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A generalized area integral estimate and applications

by

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Abstract. We consider functions defined on the unit ball B^n in C^n which are harmonic with respect to the Laplace–Beltrami operator of the Bergman metric. We estimate the area integral of such functions and obtain, as applications of such estimates: the L^p-growth of the area integral being bounded by L^p-growth of non-tangential maximal function for $0 < p < \infty$ and other results.

§0. Introduction and notations. In this paper we introduce a method to estimate area integral in terms of the L^p-growth of a harmonic function. This method avoids the repeated use of the distribution function in dealing with this type of problems. As an example in Theorems 1, 2 and 3 below we extend some classical results on boundary behavior of harmonic functions defined on upper-half space ([4], [10], [11]) to the case of the unit ball in C^n.

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Let B^n denote the unit ball in C^n, i.e. $B^n = \{z \in C^n, |z| < 1\}$. Write $z \in B^n$ as $(z_1, \ldots, z_n)$, $z_k = x_k + iy_k$, $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$, $\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$ and $\delta(z) = 1 - |z|^2$. The *Bergman metric* on B^n is

$$ds^2 = \sum_{i,j} g_{ij}\, dz_i\, d\bar{z}_j, \quad \text{where} \quad g_{ij} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \frac{1}{\big(\delta(z)\big)^{n+1}}.$$

Let (g^{ij}) be the inverse matrix of (g_{ij}), then the *Laplace–Beltrami operator* on B^n is defined to be

$$\Delta F = 4 \sum_{i,j} g^{ij} \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j},$$

and we call a function F *harmonic* if $\Delta F = 0$. The gradient with respect

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to the Bergman metric is given by

$$\nabla F = 2\sum g^{ij}\left\{\frac{\partial F}{\partial \bar z_i}\frac{\partial}{\partial z_j}+\frac{\partial F}{\partial z_j}\frac{\partial}{\partial \bar z_i}\right\},$$

and the square norm of ∇F is

$$|\nabla F|^2 = 2\left(\sum_{i,j} g^{ij}\frac{\partial F}{\partial z_i}\frac{\overline{\partial F}}{\partial z_j}+\sum_{i,j} g^{ij}\frac{\partial F}{\partial \bar z_i}\frac{\overline{\partial F}}{\partial \bar z_j}\right).$$

The volume element $d\Omega$ induced by the metric is

(1) $$d\Omega(z) = \det(g_{ij})\,d\omega(z) = (n+1)^n\big(\delta(z)\big)^{-(n+1)}d\omega(z),$$

where $d\omega = dx_1, \ldots, dx_n dy_1, \ldots, dy_n$ is the Euclidean element of volume.

The *Poisson–Szegö kernel* $P(z, \zeta)$ for $z \in \boldsymbol{B}^n$, $\zeta \in \partial \boldsymbol{B}^n$ is given by

$$P(z, \zeta) = \frac{(n-1)!}{2\pi^n}\,\frac{(1-|z|^2)^n}{|1-z\cdot\bar\zeta|^{2n}}.$$

For any function $f \in L^p(\partial \boldsymbol{B}^n)$ for $p \geqslant 1$, the Poisson integral F of f

$$F(z) = \int_{|\zeta|=1} P(z, \zeta) f(\zeta)\, d\sigma(\zeta)$$

where $d\sigma(\zeta)$ is the induced Euclidean measure on $\partial\boldsymbol{B}^n$, satisfies $\Delta F = 0$. The kernel $P(z, \zeta)$ is a reproducing kernel in the sense that the radial limit of F exists and is equal to f a.e. To allow a more general approach region, Korányi [6] introduced the notion of admissible region: for any ζ on the boundary ($|\zeta| = 1$), and any aperture $a > 0$ let

$$\mathscr{A}_a(\zeta) = \{|z| < 1,\ |1 - z\cdot\bar\zeta| < a\delta(z)\}$$

be admissible regions at ζ. He proved that F converges to f admissibly (i.e. within some admissible region) almost everywhere. If we define the generalized Lusin area integral of F to be

$$S_a(F)(\zeta) = \Big(\int_{\mathscr{A}_a(\zeta)} |\nabla F|^2(z)\, d\Omega(z)\Big)^{1/2},$$

in Section 2 below we will prove the following theorem.

THEOREM 1. *Suppose F is a harmonic function on $\boldsymbol{B}^n$; then for each $a > 0$, $\beta > 0$ there exists constant $C_{p,a,\beta}$ depending on p, a, and β such that*

$$\|S_\beta(F)\|_p \leqslant C_{p,a,\beta}\|F_a^*\|_p$$

where $F_a^*(\zeta) = \sup_{z\in\mathscr{A}_a(\zeta)} |F(z)|$ is the non-tangential maximal function of F.

Theorem 1 in the classical case $n = 1$ was proved by Marcinkiewicz and Zygmund, and also was known for harmonic function defined on $\boldsymbol{R}_+^{n+1}$ (Stein [10], [11], Fefferman and Stein [4]). The local version of the theorem was proved by Stein [9], [10] (cf. also Putz [7]) for both analytic function defined on strictly pseudoconvex domain in $\boldsymbol{C}^n$ and classical harmonic function on $\boldsymbol{R}_+^{n+1}$. Our proof of the theorem depends strongly on estimates of [9] and [5].

The paper is organized in the following way. In Section 1, we prove Theorem 1 for the case $0 < p < 2$, following the proof of Theorem 8 in [4]. In Section 2, we apply the result in Section 1 to obtain the main estimate on area integral, generalizing the method of [1], where the special case $n = 1$, $p = 2$ was treated. In Section 3 we apply the estimate in Section 2 to finish the proof of Theorem 1, to obtain the norm-bound of Calderón's commutator operator induced by a BMO function, and to remark on some related result. A final remark is that the constants C used in various inequalities in the paper are universal constants, which may be different from each other unless otherwise specified.

§ 1. The case of $0 < p < 2$. In this section we will prove Theorem 1 for the special case $0 < p < 2$, as in [4], we first make some additional assumptions that will be removed at the end of the proof. We assume: F is the Poisson integral of an L^2 function; and the admissible region defining S_β is strictly contained in the region defining F_a^*, i.e. $\beta < a$. We write $S(F)$ for $S_\beta(F)$, F^* for F_a^*.

We let E be the closed set

$$E = \{\zeta \in \partial\boldsymbol{B}^n,\ F^*(\zeta) \leqslant \gamma\}$$

and B its complement. So, if λ_{F^*} is the distribution function of F^*, then $\lambda_{F^*(\gamma)} = |B|$. Write

$$\mathscr{R} = \bigcup_{\zeta\in E}\mathscr{A}_\beta(\zeta)\cap\{z \in \boldsymbol{B}^n,\ \delta(z) < a\}$$

where $0 < a$ is small and fixed. Let $\mathscr{R}_\varepsilon$ be an approximating family of sub-regions of $\mathscr{R}$ defined as in Kerzman's notes [5], where he modified a classical construction (e.g., [4]) to show:

(2) $\mathscr{R}$ is the increasing union of R_ε;

(3) The boundary $b\mathscr{R}_\varepsilon$ has two parts $b\mathscr{R}_\varepsilon = b_0\mathscr{R}_\varepsilon + b_1\mathscr{R}_\varepsilon$ where $b_0\mathscr{R}_\varepsilon$ is a Lipschitz hypersurface which is parametrized by

$$Q = g_\varepsilon(\zeta) = \zeta + \big(\varphi(\zeta)+\varepsilon\big)n(\zeta).$$

For $\zeta \in U$, where U is an open subset of $\boldsymbol{B}^n$ containing E, $n(\zeta)$ is the Euclidean inward unit normal to $\partial\boldsymbol{B}^n$ at ζ and $\varphi(\zeta) = 0$ if and only if $\zeta \in E$. $b_1\mathscr{R}_\varepsilon$ is an open subset of $\{z \in D,\ \delta(z) = a-\varepsilon\}$,

(4) $$d\tau_\varepsilon(Q) \leqslant C\big(\delta(Q)\big)^{-n}d\sigma(\zeta)$$

where ζ and Q are related as in (3), and $d\tau_\varepsilon$ is the area element induced on $b_0(\mathscr{R}_\varepsilon)$ by the Bergman metric on $\boldsymbol{B}^n$. C is some universal constant.

(5) $$\int_{b_0\mathscr{R}_\varepsilon} |\delta^n| d\tau_\varepsilon \leqslant C < \infty \quad \text{independent of } \varepsilon.$$

We then begin to estimate $\lambda_{S(F)}$ in terms of λ_{F^*}. We have

(6) $$\int_E (S(F))^2(\zeta)\, d\sigma(\zeta) = \int_E \int_{\mathscr{A}_\beta(\zeta)} |\nabla F(z)|^2 d\Omega(z)\, d\sigma(\zeta) \leqslant C \int_{\mathscr{R}} (\delta(z))^n |\nabla F(z)|^2 d\Omega(z) = C \lim_{\varepsilon\to 0} I_\varepsilon$$

where

$$I_\varepsilon = \int_{\mathscr{R}_\varepsilon} (\delta(z))^n |\nabla F(z)|^2 d\Omega(z).$$

We then apply Green's formula to I_ε and obtain $I_\varepsilon \leqslant T_1 + T_2 + T_3$ where

$$T_1 = \int_{\mathscr{R}_\varepsilon} |\delta|^{n+1} |F|^2 d\Omega,$$

$$T_2 = \int_{b\mathscr{R}_\varepsilon} \Big| -\delta^n \Big| \frac{\partial}{\partial n_\varepsilon} |F|^2 d\tau_\varepsilon,$$

$\dfrac{\partial}{\partial n_\varepsilon}$ is the outward unit normal in the metric

$$T_3 = \int_{b\mathscr{R}_\varepsilon} |F|^2 \left| \frac{\partial}{\partial n_\varepsilon} \delta^n \right| d\tau_\varepsilon.$$

Divide the region $\mathscr{R}_\varepsilon$ into parts $\mathscr{R}_\varepsilon = \mathscr{R}_\varepsilon^E \cup \mathscr{R}_\varepsilon^B$, where $\mathscr{R}_\varepsilon^E$ is the part above E (i.e. $z \in \mathscr{R}_\varepsilon^E$, then $z/|z| \in E$); $\mathscr{R}_\varepsilon^B$ the part above the set B. Then

(7) $$T_1 = C_n \Big(\int_{\mathscr{R}_\varepsilon^E} + \int_{\mathscr{R}_\varepsilon^B} \Big) |F|^2 d\omega(z) \quad \text{where} \quad C_n = (n+1)^n \text{ by (1)}$$
$$\leqslant C_n a \int_E |F^*(\zeta)|^2 d\sigma(\zeta) + C_n a\gamma^2 \int_B d\sigma(\zeta) \leqslant C_n a \Big(\int_0^\gamma 2t\, \lambda_{F^*(t)} dt + \gamma^2 \lambda_{F^*(\gamma)} \Big).$$

The first inequality follows since by definition of $\mathscr{R}$, $|F(z)| \leqslant a$ on $\mathscr{R}$. The second one follows since $|F^*(\zeta)| \leqslant \gamma$ on E.

The estimates on T_2, T_3 are essentially the same as in [4], where instead of using properties of usual harmonic functions defined on $\boldsymbol{R}_+^{n+1}$, we use Kerzman's estimate (2)–(5) above along with the following lemmas:

LEMMA 1. *Suppose f is in $L^p(\partial \boldsymbol{B}^n)$, $p > 1$ and F its Poisson integral. Then*

(1) *$|F(z)| \leqslant K$ for $z \in \mathscr{A}_\beta$ implies $|\nabla F(z)| \leqslant CK$ for $z \in \mathscr{A}_\beta$ and some constant C.*

(2) *$|\nabla F(z)| \to 0$ for $z \in \mathscr{A}_\beta(\zeta)$, $z \to \zeta$ for almost all ζ.*

In the case F is analytic, above lemma is proved in Stein [9], p. 61.

Proof. To prove (1), it suffices to assume f is a positive real-valued function, but in this special case we have

$$|\nabla F(z)| = \Big| \nabla_z \int_{\partial B_n} P(z,\zeta) f(\zeta)\, d\sigma(\zeta) \Big| \leqslant \int_{\partial B_n} |\nabla_z P(z,\zeta)| f(\zeta)\, d\sigma(\zeta) = n \int_{\partial B_n} P(z,\zeta) f(\zeta)\, d\sigma(\zeta) = CnF(z).$$

So (1) is direct.

The second statement follows from Lemma 7.2 of [7], if we can show that $f \in L^p(\partial \boldsymbol{B}^n)$, $p > 1$ implies that $S_\beta(F)$ is bounded almost everywhere. This latter fact follows directly from Korányi's Theorem and the main result of Putz [7].

We will now sketch the estimate on T_2 and T_3; split the region $\mathscr{R}_\varepsilon = \mathscr{R}_\varepsilon^E \cup \mathscr{R}_\varepsilon^B$ as above, we have

$$T_2 \leqslant \Big(\int_{b\mathscr{R}_\varepsilon^E} + \int_{b_1\mathscr{R}_\varepsilon^B} \Big) \delta^n \frac{\partial}{\partial n_\varepsilon} |F|^2 d\tau_\varepsilon \leqslant C \Big(\int_{b\mathscr{R}_\varepsilon^E} + \int_{b_1\mathscr{R}_\varepsilon^B} \Big) \delta^n |F|\, |\nabla F|\, d\tau_\varepsilon.$$

An application of Kerzman's estimate (3), (4) together with Lemma 1 shows that

$$\int_{b_0\mathscr{R}_\varepsilon^E} \delta^n |F|\, |\nabla F|\, d\tau_\varepsilon \leqslant \gamma \int_{b_0\mathscr{R}_\varepsilon^E} |\nabla F|\, \delta^n d\tau_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

On the boundary $b_1\mathscr{R}_\varepsilon^E$, for all $\varepsilon > 0$ one has

$$\int_{b_1\mathscr{R}_\varepsilon^E} |F|\, |\nabla F|\, \delta^n d\tau_\varepsilon \leqslant C \int_0^\gamma 2t \lambda_{F^*(t)} dt.$$

If one notices that (4) is also valid for $z \in b_0\mathscr{R}_\varepsilon$,

$$\int_{b\mathscr{R}_\varepsilon^B} \delta^n |F|\, |\nabla F|\, d\tau_\varepsilon \leqslant C\gamma^2 \int_{b\mathscr{R}_\varepsilon^B} d\sigma(\zeta) = C\gamma^2 \lambda_{F^*(\gamma)}.$$

Together we obtain

(8) $$T_2 \leqslant C \Big(\int_0^\gamma 2t \lambda_{F^*(t)} dt + \gamma^2 \lambda_{F^*(\gamma)} \Big) \quad \text{as} \quad \varepsilon \to 0.$$

Similarly one can estimate T_3 using the fact that (Stein [9], p. 66)

$$\left|\frac{\partial \delta^n}{\partial n_s}\right| \leqslant C\delta^n \quad \text{on} \quad b\mathscr{R}_s$$

and obtain the same upper bound as the estimate of T_2.

From (6), (7), (8) one has

$$\int_E (S(F))^2 d\sigma \leqslant C\left(\gamma^2 \lambda_{F^*(\gamma)} + \int_0^\gamma t\lambda_{F^*(t)}\,dt\right).$$

Thus the same argument as in Fefferman & Stein [4], p. 163, implies that $\|S(F)\|_p \leqslant C_p \|F^*\|_p$ when $0 < p < 2$.

To complete the proof of the theorem one has to remove the restrictions on F and the size of admissible region. These could again be done similarly as in [4]. The restriction on F could be removed by the following lemma.

Lemma 2. *If F is harmonic, and $F^* \in L^p$ for some $0 < p < 2$, then the function F_ϱ defined by $F_\varrho(z) = F(\varrho z)$ for $|z| \leqslant 1$, $0 < \varrho < 1$ satisfies* $\sup\limits_{r>0} \int\limits_{\partial B^n} |F_\varrho(rz)|^2 d\sigma(z) < \infty$.

Proof of Lemma 2. It suffices to show that if $F^* \in L^p$ for some $0 < p < 2$, then $|F(z)| \leqslant C\delta(z)^{-n/p}$ for all $z \in B^n$. This latter fact is an immediate consequence of the following sublemma when one takes the radius r of skew ball defined below to be $\delta(z)$.

Sublemma. *Suppose F is harmonic on B^n, $S_r(z)$ is any skew ball centered at z of radius $r \leqslant \delta(z)$ (recall $S_r(z) = \{z' \in B^n,\ |1 - z'\cdot\bar{z}| < r\}$), then*

$$|F(z)|^p \leqslant \frac{C_p}{\int\limits_{S_r(z)} d\omega(\zeta)} \int\limits_{S_r(z)} |F(\zeta)|^p d\omega(\zeta)$$

for all $z \in B^n$, $0 < p < \infty$ where C_p is some constant independent of F.

The sublemma above and the restriction on the size of admissible region could all be proved by method very similar to the corresponding result for harmonic function defined on R_+^{n+1} (upper-half space) with respect to usual potential theory ([11], [4]). We will omit the proof.

§ 2. An area integral estimate. For each $\zeta_0 \in \partial B^n$, let $B(\zeta^0, \varrho)$ be the skew ball centered at ζ_0 of radius ϱ defined by $B(\zeta^0, \varrho) = \{|\zeta| = 1, |1 - \zeta\cdot\bar{\zeta}_0| < \varrho\}$. The maximal function defined with respect to this family of balls is given by

$$M(g)(\zeta) = \sup_{\zeta\in B} \frac{1}{|B|}\int_B |g|\,d\sigma,$$

where the supremum is taken over all skew balls B containing ζ, for $\zeta \in \partial B^n$ and $|B| = \int\limits_B d\sigma$.

M is a weak type 1-1 operator, (cf. e.g. Stein [9], p. 10) and from which it follows that the strong maximal function $\Lambda_r(g)$ defined by $\Lambda_r(g)(\zeta) = \left(M|g|^r(\zeta)\right)^{1/r}$ satisfies $\|\Lambda_r(g)\|_p \leqslant C_{r,p}\|g\|_p$ for all $g \in L^p(\partial B^n)$ and all $1 < r < p$, where $C_{r,p}$ is a constant depending on r, p.

Fix $g \in L^p(\partial B^n)$, $p > 1$, $\beta > 0$ and some point $z_0 \in B^n$. Let

$$B = B_{z_0} = B\left(z_0/|z_0|, (\beta+1)\,\delta(z_0)\right)$$

be the skew ball associated with z_0, and let $\tilde{B}$ be the skew ball $B\left(z_0/|z_0|, c\delta(z_0)\right)$ where $c > (\beta+1)$ is some fixed constant which will be chosen later. Suppose $u = u_{z_0}$ is a function satisfying the following property

$$(9)\qquad \Big(\int\limits_{|\zeta|=1} p(z_0, \zeta)\,|u_{z_0}(\zeta)|^r d\sigma(\zeta)\Big)^{1/r} \leqslant C_{r,u},$$

and hence

$$(10)\qquad \left(\frac{1}{|\tilde{B}|}\int\limits_{\tilde{B}} |u_{z_0}(\zeta)|^r d\sigma(\zeta)\right)^{1/r} \leqslant C_{r,u}$$

for all $r > 1$ and for some constant $C_{r,u}$ depending on r and u_{z_0}.

Now let $f = u_{z_0} g$ and let F be the Poisson integral of f. For each β, $h > 0$ let $S_{\beta,h}(F) = \Big(\int\limits_{\mathscr{A}_\beta(\zeta)\cap\{z\in B^n, \delta(z)<h\}} |\nabla F|^2 d\Omega\Big)^{1/2}$ be the truncated area function. The area integral estimate we claim is the following:

$$(11)\qquad \int_B \left(S_{\beta,\delta(z_0)}(F)\right)^q(\zeta)\,d\sigma(\zeta) \leqslant C_{\alpha,q}|B|\,C^q_{u_{z_0}}\left(\inf_{\zeta\in B}\Lambda_{qs}(g)(\zeta)\right)^q$$

where q, s are any number with $1 < q < 2$, $s > 1$. Cu_{z_0} is a constant which depends only on $C_{r,u}$ for some r which is a function of q, s.

We now proceed to prove (11). Let $f_1 = \chi_B f$, $f_2 = f - f_1$ and let F_1, F_2 be Poisson integral of f_1, f_2, resp. It suffices to prove (11) for both F_1 and F_2.

To obtain (11) for F_1, we use the result that Theorem 1 holds for $0 < p < 2$ (proved in Section 1). Thus for $1 < q < 2$, one has

$$(12)\qquad \int_B \left(S_{\beta,\delta(z_0)}(F_1)\right)^q d\sigma \leqslant C_{\beta,q}\int\limits_{\partial B^n} |f_1|^q d\sigma$$

$$\leqslant C_{\beta,q}|B|\frac{1}{|\tilde{B}|}\int\limits_{\tilde{B}} \left(|u_{z_0}|^{qr} d\sigma\right)^{1/r}\left(\frac{1}{|\tilde{B}|}\int\limits_{\tilde{B}} |g|^{qs} d\sigma\right)^{1/s}$$

$$\text{by (10)}\qquad \leqslant C_{\beta,q}|B|\,C^q_{qr,u}\left(\inf_{\zeta\in B}\Lambda_{qs}(g)\right)^q$$

where r, s is any number > 1 with $1/r + 1/s = 1$. Thus (11) is established with $c u_{z_0} = c_{qr,u}$.

To prove (11) for F_2, we shall use a pointwise estimate of the gradient of F_2. First observe that if we put $d(\zeta, \eta) = |1-\zeta\cdot\bar{\eta}|^{1/2}$ for $\zeta, \eta \in \partial\boldsymbol{B}^n$, then d is a distance on $\partial\boldsymbol{B}^n$. Using this fact one can easily verify that if the size of the ball $\tilde{B}$ is suitably chosen, (i.e. choose c to depend on the aperture β), then for all points $z \in \mathscr{A}_\beta(\zeta_0)$ with $\zeta_0 \in B$, $\delta(z) \leqslant \delta(z_0)$, and for all $\zeta \in \partial\boldsymbol{B}^n \setminus \tilde{B}$ one has

$$\left|\frac{1-z\cdot\bar{\zeta}}{1-z_0\cdot\bar{\zeta}}\right| \geqslant C \quad \text{for some constant } C. \tag{13}$$

Thus for such $z \in \boldsymbol{B}^n$,

$$\begin{aligned} |\nabla F_2(z)| &\leqslant \int_{\partial\boldsymbol{B}^n\setminus\tilde{B}} |\nabla_z P(z,\zeta)|\,|u_{z_0}(\zeta)|\,|g(\zeta)|\,d\sigma(\zeta) \\ &\leqslant n \int_{\partial\boldsymbol{B}^n\setminus\tilde{B}} P(z,\zeta)\,|u_z(\zeta)|\,|g(\zeta)|\,d\sigma(\zeta) \\ &\leqslant C\big(\delta(z)\big)^n \int_{\partial\boldsymbol{B}^n\setminus\tilde{B}} \frac{|u_{z_0}(\zeta)|\,|g(\zeta)|}{|1-z_0\cdot\bar{\zeta}|^{2n}}\,d\sigma(\zeta) \\ &\leqslant C\big(\delta(z)\big)^n (1-|z_0|^2)^{-n/l}\Big(\int P(z_0,\zeta)|u_{z_0}(\zeta)|^l d\sigma(\zeta)\Big)^{1/l} \times \\ &\qquad \times |\tilde{B}|^{-1/k}\Big(|\tilde{B}| \int_{\partial\boldsymbol{B}^n\setminus B} \frac{|g(\zeta)|^k}{|1-z_0\cdot\bar{\zeta}|^{2n}}\,d\sigma(\zeta)\Big)^{1/k} \end{aligned} \tag{14}$$

where l, k are any numbers >1 with $1/l+1/k = 1$. Thus if we apply (9) and observe that $|\tilde{B}| \approx \big(\delta(z_0)\big)^n$ (i.e. $C_1\big(\delta(z_0)\big)^n \leqslant |\tilde{B}| \leqslant C_2\big(\delta(z_0)\big)^n$ for suitable constants C_1, C_2), we can show that

$$|\nabla F_2(z)| \leqslant CC_{l,u}\big(\delta(z)\big)^n |B|^{-1} \inf_{\zeta\in B}\big(\varLambda_k g(\zeta)\big).$$

Hence for $\zeta \in \tilde{B}$, by (13) applying (13) we have

$$\begin{aligned} S_{\delta(z_0)}(F_2)(\zeta) &= \Big(\int_{\mathscr{A}_\beta(\zeta)\cap\{z\in\boldsymbol{B}^n,\,\delta(z)\leqslant\delta(z_0)\}} |\nabla F_2(z)|^2 d\Omega(z)\Big)^{1/2} \\ &\leqslant C\cdot C_{u_{z_0}}\Big(\int_{\mathscr{A}_\beta(\zeta)\cap\{z\in\boldsymbol{B}^n,\,\delta(z)\leqslant\delta(z_0)\}} \big(\delta(z)\big)^{2n}\delta(z)^{-(n+1)}d\omega(z)\Big)^{1/2} |\tilde{B}|^{-1}\inf_{\zeta\in B}\varLambda_k g(\zeta) \\ &\leqslant C\cdot C_{u_{z_0}}\big(\delta(z_0)\big)^{(n-1)/2}\Big(\int_{\mathscr{A}_\beta(\zeta)\cap\{\delta(z)\leqslant\delta(z_0)\}} \delta\omega(z)\Big)^{1/2} |\tilde{B}|^{-1}\inf_{\zeta\in B}\varLambda_k g(\zeta) \\ &\leqslant C\cdot C_{u_{z_0}}\inf_{\zeta\in B}\varLambda_k g(\zeta). \end{aligned}$$

Thus inequality (11) for F_2 follows if we choose $k = qs$.

In the next section, we will apply inequality (11) to different situations by choosing suitable u_{z_0} satisfying both (9) and (10).

§3. Applications. We will first apply inequality (11) to finish the proof of Theorem 1 stated in the introduction. Since the case $0<p<2$ has been proved in Section 1, we now assume $2\leqslant p<\infty$. Suppose F is a harmonic function on $\boldsymbol{B}^n$, to prove Theorem 1 it suffices to show that if f is the boundary function of F (i.e. $f(\zeta) = \lim_{r\to1} F(r\zeta)$ for each $\zeta \in \partial\boldsymbol{B}^n$), then

$$\|S_\beta(F)\|_p \leqslant C_{p,\beta}\|f\|_p. \tag{15}$$

To apply (11), for each $z_0 \in \boldsymbol{B}^n$ we choose $u_{z_0} \equiv 1$. Thus u_{z_0} satisfy (9), (10) trivially with $C_{r,u} = 1$ for each r. Hence for $g = f = u_{z_0}g$, (11) holds for all z_0, i.e.

$$\int_{B_{z_0}} (S_{\beta,\delta(z_0)}F)^q(\zeta)\,d\sigma(\zeta) \leqslant C_{\alpha,q}|\tilde{B}_{z_0}| \inf_{\zeta\in B_{z_0}} \big(\varLambda_{qs}(f)(\zeta)\big)^q \tag{16}$$

for all $1<q<2$ and $s>1$. To establish (15), we adopt the following argument (cf. [4], p. 148). For each $\zeta \in \partial\boldsymbol{B}_n$, let $h(\zeta) = \sup\{h>0,\ S_{\beta,h}(F)(\zeta) \leqslant c_1\varLambda_{qs}(f)(\zeta)\}$ (in particular $S_{\beta,h(\zeta)}(F)(\zeta) \leqslant c_1\varLambda_{qs}(g)(\zeta)$) then for suitable constant c_1, one has by (16) for each $z_0 \in \boldsymbol{B}^n$

$$|\{\zeta \in \partial\boldsymbol{B}^n|\ \zeta \in B_{z_0},\, h(\zeta) > \delta(z_0)\}| \geqslant \tfrac{1}{2}|B_{z_0}|. \tag{17}$$

Since $z \in \mathscr{A}_\beta(\zeta)$ implies that $z \in B_{z_0}$ by our choice of the size of B_{z_0}, if we choose q, s so that $qs<p$, we have when $p = 2$

$$\begin{aligned} \|S_\beta(F)\|_2^2 &= \int_{\partial\boldsymbol{B}^n}\Big(\int_{\mathscr{A}_\beta(\zeta)} |\nabla F(z)|^2 d\Omega(z)\Big)\,d\sigma(\zeta) \\ &\leqslant C_\beta \int_{\boldsymbol{B}^n} |\nabla F(z)|^2\big(\delta(z)\big)^n d\Omega(z) \\ \text{by (17)}\quad &\leqslant C_\beta \int_{\partial\boldsymbol{B}^n}\Big(\int_{\mathscr{A}_\beta(\zeta)\cap\{\delta(z)\leqslant h(\zeta)\}} |\nabla F|^2 d\Omega(z)\Big)\,d\sigma(\zeta) \\ &\leqslant C_\beta \int_{\partial\boldsymbol{B}^n}\big(\varLambda_{qs}(f)(\zeta)\big)^2 d\sigma(\zeta) \leqslant C_\beta\|f\|_2^2. \end{aligned}$$

When $p>2$

$$\|S_\beta(F)\|_p = \|(S_\beta(F))^2\|_{p/2}^{1/2} = \sup_{k\in BL^r}\Big|\int_{\partial\boldsymbol{B}^n}(S_\beta(F))^2 k\,d\sigma\Big|^{1/2}$$

where BL^r is the unit ball of $L^r(\partial\boldsymbol{B}^n)$ and $1/r+2/p = 1$. Thus

$$\begin{aligned} \|S_\beta(F)\|_p &\leqslant \sup_{k\in BL^r}\Big|\int\Big(\int_{\mathscr{A}_\beta(\zeta)} |\nabla F(z)|^2 d\Omega(z)\Big)\,k(\zeta)\,d\sigma(\zeta)\Big|^{1/2} \\ &\leqslant \sup_{k\in BL^r}\Big(\int_{\boldsymbol{B}^n} |\nabla F(z)|^2\big(\delta(z)\big)^n d\Omega(z)\Big)\|k\|_r \\ \text{by (17)}\quad &\leqslant C_\beta\Big|\int\Big(\int_{\mathscr{A}_\beta(\zeta)\cap\{\delta(z)\leqslant h(\zeta)\}} |\nabla F(z)|^2 d\Omega(z)\Big)\,d\sigma(\zeta)\Big|^{1/2} \end{aligned}$$

$$\leqslant C_\beta \Big| \int (\mathcal{A}_{qs}(f)(\zeta))^2 d\sigma(\zeta) \Big|^{1/2}$$
$$\leqslant C_\beta \|(\mathcal{A}_{qs}(f))^2\|_{p/2}^{1/2} = C_\beta \|\mathcal{A}_{qs}(f)\|_p \leqslant C_{\beta,p} \|f\|_p .$$

We have finished the proof of Theorem 1.

If one chooses a more complicated function u_{z_0}, one can obtain other applications of the estimate (11). For example, when b is a function in BMO (cf. [3], [4]), i.e.

$$\|b\|_* = \sup_{\substack{B \text{ skew} \\ \text{balls}}} \frac{1}{|B|} \int_B |b(\zeta) - b_B| \, d\sigma < \infty \quad \text{where} \quad b_B = \frac{1}{|B|} \int_B b \, d\sigma,$$

then for each z_0, the function $u_{z_0} = b - b_{B_{z_0}}$ satisfies (9) and (10) with the constant $C_{r,u} = C_r \|b\|_*$ where C_r is a constant depending only on r for all $r > 1$. If we apply (11) and similar argument as the proof of Theorem 1 above, one can obtain the norm of the following type of Calderón commutant operator. (This result is also mentioned in [3].)

THEOREM 2. *Suppose b is an analytic function in BMO, then the operator $[b, P]$: $L^2(\partial \boldsymbol{B}^n) \to L^2(\partial \boldsymbol{B}^n)$ defined by $[b, P] = bP(g) - P(bg)$ for $g \in L^2(\partial \boldsymbol{B}^n)$ has operator norm bounded by $C\|b\|_*$. (P is the projection from $L^2(\partial \boldsymbol{B}^n)$ to $H^2(\partial \boldsymbol{B}^n)$.)*

Proof. Fixed $g \in L^2(\boldsymbol{B}^n)$, let $f = [b, P]g$, let F be the Poisson integral of f, and let $S(F) = S_\beta(F)$ with $\beta = 1$. To prove the theorem, it suffices to establish

$$\|S(F)\|_2 \leqslant C \|b\|_* \|g\|_2 . \tag{18}$$

If we apply Green's formula

$$\tfrac{1}{2} \int_{\partial \boldsymbol{B}^n} |f|^2 d\sigma - n \int_{\boldsymbol{B}^n} |F|^2 d\omega = \frac{n}{2} \int_{\boldsymbol{B}^n} |\nabla F|^2 \delta^{-1} d\omega, \tag{19}$$

then for F analytic with $F(0) = 0$ we have

$$n \int_{\boldsymbol{B}^n} |F|^2 d\omega = \int_0^1 r^{2n-1} \int_{\partial \boldsymbol{B}^n} |F(rz)|^2 d\sigma(z)$$
$$\leqslant n \int_0^1 r^{2n-1} r^2 \Big(\int_{\partial \boldsymbol{B}^n} |f|^2(\zeta) d\sigma(\zeta) \Big) dr \leqslant \frac{n}{n+1} \frac{1}{2} \int_{\partial \boldsymbol{B}^n} |f|^2 d\sigma .$$

Thus

$$\|f\|_2^2 = \frac{c_n}{2} \int_{\partial \boldsymbol{B}^n} |f|^2 d\sigma \leqslant (n+1) \cdot \frac{n}{2} c_n \iint_{\boldsymbol{B}^n} |\nabla F|^2 \delta^{-1} d\omega \leqslant C (S(F))^2$$
$$\text{by (18)} \quad \leqslant C \|b\|_*^2 \|g\|_2^2, \quad \text{where } c_n = \frac{(n-1)!}{\pi^n} .$$

That will prove the theorem.

To prove (18), we choose for each $z_0 \in \boldsymbol{B}^n$ the function $u_{z_0} = b - b_B$ where $B = B_{z_0}$. Then as indicated before, u_{z_0} satisfies both (9) and (10) with $C_{r,u} = C_r \|b\|_*$ for all $r > 0$. We then write $f = f_1 + f_2 + f_3$ where $f_1 = (b - b_B)P(g)$, $f_2 = P(\chi_{\tilde{B}}(b - b_B)g)$ and $f_3 = P(\chi_{\partial \boldsymbol{B}^n \setminus \tilde{B}}(b - b_B)g)$ and let F_1, F_2, F_3 be their Poisson integral, respectively. Apply inequalities (11) and (12) to F_1 and F_2 directly and obtain

$$\int_B (S_{\beta,\delta(z_0)}(F_1))^q(\zeta) \, d\sigma(\zeta) \leqslant C_{a,q} |B| \, \|b\|_*^q \big(\inf_{\zeta \in B} \mathcal{A}_{qs}(p(g)) \big)^q, \tag{20}$$

$$\int_B (S_{\beta,\delta(z_0)}(F_2))^q(\zeta) \, d\sigma(\zeta) \leqslant C_{a,q} |B| \, \|b\|_*^q \big(\inf_{\zeta \in B} \mathcal{A}_{qs}(g) \big)^q. \tag{21}$$

To estimate F_3, we will use a slightly different argument. Notice that

$$F_3(z) = \int_{\partial \boldsymbol{B}^n \setminus \tilde{B}} S(z, \zeta) (b(\zeta) - b_B) g(\zeta) d\sigma(\zeta)$$

where $S(z, \zeta)$ is the Szegö kernel for $\boldsymbol{B}^n$,

$$S(z, \zeta) = \frac{(n-1)!}{2\pi^n} \cdot \frac{1}{(1 - z \cdot \bar{\zeta})^n} .$$

Thus instead of estimating the gradient of the Poisson kernel as in (14), we will estimate the gradient of $S(z, \zeta)$. By a direct computation, we have

$$|\nabla S(z, \zeta)| \leqslant C \frac{(\delta(z))^{1/2}}{|1 - z\bar{\zeta}|^{n+1/2}} .$$

Thus for $z \in \mathscr{A}_\beta(\zeta_0)$ with $\zeta_0 \in B$

$$|\nabla F_3(z)| \leqslant C (\delta(z))^{1/2} \int_{\partial \boldsymbol{B}^n \setminus B} \frac{|b(\zeta) - b_B| \, |g(\zeta)|}{|1 - z \cdot \bar{\zeta}|^{n+1/2}} d\sigma(\zeta)$$
$$\text{by (13)} \quad \leqslant C(\delta(z)) \Big(\int_{\partial \boldsymbol{B}^n \setminus B} \frac{|b(\zeta) - b_B|^l}{|1 - z_0 \cdot \bar{\zeta}|^{n+1/2}} d\sigma(\zeta) \Big)^{1/l} \Big(\int_{\partial \boldsymbol{B}^n \setminus \tilde{B}} \frac{|g(\zeta)|^k}{|1 - z_0 \cdot \bar{\zeta}|^{n+1/2}} \, d\sigma \Big)^{1/k}$$
$$\leqslant C(\delta(z)) \frac{1}{\delta(z_0)^{1/2}} \|b\|_* \inf_{\zeta \in B} |\mathcal{A}_k g(\zeta)|$$

where l, k are any number with $1/l + 1/k = 1$. Hence

$$(S_{\beta,\delta(z_0)}(F_3))^2(\zeta) \, d\sigma(\zeta) \leqslant \int_{\bigcup_{\zeta \in B} \mathscr{A}_\beta(\zeta) \cap \{\delta(z) \leqslant \delta(z_0)\}} |\nabla F(z)|^2 \delta(z) \, d\omega(z) \tag{22}$$
$$\leqslant C \|b\|_*^2 |B| \inf_{\zeta \in B} |\mathcal{A}_k g(\zeta)|^2 .$$

So if we choose $qs = k < 2$, and define for suitable constant c

$$h(\zeta) = \sup \{ h > 0, \ S_{\beta,h} F(\zeta) \leqslant C \|b\|_* (\mathcal{A}_k g(\zeta) + \mathcal{A}_k P(g)(\zeta)) \},$$

then using inequalities (20), (21), (22) and applying similar arguments as in the proof of Theorem 1, we obtain the desired inequality (18). We have thus finished the proof of Theorem 2.

As another application of the estimate (11), we obtain the following result (cf. [1], [2]) which could be compared with a result of Sibony [8], Theorem 5 of similar type.

THEOREM 3. *Suppose* $f \in L^\infty(\partial \boldsymbol{B}^n)$ *with* $\|f\|_\infty \leqslant 1$, *and* F *is the Poisson integral, then*

$$\int_{\{z,|F(z)|\geqslant 1-\varrho\}} |\nabla F(z)|^2 \delta^{-1}(z)\, d\omega(z) \leqslant C_\varrho$$

where C_ϱ *is a constant,* $C_\varrho \to 0$ *as* $\varrho \to 0$.

Proof. Let G_ϱ be the region $\{z,\ |F(z)| \geqslant 1-\varrho\}$. For each $Z \in C_\varrho$, let $u_{z_0} = f - f(z_0)$. Then

$$\int_{|\zeta|=1} P(z_0,\zeta)|u_{z_0}(\zeta)|^2 d\sigma(\zeta) = \int_{|\zeta|=1} P(z_0,\zeta)|f(\zeta)-f(z_0)|^2 d\sigma(\zeta)$$
$$\leqslant \|f\|_\infty^2 - |f(z_0)|^2 \leqslant 2\varrho.$$

Hence u_{z_0} satisfies (9) for all $r > 1$ (if $r < 2$, apply Schwartz lemma; if $r > 2$, using the property $\|u_{z_0}\|_\infty \leqslant 2$) with appropriate constant C_ϱ where $C_\varrho \to 0$ as $\varrho \to 0$. Thus if we choose $g \equiv 1$ in equality (11), we have

$$\int_{B_{z_0}} (S_{\beta,\delta(z_0)}F)^2 d\sigma \leqslant C|B_{z_0}|C_\varrho \quad \text{for each } z_0 \in G_\varrho.$$

Again a similar argument as in the proof of Theorem 1 could be applied to the region G_ϱ and finish the proof of the theorem.

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