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Generalized Archimedean fields and logics with Malitz quantifiers

by

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Abstract. A characterization of Archimedean fields in a particular interpretation of the logic with Malitz quantifiers suggests a generalization of such fields. The theory of the real closed version of these generalized Archimedean fields in other interpretations of the Malitz quantifier is found to allow elimination of quantifiers.

The reader should be familiar with the model theory of first order logic. Some knowledge of ultrapowers, for example, is assumed. The notation is for the most part similar to that used in [1] or [2]. Gothic letters range over structures with the corresponding Latin letters denoting their universes: A denotes the universe of $\mathfrak{A}$, B_i denotes the universe of $\mathfrak{B}_i$, etc. Cardinals are initial von Neumann ordinals. Write Card(A) for the cardinality of A, $\boldsymbol{P}$ for the set of positive integers, $\boldsymbol{Q}$ for the set of rational numbers, and $\boldsymbol{R}$ for the set of real numbers.

Logics with Malitz quantifiers. For each positive integer n and each infinite cardinal $\varkappa$, the logic $\mathcal{Q}_\varkappa^n$ is obtained by adding a new quantifier Q^n which binds n distinct variables and the following formation rule to those of first order logic: If φ is a formula and if the variables $x_1, \ldots, x_n$ are distinct, then $Q^n x_1, \ldots, x_n \varphi$ is also a formula. The logic $\mathcal{Q}_\varkappa^{<\omega}$ is obtained from first order logic by adding all the quantifiers Q^n together with the corresponding formation rules.

The interpretation of the quantifier Q^n depends on the cardinal $\varkappa$:

$$\mathfrak{A} \models_\varkappa Q^n x_1, \ldots, x_n \varphi [\vec{a}]$$

just in case there is a subset I of A such that (i) Card$(I) = \varkappa$ and (ii) whenever $a_1, \ldots, a_n$ are distinct elements of I, then $\mathfrak{A} \models_\varkappa \varphi [a_1/x_1, \ldots, a_n/x_n, \vec{a}]$. Here the notation indicates how each of the variables $x_1, \ldots, x_n$ is to be interpreted and $\vec{a}$ is an interpretation of the free variables in $Q^n x_1, \ldots, x_n \varphi$.

The logic $\mathcal{Q}_\varkappa^1$ coincides with the logic with the cardinal quantifier, "There exist $\varkappa$ many ...". For $n \geqslant 2$, the logics $\mathcal{Q}_{\aleph_0}^n$ and $\mathcal{Q}_{\aleph_0}^{<\omega}$ are referred to as logics with Ramsey quantifiers because of the similarity between their semantics and the well-known statement of Ramsey's theorem. (For a statement of Ramsey's theorem, see [2], p. 145.) The logics $\mathfrak{L}^n_\varkappa$ and $\mathfrak{L}^{<\omega}_\varkappa$ are due to Malitz.

Another logic used below is the logic with Chang's equi-cardinality quantifier: The logic $\mathfrak{L}_c$ has the same syntax as $\mathfrak{L}^1_\varkappa$ with the quantifier Q^1 interpreted as follows. $\mathfrak{A} \models_c Q^1 \varphi[\vec{a}]$ iff there is a subset I of A such that (i) $\mathrm{Card}(I) = \mathrm{Card}(A)$ and (ii) for $a \in I$, $\mathfrak{A} \models_c \varphi[x/a, \vec{a}]$.

Generalized Archimedean fields. Let $L = \{+, \cdot, -, 0, 1, <\}$ be the language appropriate for ordered fields: $+$ and $\cdot$ denote the operations of addition and multiplication; 0 and 1 are constant symbols for the additive and multiplicative units; $-$ denotes the additive inverse operation; and $<$ denotes the ordering relation.

Recall that an *ordered field* is a linearly ordered structure, satisfying the field axioms, in which multiplication by positive elements and addition by all elements preserves the ordering. An ordered field is *Archimedean* just in case each member of the field is bounded above by some positive integer. It is known that there is no set Σ of sentences, from either the logic $\mathfrak{L}^1_{\aleph_0}$ or the logic $\mathfrak{L}_c$, which use nonlogical symbols only from L, such that $\mathfrak{A} \models \Sigma$ iff $\mathfrak{A}$ is an Archimedean field. One way to establish this fact is to note that the Tarski–Chevalley method [8] of quantifier elimination for real closed fields can be extended [3] to the logics $\mathfrak{L}^1_{\aleph_0}$ and $\mathfrak{L}_c$. This means that both the $\mathfrak{L}^1_{\aleph_0}$ and the $\mathfrak{L}_c$ theories of real closed fields are complete and thus in these logics Archimedean fields cannot be distinguished from their non-Archimedean counterparts. However, in each of the logics with Ramsey quantifiers, Archimedean fields are characterized ([3] or [4]) by the ordered field axioms together with the sentences (from the language L in each of the logics equivalent to)

[α] $$Q^2 xy(0<x \wedge 0<y \wedge |x-y| \geqslant 1)$$

and

[β] $$\neg \exists x Q^2 yz(0<y<x \wedge 0<z<x \wedge |z-y| \geqslant 1).$$

Thus an ordered field $\mathfrak{A}$ is Archimedean iff $\mathfrak{A} \models_{\aleph_0} \alpha \wedge \beta$. The motivation for the generalization of Archimedean proposed here can be seen by replacing the "$\aleph_0$" by "$\varkappa$", i.e., an ordered field is said to be $\varkappa$-Archimedean just in case $\mathfrak{A} \models_\varkappa \alpha \wedge \beta$: A subset I of an ordered field is *positive* iff $(\forall a \in I)(0<a)$ and I is *discrete* iff

$$(\forall a \in I)(\forall b \in I)(a \neq b \rightarrow |a-b| \geqslant 1).$$

An ordered field is *$\varkappa$-Archimedean* iff the field (i) has a positive discrete subset of cardinality $\varkappa$ but (ii) no positive discrete subset of cardinality $\varkappa$ is bounded above.

As noted above, an ordered field is $\aleph_0$-Archimedean iff it is Archimedean. Therefore an $\aleph_0$-Archimedean field of cardinality λ exists iff $\aleph_0 \leqslant \lambda \leqslant 2^{\aleph_0}$.

QUESTION. For which uncountable $\varkappa$ is the preceeding statement true after $\aleph_0$ is replaced by $\varkappa$? The answer to this question is not completely known (at least to the present writer). Here is a sketch of what is known:

LEMMA. *Let $\mathfrak{A}$ be a $\varkappa$-Archimedean field and suppose that I is a positive discrete subset of A which is not bounded. Then* (i) $\mathrm{Card}(I) \leqslant \varkappa$; *and* (ii) *if $\varkappa$ is regular, then* $\mathrm{Card}(I) = \varkappa$ *and furthermore there is an order preserving injection from $\varkappa$ into I.*

Proof. Part (i). Suppose $\mathrm{Card}(I) > \varkappa$. Let J be any subset of I with $\mathrm{Card}(J) = \varkappa$. Since $\mathfrak{A} \models_\varkappa \beta$, J is not bounded. Let $I_j = \{x \in I \mid x<j\}$. Then $I = \bigcup_{j \in J} I_j$. Then $\varkappa > \mathrm{Card}(I) = \mathrm{Card}(\bigcup_{j \in J} I_j) \leqslant \varkappa \cdot \varkappa = \varkappa$, a contradiction.

Part (ii). Suppose $\mathrm{Card}(I) < \varkappa$. Since $\mathfrak{A} \models_\varkappa \alpha$, let J be a positive discrete subset of A with $\mathrm{Card}(J) = \varkappa$. Let $J_i = \{x \in J \mid x<i\}$. Then $J = \bigcup_{i \in I} J_i$ which contradicts the regularity of $\varkappa$. Therefore $\mathrm{Card}(I) = \varkappa$. Define $f\colon \varkappa \to I$ by choosing $f(\lambda) \in I - I_\lambda$, where $I_0 = \emptyset$; for $\lambda \in \varkappa$, $I_{\lambda+1} = \{x \in I \mid x \leqslant f(\lambda)\}$; and for limit ordinal $\lambda \in \varkappa$, $I_\lambda = \bigcup_{\delta \in \lambda} I_\delta$. Since $\varkappa$ is regular, for each $\lambda \in \varkappa$, $\mathrm{Card}(I_\lambda) \neq \varkappa$.

PROPOSITION. *If $\mathfrak{A}$ is $\varkappa$-Archimedean field, then $\varkappa \leqslant \mathrm{Card}(A) \leqslant 2^\varkappa$.*

Proof. (This method of proof was suggested by A. Macintyre.) Let $\mathfrak{A}$ be a $\varkappa$-Archimedean field. The definition of $\varkappa$-Archimedean insures that $\varkappa \leqslant \mathrm{Card}(A)$. For each $a \in A$, the *Archimedean class* $[\![a]\!]$ containing a is the set of all $b \in A$ such that there are positive integers m and n such that $|a| < m|b|$ and $|b| < n|a|$. The set G of Archimedean classes forms a group $\mathfrak{G}$ under the multiplication inherited from $\mathfrak{A}$. The mapping $a \mapsto [\![a]\!]$ satisfies the requirements of a valuation. Thus by a result ([5] and [6]) of Kaplansky, $\mathfrak{A}$ is isomorphic to a subfield of a power series field and these power series can be injected into the set of functions from G into the reals $\boldsymbol{R}$. Since $\mathfrak{A}$ is $\varkappa$-Archimedean, $\mathrm{Card}(G) \leqslant \varkappa$. Therefore $\mathrm{Card}(A) \leqslant \mathrm{Card}(\boldsymbol{R})^{\mathrm{Card}(G)} \leqslant (2^{\aleph_0})^\varkappa = 2^\varkappa$.

An ordered field is a *real closed field* just in case each positive element has a square root and the Wierstrass Nullstensatz holds for polynomials of a single variable with coefficients from the field, i.e., if $p(x)$ is such a polynomial and if a and b are members of the field such that $a<b$, $p(a)<0$, and $p(b)>0$, then for some c between a and b, $p(c) = 0$.

PROPOSITION. *Real closed $\varkappa$-Archimedean fields of cardinality $\varkappa$ exist.*

Proof. Let $L' = L \cup \{I\}$ where I is an unary relation symbol. Any countable Archimedean real closed field is a model for the set of first order sentences from the language L which characterize real closed fields together with the following sentences of the logic $\mathfrak{L}_c$ from the language L':

(1) $$\forall x(I(x) \rightarrow 0<x),$$

(2) $$\forall x \forall y(I(x) \wedge I(y) \wedge x \neq y \rightarrow |x-y| \geqslant 1),$$

(3) $$QxI(x),$$

(4) $$\neg \exists x Qy(I(y) \wedge y<x),$$

(5) $$\forall x[0<x \rightarrow \exists y(I(y) \wedge y<x \leqslant y+1)].$$

Here the set of positive integers is the intended interpretation for I.

By an upward Löwenheim–Skolem theorem for $\mathfrak{L}_c$ [1, p. 275], the set of sentences mentioned above has a model $\mathfrak{A}$ of each infinite cardinality $\varkappa$. Sentences (1), (2), and (3) insure that $\mathfrak{A}$ has a positive discrete set of cardinality $\varkappa$. Now let J be a positive discrete set which is bounded above by a. Let $I_a = \{x \in I \mid x \leqslant a\}$. By (4),

$\mathrm{Card}(I_a)<\varkappa$. Define $f\colon J\to I$ as follows: Let $f(j)=i\in I$ where $i<j\leqslant i+1$. Sentence (5) insures that f is well-defined and one to one. So $\mathrm{Card}(j)<\varkappa$. Thus $\mathfrak{A}$ is $\varkappa$-Archimedean.

PROBLEM. For which uncountable $\varkappa$ are there $\varkappa$-Archimedean fields of cardinality $2^{\varkappa}$?

In [4] it is shown that the $\mathcal{Q}^{<\omega}_{\aleph_0}$-theory of Archimedean real closed fields allows quantifier elimination. The rest of this paper is devoted to showing that for uncountable regular $\varkappa$, the $\mathcal{Q}^{<\omega}_{\varkappa}$-theory of $\varkappa$-Archimedean real closed fields allows quantifier elimination.

Quantifier elimination for the $\mathcal{Q}^{<\omega}_{\varkappa}$-theory of $\varkappa$-Archimedean real closed fields. The following theorem is the starting point:

THEOREM (i) (Tarski–Chevalley). *Every first order formula φ of the language L is equivalent, in all real closed fields, to a quantifier free formula ψ whose free variables form a subset of those of φ.*

(ii) (The Tarski–Chevalley theorem for $\mathcal{Q}^{1}_{\varkappa}$ [3]). *For each infinite cardinal $\varkappa$, every $\mathcal{Q}^{1}_{\varkappa}$-formula φ of the language L is equivalent, in all real closed fields of cardinality at least $\varkappa$, to a quantifier free formula ψ whose free variables form a subset of those of φ.*

The goal of this paper is to prove the statement obtained from the Tarski–Chevalley theorem for $\mathcal{Q}^{1}_{\varkappa}$ by replacing "each infinite cardinal $\varkappa$" by "each regular cardinal $\varkappa$", "$\mathcal{Q}^{1}_{\varkappa}$" by "$\mathcal{Q}^{<\omega}_{\varkappa}$", and "real closed fields of cardinality at least $\varkappa$" by "$\varkappa$-Archimedean real closed fields". First some well-known facts are recalled without proof:

PROPOSITION. *For each term t of the language L, there is a polynomial p with integer coefficients, in the variables which appear in t, such that, for every real closed field $\mathfrak{A}$ and every interpretation $\vec{a}$ for the variables, $\mathfrak{A}\models p=t[\vec{a}]$.*

PROPOSITION. *Each atomic formula of the language L is equivalent, in all real closed fields to a polynomial equality or inequality, i.e., to a formula of the form $p=0$ or of the form $p>0$ where p is a polynomial in several variables with integer coefficients.*

PROPOSITION. *Each quantifier free formula of the language L is equivalent, in all real closed fields, to a disjunction of formulas of the form*

$$p_1=0\wedge\ldots\wedge p_k=0\wedge q_1>0\wedge\ldots\wedge q_n>0$$

where the p_i and q_j are polynomials with integer coefficients.

As a consequence of these propositions and the Tarski–Chevalley theorem, first order formulas can be replaced by equivalent formulas of the form $\theta_1\vee\ldots\vee\theta_k$, where each θ_i is a conjunction of polynomial equalities or inequalities. Therefore consideration must be given to formulas of the form $Q^n x_1\ldots x_n(\theta_1\vee\ldots\vee\theta_k)$. It is easy to see that Q^1 always distributes over disjunctions, but unfortunately, as the

formula $Q^2xy(x<y\vee y<x)$ illustrates, for $n\geqslant 2$, the quantifier Q^n need not distribute over disjunctions. However this problem is solved in the next lemma.

First some notation: Let $\mathcal{S}(n)$ be the symmetric group on the set $\{1,\ldots,n\}$ and for each integer $k\geqslant 2$, let $F(n,k)$ be the set of functions from $\mathcal{S}(n)$ into the set $\{1,\ldots,k\}$. For each formula φ and each $\sigma\in\mathcal{S}(n)$, let φ^{σ} be the formula obtained from φ by replacing each of the variables x_i, for $1\leqslant i\leqslant n$, by $x_{\sigma(i)}$.

LEMMA (A distributive law for Q^n over disjunction). *Let $\mathfrak{A}$ be a structure which is linearly ordered by the interpretation in $\mathfrak{A}$ of $<$.*

(i) *If* $\mathfrak{A}\models_{\aleph_0}Q^n x_1\ldots x_n(\theta_1\vee\ldots\vee\theta_k)[\vec{a}]$, *then*

$$\mathfrak{A}\models_{\aleph_0}\bigvee_{f\in F(n,k)}Q^n x_1\ldots x_n\Big(x_1<\ldots<x_n\to\bigwedge_{\sigma\in\mathcal{S}(n)}\theta^{\sigma}_{f(\sigma)}\Big)[\vec{a}]\,.$$

(ii) *For any cardinal $\varkappa$, if*

$$\mathfrak{A}\models_{\varkappa}\bigvee_{f\in F(n,k)}Q^n x_1\ldots x_n\Big(x_1<\ldots<x_n\to\bigwedge_{\sigma\in\mathcal{S}(n)}\theta^{\sigma}_{f(\sigma)}\Big)[\vec{a}]\,,$$

then

$$\mathfrak{A}\models_{\varkappa}Q^n x_1\ldots x_n(\theta_1\vee\ldots\vee\theta_k)[\vec{a}]\,.$$

Proof. Part (i). Assume that $\mathfrak{A}\models_{\aleph_0}Q^n x_1\ldots x_n(\theta_1\vee\ldots\vee\theta_k)[\vec{a}]$. Then there is an infinite subset J_0 of A such that whenever $a_1,\ldots,a_n$ are distinct elements of J_0,

$$\mathfrak{A}\models_{\aleph_0}\theta_1\ldots\theta_k[a_1/x_1,\ldots,a_n/x_n,\vec{a}]\,.$$

Let $\sigma_1,\ldots,\sigma_{n!}$ be an enumeration of $\mathcal{S}(n)$ and for any set J, let

$$[J]^n=\{k\subseteq J\mid \mathrm{Card}(k)=n\}\,.$$

For each $i\in\{1,\ldots,n!\}$, define an infinite set J_i such that for each i, $J_i\subseteq J_{i-1}$ and at the same time define a function g from $\mathcal{S}(n)$ into $\{1,\ldots,k\}$ by specifying the value of $g(\sigma_i)$ as follows: For $i\in\{1,\ldots,n!\}$ and $j\in\{1,\ldots,k\}$, assuming J_{i-1} has already been defined, let H^i_j be the set of all $\{a_1,\ldots,a_n\}\in[J_{i-1}]^n$ such that if $\{a_1,\ldots,a_n\}=\{c_1,\ldots,c_n\}$ and $c_1<\ldots<c_n$, then

$$\mathfrak{A}\models\theta^{\sigma_i}_j[c_1/x_1,\ldots,c_n/x_n,\vec{a}]\,.$$

Then $H^i_1\cup\ldots\cup H^i_k=[J_{i-1}]^n$ because $\{c_1,\ldots,c_n\}\in[J_{i-1}]^n\subseteq[J_0]^n$, and for some h,

$$\mathfrak{A}\models_{\aleph_0}\theta_h[c_{\sigma_i(1)}/x_1,\ldots,c_{\sigma_i(n)}/x_n,\vec{a}]\,;$$

so that

$$\mathfrak{A}\models_{\aleph_0}\theta^{\sigma_i}_h[c_1/x_1,\ldots,c_n/x_n,\vec{a}]\,.$$

By Ramsey's theorem, there is an m, which is taken to be the value of $g(\sigma_i)$, and an infinite subset, which is taken to be J_i, of J_{i-1} such that $[J_i]^n\subseteq H^i_m$. Now let $a_1,\ldots,a_n$ be elements of the infinite subset $J_{n!}$ of A such that $a_1<\ldots<a_n$ and let $\sigma=\sigma_j$ be an element of $\mathcal{S}(n)$. Since $J_{n!}\subseteq J_j$, $\{a_1,\ldots,a_n\}\in[J_j]^n\subseteq H^j_{g(\sigma)}$; so that

$$\mathfrak{A}\models_{\aleph_0}\theta^{\sigma}_{g(\sigma)}[a_1/x_1,\ldots,a_n/x_n,\vec{a}]\,.$$

Therefore
$$\mathfrak{A} \models_{\aleph_0} \bigvee_{f\in F(n,k)} Q^n x_1 \dots x_n (x_1<\dots<x_n \to \bigwedge_{\sigma\in\mathscr{S}(n)} \theta^\sigma_{f(\sigma)})[\vec a].$$

Part (ii). Now assume that for some $f\in F(n,k)$,
$$\mathfrak{A}\models_\varkappa Q^n x_1\dots x_n(x_1<\dots<x_n\to \bigwedge_{f\in\mathscr{S}(n)} \theta^\sigma_{f(\sigma)})[\vec a].$$

Then there is a subset I of A such that $\mathrm{Card}(I)=\varkappa$ and whenever $a_1,\dots,a_n$ are distinct elements of I,
$$\mathfrak{A}\models_\varkappa x_1<\dots<x_n \to \bigwedge_{\sigma\in\mathscr{S}(n)}\theta^\sigma_{f(\sigma)}[a_1/x_1,\dots,a_n/x_n,\vec a].$$

Let $c_1,\dots,c_n$ be distinct elements of I and let τ and ϱ be elements of $\mathscr{S}(n)$ such that $c_{\tau(1)}<\dots<c_{\tau(n)}$ and $\varrho=\tau^{-1}$. Then
$$\mathfrak{A}\models_\varkappa \theta^\varrho_{f(\varrho)}[c_{\tau(1)}/x_1,\dots,c_{\tau(n)}/x_n,\vec a];$$
so
$$\mathfrak{A}\models_\varkappa \theta^\varrho_{f(\varrho)}[c_1/x_{\varrho(1)},\dots,c_n/x_{\varrho(n)},\vec a];$$
so that $\mathfrak{A}\models_\varkappa \theta_{f(\varrho)}[c_1/x_1,\dots,c_n/x_n,\vec a]$. Thus
$$\mathfrak{A}\models_\varkappa Q^n x_1\dots x_n(\theta_1\vee\dots\vee\theta_n)[\vec a].$$

For theories of real closed fields in logics with Ramsey quantifiers, the proceeding lemma allows the replacement of formulas with the form $Q^n x_1\dots x_n(\theta_1\vee\dots\vee\theta_k)$ by formulas of the form $Q^n x_1\dots x_n(x_1<\dots<_n\to\theta)$. Here $\theta,\theta_1,\dots,\theta_k$ are conjunctions of polynomial equalities and inequalities.

In the rest of the paper, the following definitions and notation are used throughout:

Definition. Let $\mathfrak{A}$ be an ordered field and let $S\subseteq A$. An element a of A is an $\mathfrak{A}$-*limit point* for S iff for each positive $b\in A$, there is an $s\in S$ such that $|s-a|<b$.

Definition. Let λ be an ordinal. A linearly ordered set S is a λ-*enumeration* (λ^*-*enumeration*) iff there is an order preserving (reversing) bijection from λ onto S.

Notation. Let λ be an infinite cardinal. Whenever $\mathfrak{A}\models_\lambda Q^n x_1\dots x_n\varphi[\vec a]$, $I(\mathfrak{A};\varphi;\vec a)$ denotes a subset of A with cardinality λ such that if $a_1,\dots,a_n$ are distinct elements of $I(\mathfrak{A};\varphi;\vec a)$, then $\mathfrak{A}\models_\lambda\varphi[a_1/x_1,\dots,a_n/x_n,\vec a]$.

Notation. Let θ be a conjunction of polynomial equalities or inequalities. Let u,v,w, and z be variables which do not appear in θ. For $n\in\omega$, let $\Theta_n,\Gamma_n,\Delta_n,\Xi_n$, and Ψ_n be the following formulas:

Θ_0 is θ,
Γ_0 is $\forall z\exists x(z<x\wedge\theta)$,
Δ_0 is $\forall z\exists y(y<z\wedge\theta)$,
Ξ_0 is $\forall z\exists x[z<w\to(z<x<w\wedge\theta)]$, and
Ψ_0 is $\forall z\exists y[w<z\to(w<y<z\wedge\theta)]$.

Θ_1 is $x<y\to\theta$,
Γ_1 is $\exists v\forall y[x<v\wedge(v<y\to\theta)]$,
Δ_1 is $\exists u\forall x[u<y\wedge(x<u\to\theta)]$,
Ξ_1 is $\exists u\forall y[x<u<w\wedge(u<y<w\to\theta)]$, and
Ψ_1 is $\exists u\forall x[w<u<y\wedge(w<x<u\to\theta)]$.

For $n\geqslant 2$, Θ_n is $x<x_2<\dots<x_n<y\to\theta$,
Γ_n is $\exists v\forall y[x<x_2<\dots<x_n\to x_n<v\wedge(v<y\to\theta)]$,
Δ_n is $\exists u\forall x[x_2<\dots<x_r<y\to u<x_2\wedge(x<u\to\theta)]$,
$$\Xi_n \text{ is } \exists u\forall y[x<w\wedge x_2<w\wedge\dots\wedge x_n<w\wedge \wedge(x<x_2<\dots<x_n\to x_n<u<w\wedge(u<y<w\to\theta))],$$
and
$$\Psi_n \text{ is } \exists u\forall x[w<x_2\wedge\dots\wedge w<x_n\wedge w<y\wedge \wedge(x_2<\dots<x_n<y\to w<u<x_2\wedge(w<x<u\to\theta))].$$

Let Φ be any of the formulas listed above and let φ be an arbitrary formula. Then $\Phi(\varphi)$ is the formula obtained from φ by replacing θ by φ.

For $n=1$, $Q^n xx_2\dots x_n$ denotes Q^1x, $Q^{n+1}xx_2\dots x_ny$ denotes Q^2xy, and $Q^nx_2\dots x_ny$ denotes Q^1y.

Lemma 1. *Let λ be a regular cardinal and let $\mathfrak{A}$ be a real closed field.*

(i) *$\mathfrak{A}\models_\lambda Q^1x\theta[\vec a]$ and $I(\mathfrak{A};\theta;\vec a)$ can be taken to be an unbounded λ-enumeration iff there is an unbounded subset of A which is a λ-enumeration and $\mathfrak{A}\models\Gamma_0$.*

(ii) *$\mathfrak{A}\models_\lambda Q^1y\theta[\vec a]$ and $I(\mathfrak{A};\theta;\vec a)$ can be taken to be an unbounded λ^*-enumeration iff there is an unbounded subset of A which is a λ-enumeration and $\mathfrak{A}\models\Delta_0[\vec a]$.*

(iii) *$\mathfrak{A}\models_\lambda Q^1x\theta[\vec a]$ and $I(\mathfrak{A};\theta;\vec a)=I$ can be taken to be a λ-enumeration which has an $\mathfrak{A}$-limit point c larger than any element in I iff there is an unbounded subset of A which is a λ-enumeration and $\mathfrak{A}\models\Xi_0[c/w,\vec a]$.*

(iv) *$\mathfrak{A}\models_\lambda Q^1y\theta[\vec a]$ and $I(\mathfrak{A};\theta;\vec a)=I$ can be taken to be a λ^*-enumeration which has an $\mathfrak{A}$-limit point c smaller than any element in I iff there is an unbounded subset of A which is a λ-enumeration and $\Psi_0[c/w,\vec a]$.*

Proof. Since the various parts of the lemma are similar, only selected portions of the proof are presented below.

Part (i). Suppose $\mathfrak{A}\models\Gamma_0$ and $D=\{d_i|\ i\in\lambda\}$ is an unbounded λ-enumeration. Define a λ-enumeration $\{b_i|\ i\in\lambda\}$ by recursion: Choose b_0 so that $b>d_0$ and $\mathfrak{A}\models\theta[b_0/x,\vec a]$. Choose b_{i+1} so that $b_{i+1}>\max\{b_i,d_{i+1}\}$ and $\mathfrak{A}\models\theta[b_{i+1}/x,\vec a]$. For a limit ordinal $i\in\lambda$, choose b_i as follows: For each $j\in i$, let $D_j=\{x\in D|\ x<b_j\}$ and let $d_i=\bigcup_{j\in i}D_j$. Then $\mathrm{card}(D_i)<\lambda$, so there is $d\in D-D_i$. Choose b_i so that $b_i>\max\{d,d_i\}$ and $\mathfrak{A}\models\theta[b_i/x,\vec a]$. Finally, use $\{b_i|\ i\in\lambda\}$ for $I(\mathfrak{A};\theta;\vec a)$.

Part (ii). Assume $\mathfrak{A}\models_\lambda Q^1y\theta[\vec a]$ and $I(\mathfrak{A};\theta;\vec a)=I$ can be taken to be an unbounded λ^*-enumeration. Then $\{-x|\ x\in I\}$ is an unbounded λ-enumeration, and since I is not bounded below, $\mathfrak{A}\models \Delta_0[\vec{a}]$.

Part (iii). Suppose $\mathfrak{A}\models_\lambda Q^1 x\theta[\vec{a}]$ and $I(\mathfrak{A};\theta;\vec{a}) = I$ can be taken to be a λ-enumeration which has an $\mathfrak{A}$-limit point c larger than any element in I. Then $\{1/(c-x)|\ x\in I\}$ is an unbounded subset of A which is a λ-enumeration. Since c is an $\mathfrak{A}$-limit point for I and c is larger than any element in I, $\mathfrak{A}\models \Xi_0[c/w,\vec{a}]$.

Part (iv). Assume $D = \{d_i|\ i\in\lambda\}$ is an unbounded λ-enumeration and $\mathfrak{A}\models \Psi_0[c/w,\vec{a}]$. Define a λ^*-enumeration by recursion: Choose b_0 so that $c<b_0<c+|d_0|^{-1}$ and $\mathfrak{A}\models\theta[b_0/y,\vec{a}]$. Choose b_{i+1} so that

$$c<b_{i+1}<\min\{b_i,\ c+|d_{i+1}|^{-1}\}$$

and $\mathfrak{A}\models\theta[b_{i+1}/y,\vec{a}]$. For a limit ordinal $i\in\lambda$, choose b_i as follows: For each $j\in i$, let $D_j = \{x\in D|\ x<(b_j-c)^{-1}\}$ and let $D_i = \bigcup_{j\in i} D_i$. Then $\operatorname{card}(d_i)<\lambda$, so there is $d\in D-D_i$. Choose b_i so that $c<b_i<\min\{c+|d|^{-1},\ c+|d_i|^{-1}\}$ and $\mathfrak{A}\models\theta[b_i/y,\vec{a}]$. Finally use $\{b_i|\ i\in\lambda\}$ for $I(\mathfrak{A};\theta;\vec{a})$.

LEMMA 2. *Let λ be an infinite cardinal, let $n\geqslant 1$, and let $\mathfrak{A}$ be a real closed field such that $\mathfrak{A}\models_\lambda Q^{n+1}xx_2\dots x_n y\Theta_n[\vec{a}]$.*

(i) *If $I(\mathfrak{A};\Theta_n;\vec{a})$ can be taken to be an unbounded λ-enumeration, then $\mathfrak{A}\models_\lambda Q^n xx_2\dots x_n\Gamma_n[\vec{a}]$ and $I(\mathfrak{A};\Gamma_n;\vec{a})$ can be taken to be the same as $I(\mathfrak{A};\Theta_n;\vec{a})$.*

(ii) *If $I(\mathfrak{A};\Theta_n;\vec{a})$ can be taken to be an unbounded λ^*-enumeration, then $\mathfrak{A}\models_\lambda Q^n x_2\dots x_n y\Delta_n[\vec{a}]$ and $I(\mathfrak{A};\Delta_n;\vec{a})$ can be taken to be the same as $I(\mathfrak{A};\Theta_n;\vec{a})$.*

(iii) *If $I(\mathfrak{A};\Theta_n;\vec{a})$ can be taken to be a λ-enumeration which has an $\mathfrak{A}$-limit point c larger than any element in $I(\mathfrak{A};\Theta_n;\vec{a})$, then $\mathfrak{A}\models_\lambda Qxx_2\dots x_n\Xi_n[c/w,\vec{a}]$ and $I(\mathfrak{A};\Xi_n;c/w,\vec{a})$ can be taken to be the same as $I(\mathfrak{A};\Theta_n;\vec{a})$.*

(iv) *If $I(\mathfrak{A};\Theta_n;\vec{a})$ can be taken to be a λ^*-enumeration which has an $\mathfrak{A}$-limit point c smaller than any element in $I(\mathfrak{A};\Theta_n;\vec{a})$, then $\mathfrak{A}\models_\lambda Q^n x_2\dots x_n y\Psi_n[c/w,\vec{a}]$ and $I(\mathfrak{A};\Psi_n;c/w,\vec{a})$ can be taken to be the same as $I(\mathfrak{A};\Theta_n;\vec{a})$.*

Proof. For example, consider parts (i) and (vi); proofs for parts (ii) and (iii) are similar. Let Θ be

$$p_1 = 0\wedge\dots\wedge p_h = 0\wedge q_1>0\wedge\dots\wedge q_k>0$$

where the p_i and q_j are polynomials.

Part (i). Suppose $I(\mathfrak{A};\Theta_n;\vec{a}) = I$ is an unbounded λ-enumeration. Let $c_1<\dots<c_n\in I$. Then the set $J = \{x\in I|\ x>c_n\}$ is infinite and not bounded from above. For each $c\in J$ and each i and j,

$$\mathfrak{A}\models p_i = 0\wedge q_j>0[c_1/x,\ c_2/x_2,\ \dots,\ c_n/x_n,\ c/y,\ \vec{a}].$$

Since the only polynomial, with coefficients from A, in one variable, and with infinitely many zeros, is the zero-polynomial;

$$\mathfrak{A}\models\forall y(p_i = 0)[c_1/x,\ c_2/x_2,\ \dots,\ c_n/x_n,\ \vec{a}],$$

Since non-zero polynomials, in one variable and with coefficients from A, have only finitely many zeros, by the Weierstrass Nullstellensatz for polynomials, there is an element d_j of A such that $d_j>c_n$ and

$$\mathfrak{A}\models\forall y(v<y\to q_j>0)[d_j/v,\ c_1/x,\ c_2/x_2,\ \dots,\ c_n/x_n,\ \vec{a}].$$

Let $d = \max\{d_j|\ 1\leqslant j\leqslant k\}$. Then

$$\mathfrak{A}\models\forall y(v<y\to\theta)[d/v,\ c_1/x,\ c_2/x_2,\ \dots,\ c_n/x_n,\ \vec{a}].$$

Therefore $\mathfrak{A}\models_\lambda Q^n xx_2\dots x_n\Gamma_n[\vec{a}]$ and $I(\mathfrak{A};\Gamma_n;\vec{a})$ can be taken to be I.

Part (iv). Assume $I(\mathfrak{A};\Theta_n;\vec{a}) = I$ is a λ^*-enumeration which has an $\mathfrak{A}$-limit point c which is smaller than any element in I. Let $b_1<\dots<b_n\in I$. Then the set $J = \{b\in I|\ b<b_n\}$ is infinite. For each $b\in J$ and each i and j,

$$\mathfrak{A}\models(p_i = 0\wedge q_j>0)[b/x,\ b_1/x_2,\ \dots,\ b_n/y,\ \vec{a}].$$

Then $\mathfrak{A}\models\forall x(p_i = 0)[b_1/x_2,\ \dots,\ b_n/y,\ \vec{a}]$ and there is an element d_j of A such that $c<d_j<b_1$ and

$$\mathfrak{A}\models\forall x(w<x<u\to q_j>0)[c/w,\ d_j/u,\ b_1/x_2,\ \dots,\ b_n/y,\ \vec{a}].$$

Let $d = \min\{d_j|\ 1\leqslant j\leqslant k\}$. Then

$$\mathfrak{A}\models\forall x(w<x<u\to\theta)[c/w,\ d/u,\ b_1/x_2,\ \dots,\ b_t/y,\ \vec{a}].$$

Therefore $\mathfrak{A}\models_\lambda Q^n x_2\dots x_n y\Psi_n[c/w,\vec{a}]$ and $I(\mathfrak{A};\Psi_n;c/w,\vec{a})$ can be taken to be I.

LEMMA 3. *Let λ be a regular cardinal and let $\mathfrak{A}$ be a real closed field.*

(i) *If $\mathfrak{A}\models_\lambda Q^n xx_2\dots x_n\Gamma_n[\vec{a}]$ and $I(\mathfrak{A};\Gamma_n;\vec{a})$ can be taken to be an unbounded λ-enumeration, then $\mathfrak{A}\models_\lambda Q^{n+1}xx_2\dots x_n y\Theta_x[a]$ and $I(\mathfrak{A};\Theta_n;\vec{a})$ can also be taken to be an unbounded λ-enumeration.*

(ii) *If $\mathfrak{A}\models_\lambda Q^n x_2\dots x_n y\Delta_n[\vec{a}]$ and $I(\mathfrak{A};\Delta_n;\vec{a})$ can be taken to be an unbounded λ^*-enumeration, then $\mathfrak{A}\models_\lambda Q^{n+1}xx_2\dots x_n y\Theta_n[\vec{a}]$ and $I(\mathfrak{A};\Theta_n;\vec{a})$ can also be taken to be an unbounded λ^*-enumeration.*

(iii) *If there is $c\in A$ such that $\mathfrak{A}\models_\lambda Q^n xx_2\dots x_n\Xi_n[c/w,\vec{a}]$ and $I(\mathfrak{A};\Xi_n;c/w,a)$ can be taken to be a λ-enumeration which has c for an $\mathfrak{A}$-limit point, then $\mathfrak{A}\models_\lambda Q^{n+1}xx_2\dots x_n y\Theta_n[\vec{a}]$ and $I(\mathfrak{A};\Theta_n;\vec{a}) = I$ can be taken to be a λ-enumeration with c as an $\mathfrak{A}$-limit point larger than any element in I.*

(iv) *If there is $c\in A$ such that $\mathfrak{A}\models_\lambda Q^n x_2\dots x_n y\Psi_n[c/w,\vec{a}]$ and $I(\mathfrak{A};\Psi_n;c/w,\vec{a})$ can be taken to be a λ^*-enumeration which has c for an $\mathfrak{A}$-limit point, then $\mathfrak{A}\models_\lambda Q^{n+1}xx_2\dots x_n y\Theta_n[\vec{a}]$ and $I(\mathfrak{A};\Theta_n;\vec{a}) = I$ can be taken to be a λ^*-enumeration with c as an $\mathfrak{A}$-limit point smaller than any element in I.*

Proof. For example, consider parts (ii) and (iii); proofs for parts (i) and (iv) are similar.

Part (ii). Suppose $\mathfrak{A} \models_\lambda Q^n x_2 \dots x_n y \Delta_n[\vec{a}]$ and $I(\mathfrak{A}; \Delta_n; \vec{a}) = I = \{d_i | \ i \in \lambda\}$ is an unbounded λ^*-enumeration. For $i \in \lambda$, define $b_i \in I$ and $u_i \in A$ by recursion: First arbitrarily pick $b_0 > \dots > b_{n-1} \in I$ and $u_0, \dots, u_{n-1} \in A$. Let S_i be the set of all subsets s of $\{b_j | \ j \in i\}$ such that $\operatorname{card}(s) = n$. Then $\operatorname{card}(S_i) < \lambda$. For $s \in S_i$, let $s = \{c_j | \ 1 \leqslant j \leqslant n\}$ where $c_1 < \dots < c_n$. Now choose u so that

$$\mathfrak{A} \models \forall x [u < x_2 \wedge (x < u \to \theta)] [c_1/x_2, \dots, c_n/y, u_s/u, \vec{a}].$$

The formula Δ_n asserts that such a u_s exists. Since λ is regular and $\operatorname{card}(S_i) < \lambda$, it is possible to find $u_i \in A$ and $d \in I$ such that for each $s \in S_i$, $u_i < u_s$ and for each $j \in i$, $d < b_j$. Now choose $b_i \in I$ so that $b_i < \min\{u_i, d, d_i\}$. Then $\{b_i | \ i \in \lambda\}$ is an unbounded λ^*-enumeration. Finally it is shown that $\{b_i | \ i \in \lambda\}$ can be used for $I(\mathfrak{A}; \Theta_n; \vec{a})$: Let $c_1 < \dots < c_{n+1}$ be elements from $\{b_i | \ i \in \lambda\}$ with $c_1 = b_j$ and $s = \{c_2, \dots, c_{n+1}\}$. Then $c_1 < u_j < u_s$ and $s \subseteq I$; thus

$$\mathfrak{A} \models \Delta_n[c_2/x_2, \dots, c_{n+1}/y, \vec{a}], \ \mathfrak{A} \models \forall x [u < x_2 \wedge (x < u \to \theta)] [c_2/x_2, \dots, c_{n+1}/y, u_s/u, \vec{a}],$$

and $\mathfrak{A} \models \theta [c_1/x, \dots, c_{n+1}/y, \vec{a}]$.

Part (iii). Assume there is $c \in A$ such that $\mathfrak{A} \models_\lambda Q^n x x_2 \dots x_n \Xi_n[c/w, a]$ and $I(\mathfrak{A}; \Xi_n; c/w, \vec{a}) = D = \{d_i | \ i \in \lambda\}$ is a λ-enumeration which has c for an $\mathfrak{A}$-limit point. For $i \in \lambda$, define $b_i \in D$ and $v_i \in A$ by recursion: First arbitrarily pick $b_0 < \dots < b_{n-1} \in D$ and $v_0, \dots, v_{n-1} \in A$. Let S_i be the set of all subsets s of $\{b_j | \ j \in i\}$ such that $\operatorname{card}(s) = n$. Then $\operatorname{card}(S_i) < \lambda$. For $s \in S_i$, let $s = \{c_j | \ 1 \leqslant j \leqslant n\}$ where $c_1 < \dots < c_n$. Now choose v_s so that $\mathfrak{A} \models \forall y [x_n < v < w \wedge (v < y < w \to \theta)] [c/w, c_1/x, \dots, c_n/x_n, v_s/v, \vec{a}]$. The formula Ξ_n asserts that such a v_s exists. Since λ is regular and $\operatorname{card}(S_i) < \lambda$, it is possible to find $v_i \in A$ and $d \in I$ such that for each $s \in S_i$, $c > v_i > v_s$ and for each $j \in i$, $c > d > b_j$. Now choose $b_i \in I$ so that $c > b_i > \max\{v_i, d, c - d_i\}$. Then $\{b_i | \ i \in \lambda\}$ is a λ-enumeration, with c for an $\mathfrak{A}$-limit point, which can be used for $I(\mathfrak{A}; \Theta_n; \vec{a})$.

LEMMA 4. *Let φ be a quantifier free formula and let w be a variable which does not occur in φ. For each $n \in \boldsymbol{P}$ there are quantifier free formulas $\eta_n(\varphi)$ and $\mu_n(\varphi)$ whose free variables other than w form a sub et of those of $Q^n x_1 \dots x_n \varphi$ such that for any real closed field*

(i) *$\mathfrak{A} \models_{\aleph_0} Q x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; \vec{a}) = I$ can be taken to be an ω-enumeration with an $\mathfrak{A}$-limit point c larger than any element in I iff there is an unbounded subset of A which is an ω-enumeration and $\mathfrak{A} \models \eta_n(\varphi) c/w, \vec{a}]$.*

(ii) *$\mathfrak{A} \models_{\aleph_0} Q x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; \vec{a}) = I$ can be taken to be an ω^*-enumeration with an $\mathfrak{A}$-limit point c smaller than any element in I iff there is an unbounded subset of A which is an ω-enumeration and $\mathfrak{A} \models \mu_n(\varphi)[c/w, \vec{a}]$.*

Proof. Since the two parts of the lemma are similar, only the proof for part (i) is presented below.

Part (i). The formula φ is equivalent in all real closed fields to a formula of the form $\theta_1 \vee \dots \vee \theta_k$ where each θ_i is a conjunction of polynomial equalities or

inequalities. Now proceed by induction on n: Let $\eta_1(\varphi)$ be a quantifier free formula equivalent in all real closed fields to $\Xi_0(\theta_1) \vee \dots \vee \Xi_0(\theta_k)$. Then by Lemma 1, $\eta_1(\varphi)$ has the desired properties.

Now assume that part (i) holds for all positive integers less than $n+1$. By the distributive law for Q^{n+1} over disjunction, there are conjunctions $\bar{\theta}_1, \dots, \bar{\theta}_h$ of polynomial equalities or inequalities such that

$$\mathfrak{A} \models_{\aleph_0} Q^{n+1} x x_2 \dots x_n y \varphi[\vec{a}] \quad \text{iff}$$
$$\mathfrak{A} \models_{\aleph_0} Q^{n+1} x x_2 \dots x_n y \Theta_n(\bar{\theta}_1) \vee \dots \vee Q^{n+1} x x_2 \dots x_n y \Theta_n(\bar{\theta}_h)[\vec{a}].$$

By Lemmas 2 & 3, $\mathfrak{A} \models_{\aleph_0} Q^{n+1} x x_2 \dots x_n y \Theta_n(\bar{\theta}_i)[\vec{a}]$ iff $\mathfrak{A} \models_{\aleph_0} Q^n x x_2 \dots x_n \Xi_n(\bar{\theta}_i)[c/w, \vec{a}]$ and $I(\mathfrak{A}; \Xi_n(\bar{\theta}_i); c/w, \vec{a})$ has c for an $\mathfrak{A}$-limit point. Let $\xi(\bar{\theta}_i)$ be a quantifier free formula equivalent in all real closed fields to $\Xi_n(\bar{\theta}_i)$ and let z be a variable which does not occur in $\xi(\bar{\theta}_i)$. The induction hypothesis gives a quantifier free formula $\eta_n(\xi(\bar{\theta}_i))$ such that $\mathfrak{A} \models_{\aleph_0} Q x x_2 \dots x_n \xi(\bar{\theta}_i)[c/w, \vec{a}]$ iff $\mathfrak{A} \models \eta_n(\xi(\bar{\theta}_i))[c/w, c/z, \vec{a}]$ and there is an unbounded subset of A which is an ω-enumeration. Since w and z have the same interpretation, there is no harm in replacing z in $\eta_n(\xi(\bar{\theta}_i))$ by w. Finally, let $\eta_{n+1}(\varphi)$ be $\eta_n(\xi(\bar{\theta}_1)) \vee \dots \vee \eta_n(\xi(\bar{\theta}_h))$.

THEOREM 1. *Let φ be quantifier free. For each $n \in \boldsymbol{P}$ there are quantifier free ormulas $\gamma_n(\varphi)$, $\delta_n(\varphi)$, $\xi_n(\varphi)$, and $\psi_n(\varphi)$ whose free variables form a subset of those of $Q^n x_1 \dots x_n \varphi$ such that for any real closed field $\mathfrak{A}$,*

(i) *$\mathfrak{A} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; \vec{a})$ is an unbounded ω-enumeration iff $\mathfrak{A} \models \gamma_n(\varphi)[\vec{a}]$ and there is an unbounded subset of A which is an ω-enumeration.*

(ii) *$\mathfrak{A} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; \vec{a}]$ is an unbounded ω^*-enumeration iff $\mathfrak{A} \models \delta_n(\varphi)[\vec{a}]$ and there is an unbounded subset of A which is an ω-enumeration.*

(iii) *$\mathfrak{A} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; \vec{a}) = I$ is an ω-enumeration with an $\mathfrak{A}$-limit point larger than any element in I iff $\mathfrak{A} \models \xi_n(\varphi)[\vec{a}]$ and there is an unbounded subset of A which is an ω-enumeration.*

(iv) *$\mathfrak{A} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; a) = I$ is an ω^*-enumeration with an $\mathfrak{A}$-limit point smaller than any element in I iff $\mathfrak{A} \models \Psi_n(\varphi)[\vec{a}]$ and there is an unbounded subset of A which is an ω-enumeration.*

Proof. The proofs for parts (i) and (ii) are similar to the proof of Lemma 4. The proof for part (iii) is similar to the proof of part (iv) given below.

Part (iv). By Lemma 4, $\exists w \mu_n(\varphi)$ has all the desired properties except for being quantifier free. The Tarski–Chevalley theorem removes this defect.

There are two corollaries to Theorem 1. The first is included for completeness and duplicates a result in [4]. The second requires more information given in the form of two lemmas.

COROLLARY 1. *For each quantifier free formula φ there is another quantifier ree formula Φ, whose free variables form a subset of those of $Q^n x_1 \dots x_n \varphi$, such that or any Archimedean real closed field $\mathfrak{A}$, $\mathfrak{A} \models_{\aleph_0} Q x_1 \dots x_n \Phi[\vec{a}]$ iff $\mathfrak{A} \models \Phi[\vec{a}]$.*

Proof. Let Φ be $\gamma_n(\varphi)\vee\delta_n(\varphi)\vee\xi_n(\varphi)\vee\psi_n(\varphi)$. If $\mathfrak{A}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}]$ and $I(\mathfrak{A};\varphi;\vec{a})=I$ is not bounded, then by Theorem 1, $\mathfrak{A}\models\Phi[\vec{a}]$. If $\mathfrak{A}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}]$ and I is bounded, then since $\mathfrak{A}$ is a substructure of $\boldsymbol{R}$, $\boldsymbol{R}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}]$ and I has a limit point. Thus by Theorem 1, $\boldsymbol{R}\models\Phi[\vec{a}]$. Since $\mathfrak{A}$ is a substructure of $\boldsymbol{R}$ and Φ is quantifier free, $\mathfrak{A}\models\Phi[\vec{a}]$. Since $\boldsymbol{P}$ is unbounded in A, if $\mathfrak{A}\models\Phi$, then

$$\mathfrak{A}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}].$$

LEMMA 5. *For an uncountable regular cardinal $\varkappa$, let $\mathfrak{A}$ be a $\varkappa$-Archimedean real closed field, and let φ, $\gamma_n(\varphi)$, $\delta_n(\varphi)$, $\xi_n(\varphi)$, and $\psi_n(\varphi)$ be as described in Theorem 1.*

(i) *If $\mathfrak{A}\models\gamma_n(\varphi)[\vec{a}]$, then $\mathfrak{A}\models_\varkappa Q^n x_1\ldots x_n\varphi[\vec{a}]$ and $I(\mathfrak{A};\varphi;\vec{a})$ can be taken to be an unbounded $\varkappa$-enumeration.*

(ii) *If $\mathfrak{A}\models\delta_n(\varphi)[\vec{a}]$, then $\mathfrak{A}\models_\varkappa Q^n x_1\ldots x_n\varphi[\vec{a}]$ and $I(\mathfrak{A};\varphi;\vec{a})$ can be taken to be an unbounded $\varkappa^*$-enumeration.*

(iii) *If $\mathfrak{A}\models\xi_n(\varphi)[\vec{a}]$, then $\mathfrak{A}\models_\varkappa Q^n x_1\ldots x_n\varphi[\vec{a}]$ and $I(\mathfrak{A};\varphi;\vec{a})=I$ can be taken to be a $\varkappa$-enumeration with an $\mathfrak{A}$-limit point larger than any element in I.*

(iv) *If $\mathfrak{A}\models\psi_n(\varphi)[\vec{a}]$, then $\mathfrak{A}\models_\varkappa Q^n x_1\ldots x_n\varphi[\vec{a}]$ and $I(\mathfrak{A};\varphi;\vec{a})=I$ can be taken to be a $\varkappa^*$-enumeration with an $\mathfrak{A}$-limit point smaller than any element in I.*

Proof. Part (iii). (Proofs for parts (i), (ii), and (iv) are similar.) Assume $\mathfrak{A}\models\xi_n(\varphi)[\vec{a}]$. Then using the notation developed in Lemma 4 and its proof, $\mathfrak{A}\models\exists w\eta_n(\varphi)[\vec{a}]$; so there is $c\in A$ such that $\mathfrak{A}\models\eta_n(\varphi)[c/w,\vec{a}]$. Now use induction on n: If $n=1$, then $\mathfrak{A}\models\Xi_0(\theta_1)\vee\ldots\vee\Xi_0(\theta_k)[c/w,\vec{a}]$ and by Lemma 1, $\mathfrak{A}\models_\varkappa Q^1x\varphi[\vec{a}]$ and I has the desired properties. If $n=j+1$, then $\mathfrak{A}\models\eta_j(\xi(\bar{\theta}_1))\vee\ldots\vee\eta_j(\xi(\bar{\theta}_h))$. Then by the induction hypothesis,

$$\mathfrak{A}\models_\varkappa Q^j xx_2\ldots x_j\Xi_j(\bar{\theta}_1)\vee\ldots\vee Q^j xx_2\ldots x_j\Xi_j(\bar{\theta}_h)[c/w,\vec{a}].$$

By Lemma 3,

$$\mathfrak{A}\models_\varkappa Q^{j+1}xx_2\ldots x_jy\Theta_j(\bar{\theta}_1)\vee\ldots\vee Q^{j+1}xx_2\ldots x_jy\Theta_j(\bar{\theta}_h);$$

thus $\mathfrak{A}\models_\varkappa Qxx_2\ldots x_jy\varphi[\vec{a}]$ and I has the desired properties.

LEMMA 6. *For an uncountable regular cardinal $\varkappa$, let $\mathfrak{A}$ be a $\varkappa$-Archimedean real closed field and let φ be a quantifier free formula such that $\mathfrak{A}\models_\varkappa Q^n x_1\ldots x_n\varphi[\vec{a}]$ where $\vec{a}=\langle a_1,\ldots,a_l\rangle$.*

(i) *If $I(\mathfrak{A};\varphi;\vec{a})$ is not bounded, then there is a real closed field $\mathfrak{B}$ such that $\{a_1,\ldots,a_l\}\subseteq B\subseteq A$, $\mathfrak{B}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}]$, and $\{|x|\mid x\in I(\mathfrak{B};\varphi;\vec{a})\}$ can be taken to be an ω-enumeration.*

(ii) *If $I(\mathfrak{A};\varphi;\vec{a})$ is bounded, then there are real closed fields $\mathfrak{B}$ and $\mathfrak{C}$ such that there is an unbounded subset of C which is an ω-enumeration, $\{a_1,\ldots,a_l\}\subseteq B\subseteq A\cap C$, $\mathfrak{C}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}]$ and $I(\mathfrak{C};\varphi;\vec{a})$ has a $\mathfrak{C}$-limit point.*

Proof. Part (i). Suppose $I(\mathfrak{A};\varphi;\vec{a})$ is not bounded. Let D be a positive discrete subset of A which is not bounded. Then $\mathrm{card}(D)=\varkappa$. Let $b_0\in I(\mathfrak{A};\varphi;\vec{a})$, let $A_0=\boldsymbol{Q}\cup\{a_1,\ldots,a_l,b_0\}$, and let $\mathfrak{B}_0$ be a countable real closed field with $A_0\subseteq B_0\subseteq A$. Now assume that a countable real closed field $\mathfrak{B}_i$ with $B_i\subseteq A$ has been obtained. Since B_i is countable and $\varkappa$ is both regular and uncountable, there must be $d\in D$ such that $(\forall b\in B_i)(|b|<d)$. Choose $b_{i+1}\in I(\mathfrak{A};\varphi;\vec{a})$ so that $d<|b_{i+1}|$. Let $A_i=B_i\cup\{b_{i+1}\}$ and let $\mathfrak{B}_{i+1}$ be a countable real closed field such that $A_{i+1}\subseteq B_{i+1}\subseteq A$. Finally let $\mathfrak{B}=\bigcup_{i\in\omega}\mathfrak{B}_i$. Then $\mathfrak{B}$ is a countable real closed field with $\{a_1,\ldots,a_l\}\subseteq B\subseteq A$ and $\{b_i\mid i\in\omega\}$ is an unbounded set which can be used for $I(\mathfrak{B};\varphi;\vec{a})$.

Part (ii). Now assume that $I(\mathfrak{A};\varphi;\vec{a})=I$ is bounded. Let D^+ be a maximal positive discrete subset of A with $P\subseteq D^+$, let $D^-=\{x\in A\mid -x\in D^+\}$, and let $D=D^+\cup\{0\}\cup D^-$. Then $\mathrm{card}(D)=\varkappa$. Since I is bounded, there is a $d\in D^+$ so that $(\forall x\in I)(|x|<d)$. Let $D_0=\{x\in D\mid |x|\leqslant d\}$. Then $\mathrm{card}(D_0)<\varkappa$. Since D is maximal, for each $x\in A$, the set $\{y\in D\mid |x-y|<1\}$ has either one or two elements. Let $f_0\colon I\to D_0$ be given by $f_0(x)=\min\{y\in D_0\mid |x-y|<1\}$. Since $\varkappa$ is regular, there is a $b_0\in D_0$ such that $\mathrm{card}(\{x\in I\mid f_0(x)=b_0\})=\varkappa$. Let

$$I_0=\{x\in I\mid f_0(x)=b_0\}.$$

Then $\mathrm{card}(I_0)=\varkappa$ and each element of I_0 is between b_0-1 and b_0+1. Choose $c_0\in I_0$. Let $d_0=1$ and let $A_0=\boldsymbol{Q}\cup\{a_1,\ldots,a_l,b_0,c_0\}$. Let $\mathfrak{B}_0$ be a countable real closed field such that $A_0\subseteq B_0\subseteq A$. Assume that $I_i,b_i,d_i,c_0,\ldots,c_i$, and $\mathfrak{B}_i$ have been obtained such that $I_i\subseteq I$, $\mathrm{card}(I_i)=\varkappa$, each element of I_i is between $b_i-d_i^{-1}$ and $b_i+d_i^{-1}$, $b_i\in B_i$, $d_i\in D^+$, $\mathfrak{B}_i$ is a countable real closed field with $B_i\subseteq A$, and $c_0,\ldots,c_i$ are distinct members of I. Since $\mathfrak{B}_i$ is countable, there is $d_{i+1}\in D^+$ so that $(\forall x\in B_i)(|x|<d_{i+1})$. Let D_{i+1} be a maximal subset of A such that

$$(\forall x\in D_{i+1})(b_i-d_i^{-1}\leqslant x\leqslant b_i+d_i^{-1})$$

and

$$(\forall x\in D_{i+1})(\forall y\in D_{i+1})(x\neq y\to|x-y|\geqslant d_{i+1}^{-1}).$$

Then $\mathrm{card}(D_{i+1})<\varkappa$. Let $f_{i+1}\colon I_i\to D_{i+1}$ be given by

$$f_{i+1}(x)=\min\{y\in D_{i+1}\mid |x-y|<d_{i+1}^{-1}\}.$$

Then there is $b_{i+1}\in D_{i+1}$ such that $\mathrm{Card}(\{x\in I_i\mid f_{i+1}(x)=b_{i+1}\})=\varkappa$. Let $I_{i+1}=\{x\in I_i\mid f_{i+1}(x)=b_{i+1}\}$. Then $\mathrm{Card}(I_{i+1})=\varkappa$ and each element of I_{i+1} is between $b_{i+1}-d_{i+1}^{-1}$ and $b_{i+1}+d_{i+1}^{-1}$. Choose $c_{i+1}\in I_{i+1}-\{c_0,\ldots,c_i\}$. Let $A_{i+1}=B_i\cup\{b_{i+1},c_{i+1},d_{i+1}\}$ and let $\mathfrak{B}_{i+1}$ be a countable real closed field such that $A_{i+1}\subseteq B_{i+1}\subseteq A$. Finally let $\mathfrak{B}=\bigcup_{i\in\omega}\mathfrak{B}_i$. Then $\mathfrak{B}$ is a countable real closed field with $\{a_1,\ldots,a_l\}\subseteq B\subseteq A$, $\mathfrak{B}\models_{\aleph_0}Q^n x_1\ldots x_n\varphi[\vec{a}]$, $\{c_i\mid i\in\omega\}$ can be used for $I(\mathfrak{B};\varphi;\vec{a})$, and $\{d_i\mid i\in\omega\}$ is an unbounded subset of B.

Let $\mathscr{F}$ be an ultrafilter on ω containing no finite subsets of ω. Let $\mathfrak{M}$ be the ultrapower of $\mathfrak{B}$ with respect to $\mathscr{F}$. Let $\mathfrak{N}$ be the ordered ring obtained from $\mathfrak{M}$ by taking $N = \{x \in M|\ (\exists b \in B)(|x| < b)\}$. Then $\mathscr{I} = \{x \in N|\ (\forall b \in B)(|x| < b)\}$ is a maximal ideal of $\mathfrak{N}$. Finally let $\mathfrak{C}$ be the real closure of the field $\mathfrak{N}$ (mod $\mathscr{I}$). Since $\mathfrak{C}$ can be viewed as a real closed field with $B \subseteq C$, $\mathfrak{C} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$ and $\{c_i|\ i \in \omega\}$ can be used for $I(\mathfrak{C}; \varphi; \vec{a})$. All that remains to be shown is that $\{c_i|\ i \in \omega\}$ has a $\mathfrak{C}$-limit point: Let $\bar{g}$ be the element of M represented by the function $g\colon \omega \to B$ given by $g(i) = c_i$. Since all the c_i are between $b_0 - 1$ and $b_0 + 1$, $\bar{g}$ is in fact an element of N. Let c be a positive element of C. Then there is d_j such that $2c^{-1} < d_j$. Since for $i \geqslant j$, c_i is between $b_j - d_j^{-1}$ and $b_j + d_j^{-1}$, the set $\{i \in \omega|\ |g(i) - c_j| \leqslant 2d_j^{-1}\}$ is in the ultrafilter $\mathscr{F}$. Thus in $\mathfrak{N}$, $|\bar{g} - c_j| \leqslant 2d_j^{-1}$. Let $h = \bar{g} + \mathscr{I} \in C$. Then $|h - c_j| \leqslant 2d_j^{-1} < c$ in $\mathfrak{C}$. Therefore h is a $\mathfrak{C}$-limit point for $\{c_i|\ i \in \omega\}$.

COROLLARY 2. *Let $\varkappa$ be an uncountable regular cardinal. For each quantifier free formula φ there is another quantifier free formula Φ, whose free variables form a subset of those of $Q^n x_1 \dots x_n \varphi$, such that for any $\varkappa$-Archimedean real closed field $\mathfrak{A}$, $\mathfrak{A} \models_\varkappa Q^n x_1 \dots x_n \varphi[\vec{a}]$ iff $\mathfrak{A} \models \Phi[\vec{a}]$.*

Proof. Let Φ be $\gamma_n(\varphi) \vee \delta_n(\varphi) \vee \xi_n(\varphi) \vee \psi_n(\varphi)$. If $\mathfrak{A} \models_\varkappa Q^n x_1 \dots x_n \varphi[\vec{a}]$ and $I(\mathfrak{A}; \varphi; \vec{a}) = I$ is not bounded, then by Lemma 6, there is a real closed field $\mathfrak{B}$ such that $B \subseteq A$, $\mathfrak{B} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$, and $I(\mathfrak{B}; \varphi; \vec{a})$ is either an unbounded ω-enumeration or an unbounded ω^*-enumeration. Thus by Theorem 1, $\mathfrak{B} \models \Phi$. Since $B \subseteq A$, $\mathfrak{A} \models \Phi$. If $\mathfrak{A} \models_\varkappa Q^n x_1 \dots x_n \varphi[\vec{a}]$ and I is bounded, then by Lemma 6, there are real closed fields $\mathfrak{B}$ and $\mathfrak{C}$ such that $B \subseteq A \cap C$, $\mathfrak{C} \models_{\aleph_0} Q^n x_1 \dots x_n \varphi[\vec{a}]$, and $I(\mathfrak{C}; \varphi; \vec{a})$ is either an ω-enumeration which has a $\mathfrak{C}$-limit point larger than any element in $I(\mathfrak{C}; \varphi; \vec{a})$ or an ω^*-enumeration which has a $\mathfrak{C}$-limit point smaller than any element in $I(\mathfrak{C}; \varphi; \vec{a})$. By Theorem 1, $\mathfrak{C} \models \Phi$. Since $B \subseteq A \cap C$, $\mathfrak{B} \models \Phi$ and $\mathfrak{A} \models \Phi$. Finally by Lemma 5, if $\mathfrak{A} \models \Phi$, then $\mathfrak{A} \models_\varkappa Q^n x_1 \dots x_n \varphi$.

THEOREM 2. *For each regular $\varkappa$, every $\mathscr{Q}_\varkappa^{<\omega}$-formula φ of the language L is equivalent, in all $\varkappa$-Archimedean real closed fields, to a quantifier free formula ψ whose variables form a subset of those of φ.*

Proof. Use Corollary 2 and proceed by induction on the formation in $\mathscr{Q}_\varkappa^{<\omega}$ of the formula φ.

QUESTION. Does Theorem 2 remain a theorem if $\varkappa$ is a singular cardinal?

The logic $\mathscr{Q}_c^{<\omega}$ has the same syntax as the logic $\mathscr{Q}_\varkappa^{<\omega}$ but is given the equi-cardinal interpretation: $\mathfrak{A} \models_c Q^n x_1 \dots x_n \varphi[\vec{a}]$ iff there is a subset I of A such that (i) $\mathrm{Card}(I) = \mathrm{Card}(A)$ and (ii) for distinct $a_1, \dots, a_n \in I$, $\mathfrak{A} \models_c \varphi[a_1/x_1, \dots, a_n/x_n, \vec{a}]$. A c-Archimedean field is a $\varkappa$-Archimedean field where $\mathrm{Card}(A) = \varkappa$.

QUESTION. Does Theorem 2 remain a theorem when $\varkappa$ is replaced by c?

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