Borel-approachable functions

by

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Abstract. The operation of approach is defined and applied to the Borel-measurable functions to obtain the Borel-approachable (BA) functions. The BA functions constitute a class closed under approach and strictly larger than the Borel-measurable functions, but with many of the same properties. It is at least as large as the class of Borel-programmable (BP) functions defined by Blackwell. Like the BP-sets, the class of BA-sets (those with BA indicators) is a σ-field of absolutely measurable sets and is closed under operations (A). Moreover, if $f(x, y)$ is a real valued BA-function of two real variables, then $\int f(x, y)dy$ is a BA-function of x on the set where the integral is defined

1. Orientation. It often occurs in the analysis of problems involving probability. that the σ-field $\mathscr{J}$ of the original probability space is found to be deficient, i.e., at some point in the analysis a function on the probability space is constructed which is not measurable with respect to this σ-field. Such a function may be the first time a stochastic process hits a set (Dellacherie [6], Meyer [10]) or the optimal cost function or a nearly optimal policy for a dynamic programming model (Blackwell, Freedman and Orkin [4], Bertsekas and Shreve [1]). At this point probability theory and descriptive set theory meet. The determination of a suitable replacement $\hat{\mathfrak{J}}$ for $\mathfrak{J}$ is not merely a matter of finding a σ-field with respect to which the offending function is measurable. The σ-field $\hat{\mathfrak{J}}$ should be judiciously chosen so that no further revisions of the probability space are necessary. For example, let X be a Polish space with Borel σ-field $\mathscr{B}_X$ and Y another Polish space (1). Suppose $B \subset X \times Y$ is $\mathscr{B}_X \times \mathscr{B}_Y$ measurable and $\mathrm{proj}_X(B)$ is the projection of B on X. It is useful to know that there is a function φ: $\mathrm{proj}_X(B) \to Y$ with graph in B, and the modern version of the Jankov–von Neumann selection theorem [7, 11, 1] guarantees that such a φ which is analytically measurable can be found. Analytic measurability is measurability with respect to the σ-field $\mathscr{A}_X$ generated by the analytic sets in X and the σ-field $\mathscr{B}_Y$. It is possible for $\mathrm{proj}_X(B)$ to equal X and a Borel measurable φ still not exist [2], so $\mathscr{B}_X$ is deficient for selection purposes, and we might be tempted to replace it by $\mathscr{A}_X$. However, if X, Y and Z are Polish spaces and f: $X \to Y$ and g: $Y \to Z$ are analytically measur-

(1) We use $\mathscr{B}$ throughout, with or without subscript, to denote the Borel σ-field of a Polish space.

able, $g \circ f$ can fail to be, so analytic measurability loses much of its appeal. The proof of a selection result of Brown and Purves [5, Theorem 2], for instance, composes two analytically measurable functions.

What, then, are desirable properties of the replacement $\hat{\mathfrak{J}}$ for $\mathfrak{J}$? If $\mathfrak{J}$ is the Borel σ-field of some Polish space, four properties suggest themselves:

(1) $\hat{\mathfrak{J}}$ should contain the analytic sets;

(2) $\hat{\mathfrak{J}}$ should be contained in the σ-field of absolutely measurable sets;

(3) the composition of two $\hat{\mathfrak{J}}$-measurable functions should be $\hat{\mathfrak{J}}$-measurable (²);

(4) if A is an $\hat{\mathfrak{J}}$-measurable subset of the Polish space X and $P(X)$ is the space of probability measures on $(X, \mathscr{B}_X)$ with the weak topology [12, 1], then the mapping $p \to p(A)$ from $P(X)$ to $[0, 1]$ should be $\hat{\mathfrak{J}}$-measurable.

Property (4) is discussed at some length in [13]. Although not necessary, properties (3) and (4) are sufficient to guarantee that if f is a real-valued $\hat{\mathfrak{J}}$-measurable function of two real variables, then $\int_{-\infty}^{\infty} f(x, y)\,dy$ is an $\hat{\mathfrak{J}}$-measurable function of x on the set (itself $\hat{\mathfrak{J}}$-measurable) where the integral exists (see Corollary 5.1). Some σ-fields in Polish spaces, in increasing order, are the Borel σ-field $\mathscr{B}$, the analytic σ-field $\mathscr{A}$, the σ-field of C-sets $\mathscr{C}$, the Borel-programmable σ-field BP, the Borel-approachable σ-field BA, and the σ-field $\mathscr{U}$ of absolutely measurable sets. Properties (2)–(4) are had by $\mathscr{B}$, properties (1), (2) and (4) by $\mathscr{A}$, properties (1)–(4) by $\mathscr{C}$ [13], properties (1)–(3) by BP [3], properties (1)–(4) by BA (Theorems 3–5), and properties (1)–(4) by $\mathscr{U}$. It is unknown if BP has property (4).

2. Definitions. Throughout this article, X, Y and Z will denote Polish spaces. Given a bounded, closed, real interval I, we denote by I^∞ the product of countably many copies of I. With the product topology, I^∞ is a Polish space. A partial ordering on I^∞ is given by

$$(u_1, u_2, \ldots) \leqslant (v_1, v_2, \ldots) \Leftrightarrow u_i \leqslant v_i \quad \forall i .$$

We will also have occasion to consider $\{0, 1\}^\infty = \{0, 1\} \times \{0, 1\} \times \ldots$, to which the above remarks also apply. The ith component of a function f with range in I^∞ will be denoted f^i. We denote by ω_1 the first uncountable ordinal.

A collection of functions $\{g_\alpha |\ \alpha < \omega_1\}$ from X to I^∞ is called an *approach* if

(1) $$\alpha \leqslant \beta \Rightarrow g_\alpha(x) \leqslant g_\beta(x) \quad \forall x \in X ,$$

(2) $$g_\alpha(x) = g_{\alpha+1}(x) \Rightarrow g_\alpha(x) = g_\beta(x) \quad \forall \beta \geqslant \alpha .$$

The function

(3) $$g(x) = \sup_{\alpha < \omega_1} g_\alpha(x)$$

is said to be approached by $\{g_\alpha\}$. We observe that if $g_\alpha = (g_\alpha^1, g_\alpha^2, \ldots)$, where each g_α^i maps into I, then for fixed x and i, there can be only countably many α for which

(²) We assume a definition for $\hat{\mathfrak{J}}$ independent of the particular Polish space involved. Thus, we use "$\hat{\mathfrak{J}}$" in the same way we use "Borel σ-field" or "σ-field of absolutely measurable sets".

$g_\alpha^i(x) < g_{\alpha+1}^i(x)$. Thus we can choose $\alpha(x) < \omega_1$ such that $g_{\alpha(x)}(x) = g_{\alpha(x)+1}(x)$. From (2) and (3) we then have $g_{\alpha(x)}(x) = g(x)$. If each g_α is Borel, we say the approach is Borel and g is strictly Borel-approachable (SBA). We call Borel-approachable (BA) any function $d \circ g$, where $d = I^\infty \to Y$ is Borel and g is SBA.

It is easily seen that every Borel function is BA. If $f\colon X \to Y$ is Borel, use Urysohn's theorem to choose a homeomorphism φ from Y into I^∞ and define $g_\alpha = \varphi \circ f$, $d = \varphi^{-1}$. Let g be given by (3) and observe that $f = d \circ g$.

To see that there are more BA functions that Borel functions, we compare the BA functions to the Borel-programmable (BP) functions of [3]. We repeat the definition of the BP functions here. Consider $\{0, 1\}^\infty = \{0, 1\} \times \{0, 1\} \times \ldots$, which, with the product topology, is a subspace of $[0, 1]^\infty$. A function $p\colon \{0, 1\}^\infty \to \{0, 1\}^\infty$ is a *program* if

(4) $$p(u) \geqslant u \quad \forall u \in \{0, 1\}^\infty .$$

Given a program, define

(5) $$p_0(u) = p(u) ,$$

(6) $$p_\alpha(u) = p\big(\sup_{\beta < \alpha} p_\beta(u)\big) .$$

If p is Borel, the function p_{ω_1} is BP, as is any function $d \circ p_{\omega_1} \circ e$, where $e\colon X \to \{0, 1\}^\infty$ and $d\colon \{0, 1\}^\infty \to Y$ are Borel. It is easily verified in such a case that $g_\alpha = p_\alpha \circ e$ is Borel for $\alpha < \omega_1$ and $\{g_\alpha |\ \alpha < \omega_1\}$ is a Borel approach to $p_{\omega_1} \circ e$. It follows that every BP function is BA. In particular, there are BA functions which are not Borel.

A function $g\colon X \to I^\infty$ is a *signifier* if $[0, 1] \subset I$ and there is a set $A \subset X$ such that

$$g(x) = \begin{cases} (1, 1, \ldots) & \text{if } x \in A , \\ (0, 0, \ldots) & \text{if } x \in X - A . \end{cases}$$

The set A is said to be *signified by* g. A function $f\colon X \to I$ is an *indicator* if there is a set $A \subset X$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in A , \\ 0 & \text{if } x \in X - A . \end{cases}$$

The set A is *indicated* by f. A set $A \subset X$ is defined to be BA if A has an SBA signifier. We subsequently show that A is BA if and only if A has a BA indicator. This result relates the BA sets to the BP sets, which are defined to be those with BP indicators

3. Properties of BA sets and functions. As mentioned earlier, the BA functions share many properties with the BP functions. Our proofs of some of these shared properties are only slight modifications of proofs found in [3]. We begin with two lemmas needed to explore the relation between the BA sets and the BA functions.

LEMMA 1. *A set A is* BA *if and only if given any two bounded sequences $\{r_i\}$, $\{s_i\}$ of real numbers, there is an* SBA *function f such that*

$$f^i(x) = \begin{cases} r_i & \text{if } x \in A , \\ s_i & \text{if } x \notin A . \end{cases}$$

Proof. Suppose A is a BA set signified by g. Let $\{g_\alpha|\ \alpha<\omega_1\}$ be a Borel approach to g. We may assume without loss of generality that each g_α takes values in $[-1,1]^\infty$ and $r_i\geqslant 0$, $s_i\geqslant 0$ for every i. For each nonlimit ordinal $\alpha+1$ and positive integer i, define

$$f^i_{\alpha+1}(x)=\begin{cases} r_i & \text{if } g_\alpha(x)=g_{\alpha+1}(x)=(1,1,\dots),\\ s_i & \text{if } g_\alpha(x)=g_{\alpha+1}(x)=(0,0,\dots),\\ \tfrac12 g^i_{\alpha+1}(x)-\tfrac12 & \text{otherwise.}\end{cases}$$

For each limit ordinal α, define

$$f^i(x)=\sup_{\beta<\alpha} f^i_\beta(x),$$

where we take

$$f_0(x)=\tfrac12 g_0(x)-\tfrac12.$$

It is easily verified that $\{f_\alpha|\ \alpha<\omega_1\}$ is a Borel approach to the desired function f. Q.E.D.

Lemma 2. *For $k=1,2,\dots$, let $g_k\colon X\to I^\infty$ be an* SBA *function, and define f by*

$$f^i(x)=\sup_k g^i_k(x).$$

Then f is SBA.

Proof. For each k, let $\{g_{k\alpha}|\ \alpha<\omega_1\}$ be a Borel approach to g_k. Let $I=[a,b]$. There is some interval $I_k=[a_k,b_k]$ such that $g_{k\alpha}\colon X\to I_k^\infty$ for every α. We may assume that $b_k=b$. If $a_k<a-1$, we may replace $g_{k\alpha}$ by

$$g'_{k\alpha}(x)=\begin{cases} g_{k\alpha}(x) & \text{if } g_{k\alpha}(x)\geqslant a,\\ a-\dfrac{a-g_{k\alpha}(x)}{a-a_k} & \text{if } g_{k\alpha}(x)<a,\end{cases}$$

and thereby obtain a Borel approach $\{g'_{k\alpha}|\ \alpha<\omega_1\}$ to g_k with $g'_k\colon X\to[a-1,b]^\infty$ for every α.

We may therefore assume without loss of generality that $g_{k\alpha}\colon X\to[a-1,b]^\infty$ for every k and α. For each nonlimit ordinal $\alpha+1$ and each positive integer i, define

$$f^i_{\alpha+1}=\begin{cases} \sup_k g^i_{k,\alpha+1}(x) & \text{if } g_{k\alpha}(x)=g_{k,\alpha+1}(x)\quad \forall k,\\ \displaystyle\sum_{k=1}^{\infty}\frac{1}{2^k}g^i_{k,\alpha+1}(x) & \text{otherwise.}\end{cases}$$

For each limit ordinal α and each positive integer i, define

$$f^i_\alpha(x)=\sup_{\beta<\alpha} f^i_\beta(x).$$

Then $\{f_\alpha|\ \alpha<\omega_1\}$ is a Borel approach to f. Q.E.D.

Theorem 1. *The* BA *sets form a σ-field.*

Proof. It is apparent from Lemma 1 that the class of BA sets is closed under complementation. We show it is closed under countable unions. Let $A_1, A_2, \dots$ be a sequence of sets signified by the SBA functions $f_1, f_2, \dots$, respectively. The function f defined by

$$f^i(x)=\sup_k f^i_k(x)$$

is SBA by Lemma 2, and since f signifies $\bigcup_k A_k$, this set is BA. Q.E.D.

Lemma 3. *If g is* SBA *and B is a Borel subset of I^∞, then $g^{-1}(B)$ is a* BA *set.*

Proof. Suppose $\{g_\alpha|\ \alpha<\omega_1\}$ is a Borel approach to g. Assume without loss of generality that $I=[0,1]$. For each nonlimit ordinal $\alpha+1$, define

$$f_{\alpha+1}(x)=\begin{cases}(1,1,\dots) & \text{if } g_\alpha(x)=g_{\alpha+1}(x)\in B,\\ (0,0,\dots) & \text{if } g_\alpha(x)=g_{\alpha+1}(x)\notin B,\\ \tfrac12 g_{\alpha+1}(x)-\tfrac12 & \text{otherwise,}\end{cases}$$

and for each limit ordinal α, define f_α by

$$f^i_\alpha(x)=\sup_{\beta<\alpha} f^i_\beta(x).$$

Then $\{f_\alpha|\ \alpha<\omega_1\}$ is a Borel approach to the signifier of $g^{-1}(B)$. Q.E.D.

Theorem 2. *A function $g\colon X\to I^\infty$ is* SBA *if and only if it is measurable with respect to the σ-field of* BA *sets, i.e., $g^{-1}(B)$ is* BA *for every Borel B.*

Proof. In light of Lemma 3, we need only show that every BA measurable function is SBA. Assume without loss of generality that $I=[0,1]$. Given a positive integer k and rational numbers $r_1,\dots,r_k$ in $[0,1]$, the set

$$\{x|\ g(x)\geqslant(r_1,r_2,\dots,r_k,0,0,\dots)\}$$

is BA when g is BA measurable. By Lemma 1 there is an SBA function $I_{r_1,\dots,r_k}$ such that

$$I_{r_1,\dots,r_k}(x)=\begin{cases}(r_1,\dots,r_k,0,0,\dots) & \text{if } g(x)\geqslant(r_1,\dots,r_k,0,0,\dots),\\ (0,0,\dots) & \text{otherwise.}\end{cases}$$

Since

$$g^i(x)=\sup_{\substack{k\\ r_1,\dots,r_k}} I^i_{r_1,\dots,r_k}(x),$$

g is SBA by Lemma 2. Q.E.D.

Corollary 2.1. *Every* BA *function from X to I^∞ is* SBA.

Corollary 2.2. *A function $f\colon X\to Y$ is* BA *if and only if it is measurable with respect to the σ-field of* BA *sets.*

Proof. If f is BA, then by Theorem 2 it is measurable with respect to the BA σ-field. If f is measurable with respect to the BA σ-field, then by Corollary 2.1 $\varphi\circ f$ is SBA, where $\varphi\colon Y\to[0,1]^\infty$ is the homeomorphism of Urysohn's theorem. It follows that $f=\varphi^{-1}\circ\varphi\circ f$ is BA. Q.E.D.

COROLLARY 2.3. *A function* $f\colon X\to[a,b]$ *is* BA *if and only if there exists a collection* $\{f_\alpha |\ \alpha<\omega_1\}$ *of Borel functions from* X *to* $[a-1,b]$ *such that*

(7) $$\alpha\leqslant\beta \Rightarrow f_\alpha(x)\leqslant f_\beta(x)\quad \forall x\in X,$$

(8) $$f_\alpha(x)=f_{\alpha+1}(x) \Rightarrow f_\alpha(x)=f_\beta(x)\quad \forall\beta\geqslant\alpha,$$

(9) $$f(x)=\sup_{\alpha<\omega_1} f_\alpha(x).$$

Proof. Suppose f is BA. Let $g\colon X\to[a,b]^\infty$ be given by $g=(f,f,\ldots)$. Corollary 2.2 implies that g is BA, and Corollary 2.1 then implies that g is SBA. Let $\{g_\alpha |\ \alpha<\omega_1\}$ be a Borel approach to g. As in the proof of Lemma 2, we may assume $g_\alpha\colon X\to[a-1,b]^\infty$ for every α. If we define

$$f_\alpha(x)=\sum_{i=1}^{\infty}\frac{1}{2^i}g_\alpha^i(x),$$

then conditions (7)–(9) follow from (1)–(3).

Now suppose conditions (7)–(9) are given. Define $g_\alpha=(f_\alpha,f_\alpha,\ldots)$ so that $\{g_\alpha |\ \alpha<\omega_1\}$ is a Borel approach to $g=(f,f,\ldots)$. Compose the SBA function g with projection on the first coordinate to obtain the BA function f. Q.E.D.

As a further consequence of Corollary 2.2, we have that the limit of a sequence of BA functions is BA, as is the sum and product of real-valued BA functions.

THEOREM 3. *The* BA *sets and functions are absolutely measurable.*

Proof. Let $P(X)$ be the space of probability measures on the Borel subsets of X. Under the weak topology, $P(X)$ is also a Polish space. Let $\{g_\alpha |\ \alpha<\omega_1\}$ be a Borel approach on X to an SBA function g. Define $h_\alpha\colon P(X)\to I^\infty$ by

(10) $$h_\alpha(p)=\int g_\alpha dp,$$

where the integration is componentwise. The functions h_α are Borel and are nondecreasing with increasing α. For each p, there must therefore exist $\alpha(p)<\omega_1$ such that

$$h_{\alpha(p)}(p)=h_{\alpha(p)+1}(p).$$

This implies $g_{\alpha(p)}(x)=g_{\alpha(p)+1}(x)$ for p-almost every x, so by (2), (3),

$$g_{\alpha(p)}(x)=g(x)\quad p\text{-almost every } x.$$

It follows that g is absolutely measurable. Q.E.D.

Note that $\{h_\alpha |\ \alpha<\omega_1\}$ defined by (10) is actually a Borel approach to

$$h(p)=\int g\,dp.$$

This observation is crucial for the next result, which motivated our inquiry into BA functions. The analogous result for BP functions is unknown.

THEOREM 4. *Let* $f\colon X\to I$ *be* BA. *Then* $h\colon P(X)\to I$ *defined by* $h(p)=\int f\,dp$ *is also* BA.

Proof. Let $\{f_\alpha |\ \alpha<\omega_1\}$ be a collection of Borel functions satisfying (7)–(9) and define $h_\alpha(p)=\int f_\alpha dp$. Then $\{h_\alpha |\ \alpha<\omega_1\}$ satisfies (7), (8) and $h(p)=\sup_{\alpha<\omega_1}h_\alpha(p)$. Q.E.D.

THEOREM 5. *The composition of* BA *functions is* BA.

Proof. Let $f\colon X\to Y$ and $\hat f\colon Y\to Z$ be BA. Then $f=d\circ g$, where $d\colon I^\infty\to Y$ is Borel and there is a Borel approach $\{g_\alpha |\ \alpha<\omega_1\}$ to g. Likewise, $\hat f=\hat d\circ\hat g$, where $\hat d\colon \hat I^\infty\to Z$ is Borel and $\hat g$ has a Borel approach $\{\hat g_\alpha |\ \alpha<\omega_1\}$. We assume without loss of generality that $I=\hat I=[0,1]$. For $x\in X$, let

$$\alpha(x)=\min\{\alpha |\ g_\alpha(x)=g_{\alpha+1}(x)\}.$$

Then $\alpha(x)<\omega_1$, and for any $\alpha<\omega_1$, the set $\{x\in X |\ \alpha(x)\leqslant\alpha\}$ is Borel. For $x\in X$ and $\alpha<\alpha(x)$, define

$$G_\alpha(x)=(g_\alpha^1(x),0,g_\alpha^2(x),0,\ldots).$$

For $\alpha=\alpha(x)+\beta$, where $\beta\geqslant 0$, define

$$G_\alpha(x)=\{g^1_{\alpha(x)}(x),\hat g^1_\beta[d(g_{\alpha(x)}(x))],g^2_{\alpha(x)}(x),\hat g^2_\beta[d(g_{\alpha(x)}(x))],\ldots\}.$$

Then G is a Borel approach to

$$G(x)=(g^1(x),\hat g^1[f(x)],g^2(x),\hat g^2[f(x)],\ldots).$$

Let $D\colon[0,1]^\infty\to Z$ be given by $D(y_1,\hat y_1,y_2,\hat y_2,\ldots)=\hat d(\hat y_1,\hat y_2,\ldots)$. Then $\hat f\circ f=D\circ G$ is BA. Q.E.D.

As a consequence of Theorem 5, we see that if $f\colon X\to Y$ is a BA function and $A\subset Y$ is a BA set, then since the indicator of $f^{-1}(A)$ is the composition of BA functions, $f^{-1}(A)$ is also a BA set. From Theorems 4 and 5 we also have a Fubini result.

COROLLARY 5.1. *Let* $f\colon X\times Y\to(-\infty,\infty)$ *and* $\varphi\colon X\to P(Y)$ *be* BA, *where* $P(Y)$ *is the space of probability measures on* Y. *Then the function*

$$x\mapsto\int f(x,y)\varphi(x)(dy)$$

is BA *on the* BA *set where it is defined.*

Proof. For $x\in X$, let p_x be the probability measure assigning unit point mass to x. Let $\psi(x)$ be the product of p_x and $\varphi(x)$. Since ψ is the composition of the BA function $x\mapsto(p_x,\varphi(x))$ and the continuous (in the weak topology) function taking two probability measures into their product, ψ is BA. If p is a probability measure on $X\times Y$ and f is bounded, let $h(p)=\int f\,dp$. By Theorem 4, h is BA, and by Theorem 5,

$$(h\circ\psi)(x)=\int f(x,y)\varphi(x)(dy)$$

is BA. The general case follows from the remark following Corollary 2.3. Q.E.D.

THEOREM 6. *The collection of* BA *sets is closed under operation* ($\mathcal{A}$).

Proof. Let $\mathcal{N}$ be the set of infinite sequences and Σ the set of finite sequences of positive integers. For each $s\in\Sigma$, let $A(s)$ be a BA subset of the space X. Since every analytic set is BA, the set $A_1\subset\{0,1\}^\Sigma$ given by

$$A_1 = \{y|\ \exists(\zeta_1, \zeta_2, \ldots) \in \mathcal{N} \text{ such that } y(\zeta_1, \ldots, \zeta_n) = 1\ \forall n\}$$

is BA. By Corollary 2.2, the function $f\colon X \to \{0, 1\}^\Sigma$ whose sth component is the indicator of $A(s)$ is BA. The result of operation (A) on the system $\{A(s)|\ s \in \Sigma\}$ is

$$\bigcup_{(\zeta_1,\zeta_2,\ldots)\in\mathcal{N}} \bigcap_{n=1}^{\infty} A(\zeta_1, \ldots, \zeta_n) = f^{-1}(A_1),$$

and this is BA by the remark following Theorem 5. Q.E.D.

If p is a BP function from $\{0, 1\}^\infty$ to $\{0, 1\}^\infty$ satisfying (4), then p_{ω_1} defined by (5), (6) can fail to be BP [9]. If $\{g_\alpha|\ \alpha < \omega_1\}$ is a BA approach to g, it is not known if g can fail to be BA. It is not known whether the BA σ-field properly contains the BP σ-field, nor whether the BA σ-field is properly contained in the σ-field of absolutely measurable sets. The relation between the BP sets, the BA sets and the R sets [8] has not been determined. Indeed, the cardinality of the class of BA sets is not known. A particularly intriguing question is whether the product of the BA σ-fields in X and Y is the BA σ-field in $X \times Y$.

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A stabilization property and its applications in the theory of sections

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Abstract. We introduce a stabilization property in descriptive set theory which generalizes the topological and measure theoretical situations. An associated theory of sections for measurable sets in products is developed.

I. Preliminaries. The aim of this section is to make the text more selfcontained. We will introduce the various classical notions and properties, which are the starting point of this work. They can also be found in [12].

DEFINITION 1.1. Let E be a set. A paving on E will be a class $\mathscr{E}$ of subsets of E containing the empty set. We will call $(E, \mathscr{E})$ a *paved set*.

DEFINITION 1.2. If $(E, \mathscr{E})$ is a paved set, we denote by $c\mathscr{E}$: the class of subsets A of E such that $E \setminus A$ belongs to $\mathscr{E}$, $b\mathscr{E} = \mathscr{E} \cap c\mathscr{E}$.

$\mathscr{E}^\wedge$ (resp. $\mathscr{E}^\vee$, $\mathscr{E}^-$, $\mathscr{E}^*$): the stabilization of $\mathscr{E}$ for finite intersection (resp. finite union, finite intersection and finite union, countable intersection and countable union).

$\mathfrak{S}(\mathscr{E})$: the σ-algebra generated by $\mathscr{E}$.

DEFINITION 1.3. Let $(E_i, \mathscr{E}_i)_{i\in I}$ be a family of paved sets. The set $\mathscr{E}$ of subsets of $E = \prod_i E_i$ of the form $\prod_i A_i$, where $A_i \in \mathscr{E}_i$ for each $i \in I$, is called the *product paving* $\mathscr{E}_i \prod_i$.

PROPOSITION 1.4. *Let $(E_i, \mathscr{E}_i)_{i\in I}$ be paved sets such that $E_i \in \mathscr{E}_i$ for each $i \in I$. Then $\mathfrak{S}(\prod_i \mathscr{E}_i)$ contains the product σ-algebra $\oplus_i \mathfrak{S}(\mathscr{E}_i)$. If moreover I is countable, then $\mathfrak{S}(\prod_i \mathscr{E}_i) = \oplus_i \mathfrak{S}(\mathscr{E}_i)$.*

In fact, only finite and countable products will be involved here.

Let $(E, \mathscr{E})$ be a paved set and let $(K_i)_{i\in I}$ be a family of elements of $\mathscr{E}$. We will say that $(K_i)_{i\in I}$ has the finite intersection property provided $\bigcap_{i\in J} K_i \neq \emptyset$ whenever J is a finite subset of I.