Proof. That G is locally shrinkable follows from (Theorem 6.1, [10]) and (Theorems 4 and 5, [6]). The theorem now follows from Theorem 4.2.

Recalling that an n-manifold is a separable metric space (not necessarily connected) having the property that each point possesses a neighborhood homeomorphic to either E^n or E_+^n, we have the following results.

COROLLARY 4.1. *Let G be a decomposition of an n-manifold M such that G satisfies one of the following sets of conditions*:

(1) *locally null and locally starlike*;

(2) *locally shrinkable, monotone, 0-dimensional, and* usc; *or*

(3) *locally star-0-dimensional and* usc.

Then M/G is homeomorphic to M and G is a shrinkable decomposition of M.

Remark. We gave two examples to illustrate Corollary 4.1 in Section 1.

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Toroidal decompositions of S^3 and a family of 3-dimensional ANR's (AR's)

by

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Abstract. It is shown that there exist an ANR X satisfying (1) $X\times S^1\approx S^3\times S^1$, (2) X does not contain any proper ANR of dimension larger than 1, and (3) the homeomorphism group of X is the trivial group; furthermore, there are uncountably many topologically distinct ANR's with these properties. It follows that the family of 3-dimensional AR's satisfying the properties (2) and (3), as above, is also uncountable. These ANR's are constructed as cell-like images of S^3, and hence, they are generalized manifolds and possess many other desirable properties. There exists a cellular image of S^3 satisfying the assertions, (1)–(3), given above (a suitable result for B^3 also holds). A problem of Bing concerning partitions of Peano continua is answered in the negative. A condition (Δ^a) is given and it is shown that a finite dimensional closed subset of an ANR $X\in(\Delta^a)$ has a locally connected ε-displacement inside X. Several other applications are also given.

1. Introduction and terminology.

(1.1) By an AR (ANR) we mean a compact metrizable absolute (neighborhood) retract in the category of metrizable spaces, see [13] and [21] for more details. An ANR X will be called *strongly irreducible* (Abbreviate: *s-irreducible*) if $X\times S^1\approx S^3\times S^1$ and X does not contain any proper ANR of dimension larger than one. Let E^n, B^n, and S^{n-1}, respectively, denote the n-dimensional Euclidean space, the closed unit ball in E^n, and the unit sphere in E^n. By $X\approx Y$ we mean X is homeomorphic to Y.

A method for constructing s-irreducible ANR's (or AR's) is given in [32] where these ANR's are constructed as decomposition spaces corresponding to certain null cell-like but non-cellular upper semicontinuous decompositions of S^3. We prefer to consider these decompositions for S^3 rather than B^3 to avoid technicalities concerning the boundary. It is routine to construct similar decompositions for B^3 once these decompositions for S^3 are known. By an *s-irreducible* decomposition G of S^3 we mean any cell-like upper semicontinuous decomposition G of S^3 such that S^3/G is an s-irreducible ANR. The purpose of this note is to show (1) there exist cellular s-irreducible decompositions of S^3, and (2) there are uncountably many s-irreducible decompositions of S^3. Other applications will also be given.

(1.2) If A is a subset of a metric space (X, d), the *diameter* $\Delta(A)$ of A is defined by $\Delta(A) = \sup\{d(x, y):\ x, y\in A\}$. If G is an u.s.c. decomposition ("upper semicontinuous decomposition") of a space X, we shall denote by $p\colon X \to X/G$ the natural projection onto the decomposition space X/G. The union of the nondegenerate elements of an u.s.c. decomposition G of a space X will be denoted by H_G. Throughout this note we shall call a set X *uncountable* if X has the cardinality of the set of the real numbers. The group of integers will be denoted by Z. By a *torus* T we mean a solid torus with boundary $\partial(T)$. All maps between spaces will be continuous.

2. Preliminaries.

(2.1) All our tori embedded in S^3, the 3-sphere, will be assumed to be polyhedral unless otherwise so stated. By linking of two simple closed curves in S^3 we mean integral homology linking, see [11, 28]. For notation and terminology concerning toroidal decompositions and upper semicontinuous decompositions we follow [3, 30].

(2.2) Suppose T_r is a torus in S^3. If $\{T_{r1}, T_{r2}, \ldots, T_{rm_r}\}$ is a linked chain in the sense of [30; p. 229] (or a chain in the sense of [3; p. 17]) we understand, as usual, that (1) $\bigcup_{i=1}^{m_r} T_{ri} \subset \operatorname{Int}(T_r)$, and for each i, $1 \leqslant i \leqslant m_r$, the torus T_{ri} lies in a 3-cell inside T_r; see [3, 30] where definitions of many undefined terms may also be found. Our immediate goal is to describe, in the next section, the construction of a defining sequence for a toroidal decomposition.

(2.3) Suppose T_r is a torus in S^3. Find a linked chain $\{T_{r1}, T_{r2}, \ldots, T_{rm_r}\}$ of tori circling T_r n_r times. For each i_1, $1 \leqslant i_1 \leqslant m_r$, find a linked chain $\{T_{ri_11}, T_{ri_12}, \ldots, T_{ri_1m_{ri_1}}\}$ of tori circling T_{ri_1} n_{ri_1} times. For each i_1, $1 \leqslant i_1 \leqslant m_r$, and i_2, $1 \leqslant i_2 \leqslant m_{ri_1}$, find a linked chain $\{T_{ri_1i_21}, T_{ri_1i_22}, \ldots, T_{ri_1i_2m_{ri_1i_2}}\}$ of tori circling $T_{ri_1i_2}$ $n_{ri_1i_2}$ times. Let this process be continued to construct the following:

(2.3.1) Positive integers m_r, n_r; m_{ri_1}'s and n_{ri_1}'s, with $1 \leqslant i_1 \leqslant m_r$; $m_{ri_1i_2}$'s and $n_{ri_1i_2}$'s, with $1 \leqslant i_1 \leqslant m_r$ and $1 \leqslant i_2 \leqslant m_{ri_1}$; …; $m_{ri_1i_2\ldots i_k}$'s and $n_{ri_1i_2\ldots i_k}$'s with $1 \leqslant i_1 \leqslant m_r$, $1 \leqslant i_2 \leqslant m_{ri_1}$, …, and $1 \leqslant i_k \leqslant m_{ri_1i_2\ldots i_{(k-1)}}$; …; and

(2.3.2) the linked chain $\{T_{r1}, T_{r2}, \ldots, T_{rm_r}\}$ circling T_r n_r times at the first stage of the construction and a manifold $M_{r1} = \bigcup\{T_{ri_1}\colon 1 \leqslant i_1 \leqslant m_r\}$; …; a linked chain $\{T_{r\alpha 1}, T_{r\alpha 2}, \ldots, T_{r\alpha m_{r\alpha}}\}$ circling $T_{r\alpha}$ $n_{r\alpha}$ times at $(k+1)$-th stage of the construction, and a manifold $M_{r(k+1)} = \bigcup_{i=1}^{m_{r\alpha}} \bigcup_{\alpha} T_{r\alpha i}$, where $\alpha = i_1 i_2 \ldots i_k$ with $1 \leqslant i_1 \leqslant m_r$, $1 \leqslant i_2 \leqslant m_{ri_1}, \ldots, 1 \leqslant i_k \leqslant m_{ri_1\ldots i_{(k-1)}}$. Put $I_r = \bigcap_{i=1}^{\infty} M_{ri}$. It is customary to call the sequence $\{M_{ri}\}_i$, a *defining sequence*, for the set I_r. The set L_r consisting of components of I_r will be called a *link substituting for* T_r.

(2.4) In [3], Armentrout specializes the construction of L_r by requiring that the positive integers in (2.3.1) satisfy the following:

(2.4.1) $2 \leqslant m_r < 2n_r$; $2 \leqslant m_{ri_1} < 2n_{ri_1}$, with $1 \leqslant i_1 \leqslant m_r$; …; $2 \leqslant m_{ri_1i_2\ldots i_k} < 2n_{ri_1i_2\ldots i_k}$, with $1 \leqslant i_1 \leqslant m_r$, $1 \leqslant i_2 \leqslant m_{ri_1}$, …, and $1 \leqslant i_k \leqslant m_{ri_1i_2\ldots i_{(k-1)}}$; … Any link L_r constructed by satisfying (2.4.1) will be called an (m_α, n_α)*-link substituting for* T_r. We further

specialize the construction of L_r by requiring that the positive integers in (2.4.1) satisfy the following:

(2.4.2) $2 \leqslant m_r < n_r$; $2 \leqslant m_{ri_1} < n_{ri_1}$, with $1 \leqslant i_1 \leqslant m_r$; …; $2 \leqslant m_{ri_1i_2\ldots i_k} < n_{ri_1i_2\ldots i_k}$, with $1 \leqslant i_1 \leqslant m_r$, $1 \leqslant i_2 \leqslant m_{ri_1}$, …, and $1 \leqslant i_k \leqslant m_{ri_1i_2\ldots i_{(k-1)}}$; …

Any link L_r constructed in T_r by satisfying (2.4.2) will be called a *special* (m_α, n_α)*-link substituting for* T_r.

(2.5) A 1-dimensional continuum C' will be called an *s-circle* if there exists a cell-like u.s.c. decomposition G of C' such that the set $p(H_G)$ is 0-dimensional and the decomposition space $C = C'/G$ is homeomorphic to a circle. It follows from a theorem of Sher [31] (cf. [14; p. 352]) that C' has the shape of the circle S^1. The following result is an immediate consequence of [32; p. 24–27]:

(2.5.1) If C' is an s-circle in S^3, then there exists a PL simple closed curve C with rational vertices such that C links C'.

The following result is an immediate consequence of (2.5.1) and some results given in [11] and [32]:

(2.5.2) There exists a countable family $\mathcal{F}$ of PL simple closed curves in S^3 (with rational vertices), e.g., let $\mathcal{F}$ consist of all PL simple closed curves with rational vertices, such that for every s-cricle P in S^3 and each open subset U of S^3 with $P \cap U = \emptyset$, there exists a simple closed curve C belonging to $\mathcal{F}$ such that P and C are linked and $C \cap U \neq \emptyset$.

(2.5.3) Suppose a countable family $\mathcal{F}$ of simple closed curves satisfying the conclusions of (2.5.2) is given. Let $\{C_i\}_{i=1}^{\infty}$ be an enumeration of the family $\mathcal{F}$ such that each member of $\mathcal{F}$ appears in the sequence $\{C_i\}_{i=1}^{\infty}$ infinitely many times. A sequence $\{C_i\}_{i=1}^{\infty}$, constructed in this manner, will be called a *dense sequence of simple closed curves in* S^3.

(2.5.4) Suppose a dense sequence $\{C_i\}_{i=1}^{\infty}$ of simple closed curves is given. For each i, $1 \leqslant i < \infty$, choose a normal disc bundle (solid torus) neighborhood T_i of C_i in S^3 such that each normal disc has a radius less than $1/i$. The PL torus T_i has C_i as its core or centerline. The sequence $\{T_i\}_{i=1}^{\infty}$ of PL tori constructed in this manner has the property that for every s-circle P in S^3 and an open subset U of S^3 with $P \cap U = \emptyset$, there exists a torus $T_i \subset (S^3 - P)$ such that the centerline C_i of T_i is linked with P and the set $(T_i \cap U)$ contains a meridional disc of T_i. The sequence of tori $\{T_i\}_{i=1}^{\infty}$, described above, will be called a *dense sequence of tori corresponding to the dense sequence* $\{C_i\}_{i=1}^{\infty}$ *of simple closed curves*. By a CT *dense sequence* $\{(C_i, T_i)\}_{i=1}^{\infty}$ we mean a dense sequence $\{C_i\}_{i=1}^{\infty}$ of simple closed curves in S^3 and a dense sequence $\{T_i\}_{i=1}^{\infty}$ of tori corresponding to the sequence $\{C_i\}_{i=1}^{\infty}$.

3. The main construction.

(3.1) Let $\{(C_i, T_i)\}_{i=1}^{\infty}$ be a CT sequence in S^3. Find a linked chain $\{T_{11}, T_{12}, \ldots, T_{1l_1}\}$ of tori circling T_1 exactly once such that for each i, $1 \leqslant i \leqslant l_1$, the diameter $\Delta(T_{1i})$ is less than 1. For each i, $1 \leqslant i \leqslant l_1$, construct a (special) (m_α, n_α)-link L_{1i}, consisting of components of the set I_{1i}, substituting for T_{1i}. The set $W_1 = \{L_{1i}\colon 1 \leqslant i \leqslant l_1\}$ will be called a (special) (m_α, n_α)-wreath substituting for T_1. We proceed inductively. Suppose (special) (m_α, n_α)-wreaths $W_1, W_2, \ldots, W_{(n-1)}$ have been constructed. Find a linked chain $\{T_{n1}, T_{n2}, \ldots, T_{nl_n}\}$ of tori circling T_n exactly once such that for each i, $1 \leqslant i \leqslant l_n$, the diameter $\Delta(T_{ni})$ is less than $1/i$. For each i, $1 \leqslant i \leqslant l_n$, construct a (special) (m_α, n_α)-link L_{ni} consisting of components of I_{ni} where the set $\bigcup_{i=1}^{l_n} I_{ni}$ has the property that the sets $\bigcup_{i=1}^{l_1} I_{1i}, \bigcup_{i=1}^{l_2} I_{2i}, \ldots, \bigcup_{i=1}^{l_{(n-1)}} I_{(n-1)i}$, and $\bigcup_{i=1}^{l_n} I_{ni}$ are mutually disjoint. Let $W_n = \{L_{ni}\colon 1 \leqslant i \leqslant l_n\}$ denote the (special) (m_α, n_α)-wreath substituting for T_n. We continue this process to construct a sequence $\{W_i\}_{i=1}^{\infty}$ of (special) (m_α, n_α)-wreaths such that for each i, $1 \leqslant i < \infty$, W_i substitutes for T_i. We need some terminology for convenience of reference to these constructions.

(3.1.1) By a CTW *dense sequence* $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$ in S^3 we shall mean a CT dense sequence $\{(C_i, T_i)\}_{i=1}^{\infty}$ and a sequence of (m_α, n_α)-wreaths as constructed in (3.1). By a *special* CTW *dense sequence* $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$ we mean that $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$ is a CTW dense sequence such that for each i, $1 \leqslant i < \infty$, W_i is a special (m_α, n_α)-wreath substituting for T_i.

(3.2) Let $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$ be a (special) CTW dense sequence in S^3. Put

$$L = \Big(\bigcup_{i=1}^{l_1} L_{1i}\Big) \cup \Big(\bigcup_{i=1}^{l_2} L_{2i}\Big) \cup \ldots \cup \Big(\bigcup_{i=1}^{l_n} L_{ni}\Big) \cup \ldots$$

and let $|L|$ denote the union of sets in L. Define a decomposition of S^3 by

$$G = L \cup \{\{x\}\colon x \in (S^3 - |L|)\}.$$

We shall say that the decomposition G is *induced* by the (special) CTW dense sequence.

(3.3) *If G is a decomposition of S^3 induced by a (special)* CTW *dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$, then G is an* u.s.c. *decomposition.* We omit details of an elementary proof of (3.3).

4. Some basic results.

(4.1) Suppose G is u.s.c. decomposition of S^3 induced by a (special) CTW dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$. Let $p\colon S^3 \to S^3/G$ denote the projection onto the decomposition space. We now state several general propositions concerning this setting:

(4.1.1) *The image, $p(H_G)$, of the union of all the nondegenerate elements of G is a 0-dimensional subset of S^3/G.* This follows from the construction of G.

(4.1.2) *The decomposition G is a cellular* u.s.c. *decomposition of S^3, i.e., every element of G is cellular in S^3.* This follows from (2.2).

(4.1.3) *The decomposition space S^3/G has the property that $S^3/G \times S^1$ is homeomorphic to $S^3 \times S^1$.* This follows from [19].

(4.1.4) *Every s-irreducible space X is three dimensional.* This follows from (4.1.3) and [22; p. 34].

(4.1.5) *The decomposition space S^3/G is a 3-dimensional* ANR *and the map $p\colon S^3 \to S^3/G$ is a (simple and/or fine) homotopy equivalence. Furthermore, for each closed subset A of S^3/G the restriction $p|\colon p^{-1}(A) \to A$ is a shape equivalence.* See [7, 18, 20, 23, 31] and the survey article [25].

5. Decompositions and S-irreducible ANR's (AR's).

(5.1) Our immediate goal is to outline proofs of Theorems (5.1.1) and (5.1.2) which are stated below. We rely heavily on [3] for many technical details.

(5.1.1) *Suppose G is an* u.s.c. *decomposition of S^3 induced by a special* CTW *dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$ such that each nondegenerate element of G is 1-dimensional. Then, S^3/G is an s-irreducible* ANR.

(5.1.2) *Suppose G is an* u.s.c. *decomposition of S^3 induced by a* CTW *dense sequence $\{(C_i, T_i, W_i)\}_{i=0}^{\infty}$ such that each nondegenerate of G is 1-dimensional. Then, S^3/G is an s-irreducible* ANR.

We remark that the assumption, "each nondegenerate element of G is 1-dimensional", in (5.1.1) and (5.1.2) can be easily satisfied by carefully choosing defining sequences. We have made this assumption so that we may use facts concerning homological linking from [32]. The results on homological linking given in [32] can be easily extended to prove the following more general result:

(5.1.3) *Suppose G is an* u.s.c. *decomposition of S^3 induced by a (special)* CTW *dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$. Then S^3/G is an s-irreducible* ANR.

(5.2) Suppose S^3/G contains a proper AR A such that A has dimension larger than 1, i.e., A has dimension 2 or 3. Since A has a dimension larger than 1, A contains a simple closed curve C. We need the following:

(5.2.1) The set $C' = p^{-1}(C)$, where $p\colon S^3 \to S^3/G$ is the projection, is an s-circle.

Proof. It suffices to show that C' is a 1-dimensional continuum. By (4.1.1), the image $p(H_G)$ is zero dimensional. Consider the family of closed sets $\{p^{-1}(c)\}_{c \in C}$. Our proof is finished by applying Proposition G of [22; p. 90]. This is similar to an argument given in [33].

(5.2.2) Suppose U_0 is an open subset of S^3/G such that $A \subset U_0$. Then there exists a sequence $\{U_i\}_{i=0}^{\infty}$ of open subsets of S^3/G such that (1) $A = \bigcap_{i=0}^{\infty} U_i$, (2) $U_{i+1} \subset U_i$ for $0 \leqslant i < \infty$, and (3) each loop in U_{i+1} is nullhomotopic in U_i for $0 \leqslant i < \infty$. This is well known see [4, 14].

(5.2.3) Suppose a sequence $\{U_i\}_{i=0}^{\infty}$ of open subsets of S^3/G satisfying the conclusions of (5.2.2) is given. Then, the sequence $\{V_i = p^{-1}(U_i)\}_{i=0}^{\infty}$ of open subsets of S^3 has the properties (1) $A' = p^{-1}(A) = \bigcap_{i=0}^{\infty} V_i$, (2) $V_{i+1} \subset V_i$ for $0 \leqslant i < \infty$, and (3) each loop in V_{i+1} is nullhomotopic in V_i for $0 \leqslant i < \infty$ [2; Lemma 9]

Choose a sequence $\{U_i\}_{i=0}^{\infty}$ of open subsets of S^3/G satisfying the conclusion of (5.2.2) and $U_0 \cap W = \emptyset$ where $W \subset (S^3/G - A)$ and W is an open subset of S^3/G. Now, the sequence $\{V_i = p^{-1}(U_i)\}_{i=0}^{\infty}$ of open subsets of S^3 satisfies the conclusions of (5.2.3) and $V_0 \cap W' = \emptyset$ where $W' = p^{-1}(W) \subset (S^3 - A')$ is an open subset of S^3. By (5.2.1), we choose an s-circle $C' = p^{-1}(C)$ inside A' where C is a simple curve contained inside A. Since the decomposition G is induced by a CTW dense sequence $\{(C_i, T_i, W_i)\}_{i=0}^{\infty}$, there exists, see (2.5.4), an index i such that (1) the torus T_i is contained in $(S^3 - C')$, (2) the core C_i is linked with C', and (3) the set $(T_i \cap W')$ contains a meridional disc of T_i. Consider the chain $\{T_{i1}, T_{i2}, \ldots, T_{il_i}\}$ of tori circling T_i exactly once, see (3.1). Put $n = l_i$. By the "lifting" argument [7], choose a PL simple closed curve E_1 such that $E_1 \subset (S^3 - T_i)$, $E_1 \subset V_{n+2}$, and E_1 is linked with C_i. Since each loop in V_{n+2} is nullhomotopic in V_{n+1}, it follows that E_1 bounds a PL singular disc D_1 in V_{n+1}. By the usual arguments concerning the curves of intersection, we may assume that there exists a torus T_{ij}, $1 \leqslant j \leqslant n$, and a PL meridional disc D in $(T_{ij} \cap V_{n+1})$, see [32; p. 31–32] for details. For simplicity of notation we let $\alpha = ij$. We shall assume, from now on, that our decomposition is induced by special CTW dense sequence. We need the following:

(5.2.4) Suppose T is a torus in S^3 and $\{T_1, T_2, \ldots, T_m\}$ is a chain of tori circling T n times with $2 \leqslant m < n$. Suppose D_0 is a PL meridional disc in T such that D_0 and ∂T_i, $1 \leqslant i \leqslant m$, are in relative general position. Let T^* be the universal covering space of T. Then, there exists an integer i, $1 \leqslant i \leqslant m$, and two consecutive copies D_1 and D_2 of D_0 in T^* such that some copy T_i^* in T^* intersects D_1 and D_2 meridionally.

This result is an adaption of Lemma 2 of [3]. Moreover, a proof for (5.2.4) follows immediately by suitably applying the arguments of [3; Lemma 2]. We omit details.

We shall now return to our original setting in our proof. Recall that $\alpha = ij$. Consider the special (m_α, n_α)-link L_α substituting for T_α and the chain $\{T_{\alpha 1}, T_{\alpha 2}, \ldots, T_{\alpha n_\alpha}\}$ circling T_α n_α times. We may assume that the meridional disc D is in relative general position with $\partial T_{\alpha i}$ where $1 \leqslant i \leqslant m_\alpha$. Let T_α^* be the universal covering space of T_α. It is clear that the hypotheses of (5.2.4) are satisfied, and therefore, we may assume the conclusions of (5.2.4). This means that the hypotheses of Lemma 5 of [3] are satisfied with $U = V_{n+1}$. Therefore, there is a loop γ_α in $(T_\alpha \cap V_{n+1})$ such that γ_α is not nullhomotopic to zero in T_α. Now $T_\alpha = T_{ij}$, $1 \leqslant j \leqslant n$ and $n = l_i$, is linked with a torus $T_{i(j+1)}$ where indices are computed cyclically. The loop γ_α is nullhomotopic inside V_n and the core of $T_{i(j+1)}$ is linked with γ_α. It follows by an argument similar to the one used to construct $\gamma_\alpha = \gamma_{ij}$ that there exists a loop $\gamma_{i(j+1)}$ in $(T_\alpha \cap V_n)$ such that $\gamma_{i(j+1)}$ is not nullhomotopic inside $T_{i(j+1)}$. We continue in this manner to construct a linked chain $\{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}$ of loops in $(T_i \cap V_0)$. Since $(T_i \cap W')$ contains a meridional disc, it follows that the $V_0 \cap W' \neq \emptyset$. This contradicts the fact that $V_0 \cap W' = \emptyset$ and this proves that S^3/G does not contain any proper AR of dimension larger than 1. It follows from the arguments in [32] that S^3/G does not contain any proper ANR of dimension larger than 1.

6. Uncountably many S-irreducible ANR's.

(6.1) Let $\mathscr{C}$ denote the family of all the s-irreducible ANR's under the equivalence relation of "the same topological type." We shall show that the class $\mathscr{C}$ is uncountable, i.e., $\mathscr{C}$ has the cardinality of the reals see (1.2). More precisely, we shall prove the following:

(6.1.1) *There exists an uncountable set Λ such that for each $\lambda \in \Lambda$, there exists a cell-like* u.s.c. *decomposition G_λ of S^3 such that the decomposition space S^3/G_λ is an s-irreducible* ANR. *Furthermore, the mapping $\Lambda \to \mathscr{C}$ defined by $\lambda \to S^3/G_\lambda$ is* 1–1 *and hence $\mathscr{C}$ is uncountable.*

The following result is useful in this sequel:

(6.1.2) There exists a null collection $\{\alpha_i: 1 \leqslant i < \infty\}$ of mutually disjoint arcs in S^3 such that $\Pi_1(S^3 - \alpha_i)$ is not isomorphic to $\Pi_1(S^3 - \alpha_j)$ whenever $i \neq j$. A specific collection of arcs of this type is given in [29].

Suppose a collection $\{\alpha_i: 1 \leqslant i < \infty\}$ of arcs satisfying the assertions of (6.1.2) is given. Let Λ denote the set of all the infinite subsets of $\{\alpha_i: 1 \leqslant i < \infty\}$. Suppose $\lambda \in \Lambda$ is given. We let $\lambda = \{\beta_i: 1 \leqslant i < \infty\}$. Choose a CT dense sequence $\{(C_i, T_i)\}_{i=1}^{\infty}$ as described in (2.5.4). For each i, $1 \leqslant i < \infty$, we construct (special) (m_α, n_α)-wreath W_i substituting for T_i as described in (3.1) and such that the set $\bigcup \{I_{ij}: 1 \leqslant j \leqslant l_i\}$ does not meet the set $\bigcup \{\beta_i: 1 \leqslant i < \infty\}$. This can be easily accomplished by requiring that the kth stage tori, used in the construction of the set I_r inside the torus T_r, miss the set $\bigcup \{\beta_i: 1 \leqslant i \leqslant k\}$. This process yields a (special) CTW dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$ such that for each i, $1 \leqslant i < \infty$, the links of W_i do not meet the set $\bigcup \{\beta_i: 1 \leqslant i < \infty\}$. Since $\{\alpha_i: 1 \leqslant i < \infty)$ is a null collection, it is clear that $\{\beta_i: 1 \leqslant i < \infty\}$ is also a null collection. Define a decomposition G_λ of S^3 by requiring that the set of all the nondegenerate elements of G_λ is the union of the set of all the nondegenerate elements of G, where G is the induced decomposition of S^3 by the (special) CTW dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$, with the set $\{\beta_i: 1 \leqslant i < \infty\}$. It follows that G_λ is an u.s.c. decomposition of S^3. The fact that S^3/G_λ is an s-irreducible ANR is clear from discussions in (4) and (5). Suppose λ, $\nu \in \Lambda$ such that $\lambda \neq \nu$, i.e., there exists an arc α_j satisfying $\alpha_j \in \lambda$ and $\alpha_j \notin \nu$. Now

$$\Pi_1(S^3 - g) \simeq \Pi_1[S^3/G_\lambda - p(g)]$$

for $g \in G_\lambda$ and hence $\Pi_1[S^3/G_\nu - \{x\}] \not\simeq \Pi_1(S^3 - \alpha_j)$ for any $x \in S^3/G_\nu$, where base points are suppressed. We have used some facts concerning the fundamental group and cell-like decompositions (cf. [25]). Our proof for (6.1.1) is finished.

The following are some immediate corollaries of our method:

(6.1.3) *There exists an s-irreducible X such that $\Pi_1(X - \{x\}) = 0$ for each $x \in X$.*

(6.1.4) *There exist nonisomorphic groups $G_1, G_2, \ldots, G_n$ and an s-irreducible* ANR *X with points $x_1, x_2, \ldots, x_n$ such that* (1) $\Pi_1(X - \{x_i\}) = G_i$ *for* $1 \leqslant i \leqslant n$, *and* (2) $\Pi_1(X - \{x\}) = 0$ *for each* $x \in (X - \{x_i: 1 \leqslant i \leqslant n\})$.

(6.1.5) *There exist nonisomorphic groups $G_1, G_2, \ldots$ and an s-irreducible* ANR

X with points $x_1, x_2, \ldots$, *such that* (1) $\Pi_1(X-\{x_i\}) = G_i$ *for* $1 \leqslant i < \infty$, *and* (2) $\Pi_1(X-\{x\}) = 0$ *for* $x \in (X-\{x_i\colon 1 \leqslant i < \infty\})$.

(6.1.6) *There exist uncountable many s-irreducible* ANR X *such that the group of homeomorphism of* X *is the trivial group.* (This may be compared with [29].)

A finite dimensional ANR X is a *generalized n-manifold* if for each $x \in X$, $H_*(X, X-\{x\}; Z) \simeq H_*(E^n, E^n-\{0\}; Z)$. It is well known that a finite dimensional cell-like image of a closed manifold is a generalized manifold [35], and therefore, the decomposition spaces of S^3 constructed in this note are generalized 3-manifolds.

Our results can be suitably stated for decomposition of E^3 and B^3. There is a standard method of constructing a decomposition of B^3 by choosing a sequence of arcs (chords) [11, 32]. We shall state the following as a sample:

(6.1.7) *There exist uncountably many topologically distinct 3-dimensional* AR*'s which are cell-like images of* B^3 *and each of which does not contain any* ANR *of dimension larger than one.*

7. A problem of Bing.

(7.1) A set M is said to be *partitionable* if for each $\varepsilon > 0$ there is a finite collection P of mutually exclusive connected open subsets of M such that each element of P has diameter less than ε and the union of elements of P is dense in M. The collection P is called an *ε-partitioning* of M.

The following question appears in [10; p. 555]:

(7.1.1) *If M is locally simply connected, can it be partitioned into simply connected pieces?*

(7.1.2) A space X is *locally simply connected at a point* x in X if each neighborhood U of x in X contains a neighborhood V of x in X such that each loop in V is nullhomotopic in U. We say X is *locally simply connected* if X is locally simply connected at every point in X. Any ANR is locally simply connected (locally contractible) (cf. [13]).

The following provides a negative answer to (7.1.1):

(7.1.3) *There exists a Peano continuum X such that X is locally simply connected and X is not partitionable with simply connected pieces. Moreover, the required Peano continuum X can be chosen to be a simply connected* ANR *which does not have any partition P with simply connected pieces of diameter less than the diameter of X.*

Proof of (7.1.3) Let G be an u.s.c. decomposition of S^3 which is induced by a CTW dense sequence $\{(C_i, T_i, W_i)\}_{i=1}^{\infty}$. Put $X = S^3/G$. Suppose $P = \{U_1, U_2, \ldots, U_n\}$ is an arbitrarily given partitioning of X such that for each i, $1 \leqslant i \leqslant n$, U_i is simply connected. This means that the sets $p^{-1}(U_1)$, $p^{-1}(U_2)$, ..., and $p^{-1}(U_n)$ are mutually exclusive simply connected open subsets of S^3 [7]. This is impossible because of the following: Any simply connected open subset of S^3 which is saturated with respect to the decomposition G is dense in S^3, see our arguments in (5). This finishes our proof for (7.1.3).

8. ε-displacements in some ANR's.

(8.1) Aleksandrov [1; p. 7] has used ε-displacements in his study of dimension theory and homology theory of compacta. The concept has proved useful in many other situations, see for example, the proof of Theorem 2.1 in [13; p. 164]. Since an ANR may not contain enough proper ANR's, it is natural to seek ε-displacements in ANR's which have some nice local properties. More precisely, we shall show that ε-displacements of compacta into locally connected (Abbreviate: lc) compacta inside ANR's satisfying a condition (Δ^a) can, indeed, be constructed, see (8.3)–(8.6). We begin with some definitions.

(8.1.1) A closed subset A of a compact metric space X is lc-*displacable in* X if and only if for each $\varepsilon > 0$, there exists a surjective map $\varphi\colon A \to P_\varepsilon$ such that (1) $\varphi(A) = P_\varepsilon$ is an lc closed subset of X, (2) $\dim(P_\varepsilon) \leqslant \dim(A)$, and (3) $d[a, \varphi(a)] < \varepsilon$ for each $a \in A$. The set P_ε will be called an ε-lc-*displacement of A in X*. This definition is adapted from [1; p. 7], see also [22; p. 72–73], and [13; p. 164].

(8.2) A compact metric space Y *satisfies the condition* (Δ^a). (Notation: $Y \in (\Delta^a)$) if and only if for any compact metric space X the subset $\{f \in Y^X\colon \dim[f(X)] \leqslant \dim(X)\}$ of the function space Y^X is dense in the space Y^X. This definition is motivated by Borsuk's condition (Δ) and his Theorem 2.1 in [13; p. 164]. Clearly, $X \in (\Delta)$ implies $X \in (\Delta^a)$. This condition (Δ^a) may be thought of as an "approximate (Δ)".

(8.3) *Every finite dimensional closed subset of an* ANR X *such that* $X \in (\Delta^a)$ *is* lc-*displacable in* X.

Proof. Suppose X is contained in the Hilbert cube Q. Suppose A is a closed subset of X with $\dim(A) < \infty$. Let U be an arbitrary neighborhood of A in X and let $\varepsilon > 0$ be given. Choose a (compact) neighborhood V of X in Q and a retraction $r\colon V \to X$ such that $d[v, r(v)] < \frac{1}{4}\varepsilon$ for all $v \in V$. Let $W \subset V$ be a neighborhood of A in V such that $r(W) \subset U$. By [1; p. 7], let $\psi\colon A \to P'$ be a surjective map satisfying (1) $\psi(A) = P'$ is a polyhedron contained in W, (2) $\dim(A) = \dim(P')$, and (3) $d[a, \psi(a)] < \frac{1}{4}\varepsilon$ for each $a \in A$. Let us denote by $r\colon P' \to r(P')$ the restriction of $r\colon V \to X$. Since $X \in (\Delta^a)$, there exists a subset P of U and a surjective map $\xi\colon P' \to P$ such that $d[r(x), \xi(x)] < \frac{1}{4}\varepsilon$ for each $x \in P'$, and $\dim(P) \leqslant \dim(P')$. Define $\varphi\colon A \to P$ as the composite of the maps $\psi\colon A \to P'$ and $\xi\colon P' \to P$. It is now clear that $P_\varepsilon = P$ is an ε-lc-displacement of A in X and our proof is finished.

We need the following result for the further study of our decomposition spaces. This result was pointed out to us by R. J. Daverman.

(8.4) *Suppose G is a cell-like* u.s.c. *decomposition of S^n such that the image, $p(H_G)$, of the union of all the nondegenerate elements of G is a 0-dimensional subset of the decomposition space S^n/G. Then, S^n/G is an* ANR *satisfying the condition* (Δ^a).

Proof. Let $f\colon K \to S^n/G$ be a map, where K is a k-dimensional polyhedron. By the usual "lifting arguments" choose a lift $\tilde{f}\colon K \to S^n$ such that $p\tilde{f}$ and f are sufficiently close in the function space and $\dim[\tilde{f}(K)] \leqslant k$ [7] and [25]. There is no loss of generality if we assume that the lift $\tilde{f}: K \to S^n$ is PL and $[\tilde{f}(K) \cap H_G]$ is dense in the complex $\tilde{f}(K)$. It follows easily from Corollary 1 of [22; p. 46] that $\dim[\tilde{f}(K) - H_G] \leqslant (k-1)$. Now, $p\tilde{f}(K) = p[\tilde{f}(K) - H_G] \cup p[\tilde{f}(K) \cap H_G]$ where $\dim\{p[\tilde{f}(K) - H_G]\} \leqslant (k-1)$ and $\dim\{p[\tilde{f}(K) \cap H_G]\}$ equals to zero. By [22; p. 28], it follows that $\dim\{p[\tilde{f}(K)]\} \leqslant k$. By Borsuk [13; p. 164], see the proof of Theorem 2.1 in [13], our proof is finished.

We now state the following corollary of (8.4):

(8.5) *Each s-irreducible* ANR S^3/G *constructed in this note has the property that each closed subset A of S^3/G is* lc*-displacable in S^3/G: furthermore, if A is connected, then any ε-*lc*-displacement P_ε is a Peano continuum.*

Proof. By (8.4), $S^3/G \in (\Delta^a)$ and our proof is finished by (8.3).

We have also proved the following more general result:

(8.6) *Suppose G is a cell-like* u.s.c. *decomposition of S^n, $n \geqslant 0$, such that $p(H_G)$ is 0-dimensional. Then, each closed subset A of S^n/G is* lc*-displacable in S^n/G; furthermore, if A is connected, then any ε-*lc*-displacement P_ε is a Peano continuum.*

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