } A \cap [-1, 0] = \emptyset, \\ \{a^-\} & \text{if } A \cap [0, 1] = \emptyset, \\ \{a^+, a^-\} & \text{if } A \cap [-1, 0] \neq \emptyset \neq A \cap [0, 1]. \end{cases}$$

Now let $N = \{(A, B) \in 2^X \times 2^X\colon\ A \subset B\}$. It is easy to verify that the following defines a map $g\colon N \to 2^X$. Whenever $A \cap [-1, 1] \neq \emptyset$, let $t = \inf\{|s|\colon\ s \in A \cap [-1, 1]\}$. Then let

$$g(A, B) = \begin{cases} B & \text{if } A \cap [-1, 1] = \emptyset, \\ B \cup f(A) & \text{if } \tfrac{1}{2} \leqslant t \leqslant 1, \\ (B \cup f(A) \cup [2t-1, 1-2t]) \setminus (2t-1, 1-2t) & \text{if } 0 < t < \tfrac{1}{2}, \\ (B \cup [-1, 1]) \setminus (-1, 1) & \text{if } t = 0. \end{cases}$$

Note that, for every $A \in 2^X$, $J \not\subset g(A, A)$. Now define a map $h\colon \Gamma(2^X) \to \Gamma(2^X) \setminus \Gamma_Y(2^X)$ by

$$h(\beta) = \{g(\bigcap \beta, B)\colon\ B \in \beta\}.$$

It follows easily from Lemma 2.4 that, for any $\beta \in \Gamma(2^X)$, $h(\beta)$ is indeed an element of $\Gamma(2^X)$, and it follows from the definition of g that $h(\beta) \notin \Gamma_Y(2^X)$. The continuity of h follows from the continuity of g and of the map which takes β to $\bigcap \beta$ [13, (1.203.3)]. Finally, given any $\varepsilon > 0$, it is clear that J can be chosen sufficiently small that h is within ε of the identity map on $\Gamma(2^X)$. We have thus proved for this case that $\Gamma_Y(2^X)$ is a Z-set in $\Gamma(2^X)$.

Now consider the case that Y contains no free arcs in X. It is then shown in the proof of [6, Lemma 5.4] that, given $\varepsilon > 0$, there is a map $g\colon 2^X \to 2^X \setminus 2^X_Y$, where $2^X_Y = \{A \in 2^X\colon\ Y \subset A\}$, such that

(1) if $A \subset B$, then $g(A) \subset g(B)$,

(2) if $A \in C(X)$, then $g(A) \in C(X)$, and

(3) g is within ε of the identity on 2^X.

Then a map $h\colon \Gamma(2^X) \to \Gamma(2^X) \setminus \Gamma_Y(2^X)$ is defined by

$$h(\beta) = \{g(B)\colon\ B \in$$

It follows from (3) that h is within ε of the identity on $\Gamma(2^X)$. Furthermore, it follows from (2) that $h(\Gamma(C(X))) \subset \Gamma(C(X))$, so the restriction $h|_{\Gamma(C(X))}$ provides a map $\Gamma(C(X)) \to \Gamma(C(X)) \setminus \Gamma_Y(C(X))$ which is within ε of the identity on $\Gamma(C(X))$. ∎

Ideas used in the proof of the following theorem are similar to those used in [16] to give a proof of the Curtis–Schori hyperspace theorem.

5.2. THEOREM. *Let X be a Peano continuum. Then $\Gamma(2^X) \approx Q$ and, if in addition X contains no free arc, then $\Gamma(C(X)) \approx Q$.*

Proof. In view of Theorems 2.9 and 4.3, it suffices to prove the following: Given $\varepsilon > 0$, there exists a Z-map $f\colon \Gamma(2^X) \to \Gamma(2^X)$ such that, for any $\alpha \in \Gamma(2^X)$, $H^2(\alpha, f(\alpha)) < \varepsilon$.

Assume that the metric d for X is convex. For given $\varepsilon > 0$, define f to be $f(\alpha) = \{K_{2\varepsilon/3}(A)\colon\ A \in \alpha\}$. By Lemma 4.1, f is continuous. It is also clear that, for any $\alpha \in \Gamma(2^X)$, $H^2(\alpha, f(\alpha)) \leqslant 2\varepsilon/3 < \varepsilon$. Since d is convex, $f[\Gamma(C(X))] \subset \Gamma(C(X))$. Finally we will show that f and $f|\Gamma(C(X))$ are Z-maps, that is, their images are Z-sets in $\Gamma(2^X)$ and $\Gamma(C(X))$ respectively.

There exists a finite set $\{x_1, \ldots, x_n\} \subset X$ so that the set of closed balls $B_i = \{x \in X\colon\ d(x, x_i) \leqslant \varepsilon/3\}$, $i = 1, \ldots, n$, covers X. By Lemma 5.1, $\Gamma_{B_i}(2^X)$ is a Z-set in $\Gamma(2^X)$ for each i, and, if X contains no free arcs, $\Gamma_{B_i}(C(X))$ is a Z-set in $\Gamma(C(X))$. Let $\alpha \in \Gamma(2^X)$ and let $a \in \bigcap \alpha$. Since $\{B_i\}$ covers X, there exists some i such that $d(a, x_i) \leqslant \varepsilon/3$. If $x \in B_i$, then $d(x, a) \leqslant 2\varepsilon/3$, hence $B_i \subset K_{2\varepsilon/3}(\bigcap \alpha) = \bigcap f(\alpha)$. Therefore, since α was arbitrary, $f[\Gamma(2^X)] \subset \Gamma_{B_1}(2^X) \cup \ldots \cup \Gamma_{B_n}(2^X)$. In like manner $f[\Gamma(C(X))] \subset \Gamma_{B_1}(C(X)) \cup \ldots \cup \Gamma_{B_n}(2^X)$. Since a closed subset of a finite union of Z-sets is again a Z-set (see [3, Theorem 3.1]), $f[\Gamma(2^X)]$ is a Z-set in $\Gamma(2^X)$ and, if X contains no free arcs, $f[\Gamma(C(X))]$ is a Z-set in $\Gamma(C(X))$. ∎

By Theorem 4.3, if X is any Peano continuum, then $\Gamma(C(X))$ is an AR. Theorem 5.2 gives a sufficient condition for $\Gamma(C(X))$ to be a Hilbert cube. The following example indicates that $\Gamma(C(X))$ is not a Hilbert cube for every Peano continuum X.

5.3. EXAMPLE. Let S^1 be the unit circle $S^1 = \{(x, y) \in R^2\colon\ x^2 + y^2 = 1\}$. Then $\Gamma(C(S^1)) \not\approx Q$.

Proof. It is well known that, for any $q \in Q$, $\{q\}$ is a Z-set in Q. Thus it suffices to show that $\{\{S^1\}\}$ is not a Z-set in $\Gamma(C(S^1))$. The function $g\colon \Gamma(C(S^1)) \to F_1(C(S^1))$ defined by $g(\gamma) = \{\bigcap \gamma\}$ is continuous [13, (1.203.3)]. Hence, for any given $\varepsilon > 0$, there exists some $\delta > 0$ such that if $\gamma \in \Gamma(C(S^1))$ and $H^2(\gamma, \{A\}) < \delta$ for some $A \in C(S^1)$, then $H^2(g(\gamma), \gamma) < \varepsilon$. Now suppose $\{\{S^1\}\}$ is a Z-set in $\Gamma(C(S^1))$. Then, for δ as above, there exists a mapping $f\colon \Gamma(C(S^1)) \to \Gamma(C(S^1)) \setminus \{\{S^1\}\}$ such that f is within δ of the identity map on $\Gamma(C(S^1))$. Let k be the restriction of $g \circ f$ to $F_1(C(S^1))$. Note that

(1) k maps $F_1(C(S^1))$ into $F_1(C(S^1)) \setminus \{\{S^1\}\}$.

Let $\{A\} \in F_1(C(S^1))$. Since f is within δ of the identity, $H^2(f(\{A\}), \{A\}) < \delta$.

Hence, taking $\gamma = f(\{A\})$, we have from the definition of δ that $H^2(g(\gamma), \gamma) < \varepsilon$. Thus, by the triangle inequality, $H^2(k(\{A\}), \{A\}) < \varepsilon + \delta$. Hence

(2) k is within $\varepsilon+\delta$ of the identity of $F_1(C(S^1))$.

Finally, recall that

(3) $F_1(C(S^1))$ is naturally isometric to $C(S^1)$.

Since ε and δ may be chosen as small as we please, we see from (1), (2) and (3) that $\{S^1\}$ is a Z-set in $C(S^1)$. But it is well known that $C(S^1)$ is a 2-cell with S^1, as a point of $C(S^1)$, in its interior (see [13, (0.55)]). Thus $\{S^1\}$ cannot be a Z-set in $C(S^1)$ [8, VI 2, p. 75]. The contradiction proves that $\Gamma(C(S^1)) \not\approx Q$.

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A generalization of a theorem of Skala

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Abstract. Let $n \geqslant 1$, let (A, f) be some algebra of type $n+1$ satisfying

(a) $f(f(x_0, \ldots, x_n), y_1, \ldots, y_n) = f(x_0, f(x_1, y_1, \ldots, y_n), \ldots, f(x_n, y_1, \ldots, y_n))$ for any $x_0, \ldots, x_n, y_1, \ldots, y_n \in A$

and put

$$C := \{x \in A \mid f(x, x_1, \ldots, x_n) = x \text{ for any } x_1, \ldots, x_n \in A\},$$

$$S_i(M) := \{x \in A \mid f(x, x_1, \ldots, x_n) = x_i \text{ for any } x_1, \ldots, x_n \in M\} \quad (1 \leqslant i \leqslant n,\ M \subseteq A)$$

and

$$S(M) := S_1(M) \cup \ldots \cup S_n(M) \quad (M \subseteq A).$$

The following result of H. Skala (cf. [1]) is generalized:

THEOREM 1. *Let* $|C| \geqslant 3$ *and assume* $f(x_0, \ldots, x_n) \in \{x_0, \ldots, x_n\}$ *for any* $x_0, \ldots, x_n \in A$. T.f.a.e.:

(i) $a \in A \setminus C$.

(ii) $a \in S(C \cup \{a\})$.

In the following if $x \in A$ or if $x \subseteq A$ then $x(i)$ denotes the sequence $x, \ldots, x$ of length i $(1 \leqslant i \leqslant n)$.

LEMMA 1. *Let* $B \subseteq A$ *satisfying*

(b) $f(x, y, \ldots, y) = x$ *for any* $x, y \in B$

and let $a \in A$ *such that* (α) *and* (β):

(α) $f(a, x, \ldots, x) = x$ *for any* $x \in B$.

(β) $f(a, B(i-1), a, B, \ldots, B) \subseteq B \cup \{a\}$ *for any* $i = 1, \ldots, n$.

Further let $a_1, \ldots, a_n, b, b_1, \ldots, b_n \in B$ *and assume* $f(a, a_1, \ldots, a_n) = b$. *Finally, suppose* $b_i = b$ *whenever* $a_i = b$ $(1 \leqslant i \leqslant n)$. *Then* $f(a, b_1, \ldots, b_n) = b$.

Proof. We prove $c_i := f(a, b_1, \ldots, b_i, a_{i+1}, \ldots, a_n) = b$ for any $i = 0, \ldots, n$ by induction on i. $c_0 = b$ is our hypothesis. Now, let $0 < j \leqslant n$ and suppose $c_{j-1} = b$ to be already proved. If $a_j = b$ then $b_j = b = a_j$ whence $c_j = b$. If, otherwise, $a_j \neq b$ then $f(f(a, b_1, \ldots, b_{j-1}, a, a_{j+1}, \ldots, a_n), a_j, \ldots, a_j) = b$ by (a), (b) and (α) whence $f(a, b_1, \ldots, b_{j-1}, a, a_{j+1}, \ldots, a_n) = b$ by (β), (α) and (b) and therefore

$$c_j = f(f(a, b_1, \ldots, b_{j-1}, a, a_{j+1}, \ldots, a_n), b_j, \ldots, b_j) = f(b, b_j, \ldots, b_j) = b$$

by (a), (b) and (α).