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Constructions of singular measures with remarkable convolution properties

by

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Abstract. We prove for every I-group there is a probability measure μ with μ^2 singular and $\mu*\tilde{\mu}$ absolutely continuous. We also prove for all nondiscrete LCA groups the existence of a nontrivial singular measure ν such that $|\hat{\nu}|$ is the Fourier transform of an absolutely continuous measure. Our results extend some work of Connolly and Williamson who obtained the above results for the real line.

Let G be a nondiscrete LCA group with dual Γ and let $M(G)$ be the convolution measure algebra of G (cf. [6]). Given a measure μ in $\mathrm{M}(G)$, we denote by $\hat{\mu}$ the Fourier transform of μ and by $\tilde{\mu}$ the measure defined by the requirement that $\tilde{\mu}(B) = \overline{\mu(-B)}$ for all Borel sets B in G.

For the real line $\boldsymbol{R}$, D.M. Connolly and J.H. Williamson [1] construct a probability measure μ with the remarkable property that $\mu^2 = \mu*\mu$ is singular and $\mu*\tilde{\mu}$ is absolutely continuous (with respect to Haar measure). (In this connection, see also [5] and [9].) As an immediate consequence of this result, they prove the existence of a nonzero singular measure ν such that $|\hat{\nu}|$ is the Fourier transform of an absolutely continuous measure. In the present paper we shall prove that their first result holds for all I-groups, and that their second result holds for all nondiscrete groups. Recall that G is an I-group if every nonempty open set in G contains an element of infinite order.

THEOREM 1. *If G is an I-group, then there exists a probability measure μ in $M(G)$ such that μ^2 is singular and $\mu*\tilde{\mu}$ is absolutely continuous.*

THEOREM 2. *Suppose that G is a nondiscrete* LCA *group. Then there exists a nonzero singular measure ν in $M(G)$ such that $|\hat{\nu}|$ is the Fourier transform of an absolutely continuous measure.*

In order to prove the above results, we need several lemmas. They will be stated in a generality greater than actually needed in the present paper. This is because we hope that they may be useful in some future study of the subject. Now let us introduce some notation. Let $\lambda = \lambda_G$ denote the Haar measure of the nondiscrete LCA group G. When G is compact, we shall always normalize λ so that $\lambda(G) = 1$. Define $M_a = M_a(G)$

and $M_s = M_s(G)$ to be the set of all absolutely continuous measures and the set of all singular measures in $M(G)$, respectively. We shall often identify $L^1(G)$ with $M_a(G)$ through the map $f \to f\lambda$, and define $\|f+\mu\| = \|f\lambda+\mu\|$ for f in $L^1(G)$ and μ in $M(G)$. Let $\boldsymbol{T} = \boldsymbol{R}(\mathrm{mod}\, 2\pi)$ denote the circle group, and $\boldsymbol{Z}$ the group of integers identified with the dual of $\boldsymbol{T}$. For a natural number $r \geqslant 2$, let $\boldsymbol{T}(r)$ denote the cyclic subgroup of $\boldsymbol{T}$ of order r. Thus the dual of $\boldsymbol{T}(r)$ is (isomorphic to) $\boldsymbol{Z}(r) = \boldsymbol{Z}(\mathrm{mod}\, r)$. It is easy to see that the results of [1] continue to hold for $\boldsymbol{T}$.

Though our first lemma seems to be well-known, we do not know any appropriate references, and so, we give a complete proof.

LEMMA 1. *Let H be a closed subgroup of G whose annihilator Λ is a discrete subgroup of Γ. Let τ be a probability measure in $M(G)$ such that $\Lambda \cap S = \{0\}$ and $\Lambda \cap (S-S)$ is a finite set, where $S = \{\gamma \in \Gamma\colon\ \hat{\tau}(\gamma) \neq 0\}$. Then there exists a unique linear map J from $M(G/H)$ into $M(G)$ such that*

$$\text{(i)} \qquad (J\mu)\hat{\ }(\gamma) = \sum_{\alpha\in\Lambda} \hat{\mu}(\alpha)\hat{\tau}(\gamma-\alpha) \qquad (\mu \in M(G/H),\ \gamma \in (\Lambda+S)\cup T),$$

where T denotes the interior of $\Gamma \setminus (\Lambda+S)$. Moreover, J is an isometry, and preserves the positivity and singularity of measures. If, in addition, τ is in $M_a(G)$, then J also preserves the absolute continuity of measures.

Proof. The uniqueness of J is obvious since $(\Lambda+S)\cup T$ is dense in Γ. Notice also that the right-hand side of (i) is actually a finite sum for each μ in $M(G)$ and each γ in Γ, since $\Lambda\cap(S-S)$ is a finite set, and that its restriction to Λ is equal to $\hat{\mu}$, since $\Lambda \cap S = \{0\}$ and $\hat{\tau}(0) = 1$. Now let λ_Q denote the Haar measure of the compact quotient group G/H of norm one, and let $x \to \bar{x}$ denote the quotient map of G onto G/H. We normalize the Haar measures λ_G and λ_H so that

$$\text{(1)} \qquad \int_G f\, d\lambda_G = \int_{G/H} d\lambda_Q(\bar{x}) \int_H f(x+t)\, d\lambda_H(t)$$

holds for all f in $L^1(G)$ (cf. 2.7.3 of [6]).

We first deal with the case where τ is in $M_a(G)$. Let $w \in L^1_+(G)$ denote the Radon–Nikodym derivative of τ with respect to λ_G. Then we have

$$\text{(2)} \qquad \int_H w(x+t)\, d\lambda_H(t) = 1 \qquad (\lambda_Q\text{-a.a. } \bar{x} \in G/H),$$

since $\hat{w}(0) = \hat{\tau}(0) = 1$ and $\Lambda \cap S = \{0\}$. Given f in $L^1(G/H)$, let us now define $(Jf)(x) = f(\bar{x})w(x)$. Then (1) and (2) yield

$$\|Jf\| = \int_G |f(\bar{x})|\, w(x)\, d\lambda_G(x) = \int_{G/H} |f(\bar{x})|\, d\lambda_Q(\bar{x}) = \|f\|.$$

Moreover, we claim that

$$\text{(3)} \qquad (Jf)\hat{\ }(\gamma) = \sum_{\alpha\in\Lambda} \hat{f}(\alpha)\hat{\tau}(\gamma-\alpha) \qquad (\gamma \in \Gamma).$$

In fact, this is obvious when f is a trigonometric polynomial. For a general f in $L^1(G/H)$, take a sequence (f_n) of trigonometric polynomials on G/H such that $\|f_n - f\| \to 0$. Then we have $\|\hat{f}_n - \hat{f}\|_\infty \to 0$ and

$$\|(Jf_n)\hat{\ } - (Jf)\hat{\ }\|_\infty \leqslant \|J(f_n - f)\| \to 0.$$

Since for each fixed $\gamma \in \Gamma$ the index α in the right-hand side of (3) essentially ranges over a finite set independent of f, it follows that $(Jf_n)\hat{\ }$ converges to the sum in the right-hand side of (3). This establishes our claim. By a weak* argument, it is now easy to show that J extends to a norm-decreasing linear map from $M(G/H)$ into $M(G)$ for which (i) holds for all μ in $M(G/H)$ and all γ in Γ. (Notice that $\hat{\tau}$ vanishes at infinity; hence the sum in the right-hand side of (i) defines a continuous function on Γ whenever $\hat{\mu}$ is a bounded function on Λ.)

We now remove the assumption that τ is in $M_a(G)$. Take a net (τ_i) of probability measures in $M_a(G)$ such that $|\hat{\tau}_i| \leqslant |\hat{\tau}|$ on Γ for all i and such that $\hat{\tau}_i \to \hat{\tau}$ uniformly on compact subsets of Γ. For each i, let J_i denote the norm-decreasing linear map from $M(G/H)$ into $M(G)$ induced by τ_i as in the preceeding paragraph. Then one can easily check that given μ in $M(G/H)$, the net $((J_i\mu)\hat{\ })$ converges uniformly on compact subsets of $(\Lambda+S)\cup T$. Since $\|J_i\mu\| \leqslant \|\mu\|$ for all i and since $(\Lambda+S)\cup T$ is a dense open subset of Γ, it follows at once that the net $(J_i\mu)$ converges weak* to a measure $J\mu$ in $M(G)$ of norm $\leqslant \|\mu\|$, and that (i) holds. Since $(J\mu)\hat{\ } = \hat{\mu}$ on Λ and $\|J\mu\| \leqslant \|\mu\|$, J is an isometry and preserves the positivity of measures. It only remains to show that J maps $M_s(G/H)$ into $M_s(G)$. Suppose by way of contradiction that $J\mu \notin M_s(G)$ for some $\mu \in M_s(G/H)$. Then there exists a measure ν in $M_a(G)$ such that $\|J\mu - \nu\| < \|J\mu\| = \|\mu\|$ and $\operatorname{supp}\hat{\nu}$ is compact. Let P be the trigonometric polynomial on G/H with $\hat{P} = \hat{\nu}$ on Λ. Then $(\mu - P)\hat{\ } = (J\mu - \nu)\hat{\ }$ on Λ, so that

$$\|\mu - P\| \leqslant \|J\mu - \nu\| < \|\mu\|.$$

Since μ is in $M_s(G/H)$, this yields the desired contradiction. The proof is complete.

LEMMA 2. *Let $H \subset G$ and $\Lambda \subset \Gamma$ be as in Lemma 1 and let $\varrho \in M(G)$. Suppose that there are two non-negative measures τ_1 and τ_2 in $M(G)$, as in Lemma 1, such that $\hat{\tau}_2(\gamma) - \hat{\tau}_1(\gamma) = 1$ whenever $\gamma \in \Gamma$ and $\hat{\varrho}(\gamma) \neq 0$. Then there exists a unique linear map K from $M(G/H)$ into $M(G)$ such that*

$$\text{(i)} \qquad (K\mu)\hat{\ }(\gamma) = \sum_{\alpha\in\Lambda} \hat{\mu}(\alpha)\hat{\varrho}(\gamma-\alpha) \qquad (\gamma \in (\Lambda+S)\cup T),$$

where $S = \{\gamma \in \Gamma\colon\ \hat{\varrho}(\gamma) \neq 0\}$ and T is the interior of $\Gamma\setminus(\Lambda+S)$. Moreover, we have

$$\text{(ii)} \qquad \|K\| \leqslant 2\|\varrho\|(\|\tau_1\| + \|\tau_2\|).$$

Proof. Notice that the condition $\|\tau_j\| = 1$ is not assumed. Applying the Hahn decomposition theorem to $\operatorname{Re}\varrho$ and $\operatorname{Im}\varrho$, we can write $\varrho = \varrho_1 + i\varrho_2 - \varrho_3 - i\varrho_4$, where the ϱ_k are nonnegative measures in $M(G)$ with $\|\varrho_1\| + \ldots + \|\varrho_4\| \leqslant 2\|\varrho\|$. Let J_{jk} be the map induced by $\tau_j * \varrho_k$ as in Lemma 1, so that $\|J_{jk}\| = \|\tau_j\| \cdot \|\varrho_k\|$ for all $j = 1, 2$ and $k = 1, 2, 3, 4$. Then the required map K may be defined by

$$K = \sum_{j=1}^{2} \sum_{k=1}^{4} (-1)^j i^{k-1} J_{jk}.$$

It is easy to confirm (i) and (ii), and the proof is complete.

Lemma 3. *Suppose that n is a natural number and that there exists a probability measure ϱ in $M(G)$ such that $\varrho^n \in M_s$ and $\varrho * \tilde{\varrho} \in M_a$. Let $f \in L^1_+(G)$, $\|f\| = 1$, and $\varepsilon > 0$. Then there exists a probability measure μ in $M(G)$, with $\mu^{n+1} \in M_a$, such that*

$$\mu^n \in M_s, \quad \mu * \tilde{\mu} \in M_a, \quad \text{and} \quad \|f * \tilde{f} - \mu * \tilde{\mu}\| < \varepsilon.$$

Proof. Notice that $\tilde{g}$ is defined by $\tilde{g}(x) = \overline{g(-x)}$ for all x in G. Choose a neighborhood V of 0 in G so that

(1) $$\|f * \tilde{f} - \sigma * f * \tilde{f}\| < \varepsilon/2.$$

holds for every probability measure σ in $M(G)$ concentrated on V. Let W be any symmetric neighborhood of 0 such that $W + W \subset V$. By considering an appropriate translate of ϱ, we may assume that $\varrho(W) > 0$. Therefore, replacing ϱ by $\varrho(W)^{-1} \varrho \mid W$, we may also assume that ϱ is concentrated on W. Then (1) implies

(2) $$\|f * \tilde{f} - \varrho * \tilde{\varrho} * f * \tilde{f}\| < \varepsilon/2.$$

We now apply Theorem 2.4, its proof and Lemma 3.1 of [8] to find a probability measure $\nu \in M_s(G)$ such that $\nu^{n+1} \in M_a$, $\varrho^n * \nu^n \in M_s$, and

(3) $$\|\varrho * \tilde{\varrho} * f * \tilde{f} - \varrho * \tilde{\varrho} * \nu * \tilde{\nu}\| < \varepsilon/2.$$

(Notice that $\varrho * \tilde{\varrho}$ is in M_a by hypothesis.) Setting $\mu = \varrho * \nu$, we see that μ has all the required properties.

Lemma 4. *Let n be a natural number such that $\varrho^n \in M_s$ and $\varrho * \tilde{\varrho} \in M_a$ for some probability measure ϱ in $M(\boldsymbol{T})$. Then, to each $\varepsilon > 0$, there corresponds a natural number r_ε with the following property:*

For every natural number $r \geqslant r_\varepsilon$, there exist two trigonometric polynomials P and Q on the cyclic group $\boldsymbol{T}(2r+1)$ such that

(a) $P \geqslant 0$, $\hat{P}(0) = 1$, *and* $\|1 - P * \tilde{P}\| < \varepsilon$;

(b) $\hat{P}(m) = \hat{Q}(m) = 0$ *for all* $m = \pm r_\varepsilon, \pm(r_\varepsilon + 1), \ldots, \pm r \in \boldsymbol{Z}(2r+1)$;

(c) $\|Q\|_\infty \leqslant 1$ *and* $\hat{Q}(0) = 0$;

(d) $$\int_{\boldsymbol{T}(2r+1)} Q \cdot P^{(n)} \, d\lambda > 1 - \varepsilon,$$

where $P^{(n)} = P * \ldots * P$ (n *times*).

Proof. Let $\varepsilon > 0$ be given. By Lemma 3, we may assume that the measure ϱ satisfies

(1) $$\|1 - \varrho * \tilde{\varrho}\| < \varepsilon/6.$$

Since ϱ^n is a singular probability measure, there exists a trigonometric polynomial Q_1 on $\boldsymbol{T}$ such that

(2) $$\int_{\boldsymbol{T}} Q_1 \, d\varrho^n > 1 - \varepsilon,$$

(3) $$\|Q_1\|_\infty \leqslant 1 \quad \text{and} \quad \hat{Q}_1(0) = 0.$$

To find such a Q_1, it suffices to apply Doss' result in [2]. Convoluting ϱ with an appropriate trigonometric polynomial, we then obtain a trigonometric polynomial P_1 on $\boldsymbol{T}$ such that

(4) $$P_1 \geqslant 0, \quad \hat{P}_1(0) = 1, \quad \text{and} \quad \|1 - P_1 * \tilde{P}_1\| < \varepsilon/6,$$

(5) $$\int_{\boldsymbol{T}} Q_1 \cdot P_1^{(n)} \, d\lambda_{\boldsymbol{T}} > 1 - \varepsilon.$$

We define r_ε to be any even integer such that

(6) $$\hat{Q}_1(k) = \hat{P}_1(k) = 0 \quad (k \in \boldsymbol{Z};\ |k| \geqslant r_\varepsilon/2).$$

Now let r be a given natural number $\geqslant r_\varepsilon$. We apply Lemma 1 with $\Lambda = (2r+1)\boldsymbol{Z}$ to find a probability measure σ in $M(\boldsymbol{T})$ such that

(7) $$\hat{\sigma}\big((2r+1)m + k\big) = \hat{P}_1(k) \quad (m, k \in \boldsymbol{Z};\ |k| \leqslant r).$$

Notice that (7) implies that σ is concentrated on the subgroup $\boldsymbol{T}(2r+1)$ of $\boldsymbol{T}$. In fact, we have $\sigma = P\lambda$, where P is the restriction of P_1 to $\boldsymbol{T}(2r+1)$ and λ is the Haar measure of $\boldsymbol{T}(2r+1)$ regarded as an element of $M(\boldsymbol{T})$. It is obvious that $P \geqslant 0$ and $\hat{P}(0) = 1$. To confirm that $\|1 - P * \tilde{P}\| < \varepsilon$, let K_m be the mth Fejer kernel on $\boldsymbol{T}$ ($m = 0, 1, \ldots$). Setting $\tau_1 = K_{r_\varepsilon/2}$ and $\tau_2 = 2K_{r_\varepsilon}$, we see that $\hat{\tau}_2(k) - \hat{\tau}_1(k) = 1$ for all $k = 0, \pm 1, \ldots, \pm r_\varepsilon/2$. It follows from (4), (6), (7) and Lemma 2 that

$$\|1 - P * \tilde{P}\| \leqslant 2\|1 - P_1 * \tilde{P}_1\| (\|\tau_1\| + \|\tau_2\|) < \varepsilon.$$

This establishes part (a). Now define Q to be the restriction of Q_1 to $\boldsymbol{T}(2r+1)$.

Then (b) and (c) follow from (3), (6) and (7). Finally we have

$$\int_{T(2r+1)} QP^{(n)}\,d\lambda = \int_{T(2r+1)} Q\,d\sigma^n = \sum_{k=-r}^{r} \hat{Q}(-k)\hat{\sigma}^n(k)$$
$$= \sum_{k=-\infty}^{\infty} \hat{Q}_1(-k)\hat{P}_1^n(k) = \int_T Q_1 P_1^{(n)}\,d\lambda_T > 1-\varepsilon$$

by (5), (6) and (7). This establishes part (d) and the proof is complete.

LEMMA 5. *Let G be a compact abelian group of unbounded order, and let n be a natural number, as in Lemma 4. Then there exists a probability measure μ in $M(G)$ such that $\mu^n \in M_s$ and $\mu*\tilde{\mu} \in M_a$.*

Proof. By induction, we shall construct two sequences (P_m) and (Q_m) of trigonometric polynomials on G as follows. Set $P_0 = 1$ and $Q_0 = 0$, and assume that $P_0, Q_0, \ldots, P_{m-1}, Q_{m-1}$ have been defined for some natural number m. Let $S_{m-1} = \{\gamma \in \Gamma\colon \hat{R}_{m-1}(\gamma) \neq 0\}$, where $R_{m-1} = P_0P_1\ldots\ldots P_{m-1}$, and let r_m be any sufficiently large natural number. Since G is of unbounded order and since S_{m-1} is a finite set, there exists an element θ_m of Γ such that

$$\text{(1)}\qquad \{j\theta_m\colon j = \pm1, \pm2, \ldots, \pm2r_m\} \cap (S_{m-1}-S_{m-1}) = \emptyset.$$

By Lemma 4 (and its proof), there exist two trigonometric polynomials P_m, Q_m on G subject to the following conditions:

$$\text{(2)}\qquad P_m \geq 0,\ \hat{P}_m(0) = 1, \quad \text{and} \quad \|1-P_m*\tilde{P}_m\| < 2^{-m}/\|R_{m-1}\|_\infty;$$

$$\text{(3)}\qquad \operatorname{supp}(|\hat{P}_m|+|\hat{Q}_m|) \subset \{j\theta_m\colon j = 0, \pm1, \ldots, \pm r_m\};$$

$$\text{(4)}\qquad \|Q_m\|_\infty \leq 1 \quad \text{and} \quad \hat{Q}_m(0) = 0;$$

$$\text{(5)}\qquad \int_G Q_m P_m^{(n)}\,d\lambda_G > 1-2^{-m}.$$

This completes the induction.

Notice that every R_m is non-negative by (2), that (1), (2) and (3) imply

$$\text{(6)}\qquad \hat{R}_N(\gamma_0+\ldots+\gamma_m) = \hat{P}_0(\gamma_0)\ldots\hat{P}_m(\gamma_m) \quad (m \leq N)$$

whenever $\gamma_0 \in \operatorname{supp}\hat{P}_0, \ldots, \gamma_m \in \operatorname{supp}\hat{P}_m$, and that

$$\text{(7)}\qquad \hat{R}_N = 0 \quad \text{on} \quad \Gamma \setminus (\operatorname{supp}\hat{P}_0+\ldots+\operatorname{supp}\hat{P}_N).$$

It follows that the sequence (R_m) converges weak* to a probability measure μ in $M(G)$ such that

$$\text{(6)}'\qquad \hat{\mu}(\gamma_0+\ldots+\gamma_m) = \hat{P}_0(\gamma_0)\ldots\hat{P}_m(\gamma_m)$$

for all $\gamma_0 \in \operatorname{supp}\hat{P}_0, \ldots, \gamma_m \in \operatorname{supp}\hat{P}_m$ $(m = 0, 1, \ldots)$ and such that

$$\text{(7)}'\qquad \hat{\mu} = 0 \quad \text{on} \quad \Gamma \setminus \bigcup_{m=0}^{\infty} (\operatorname{supp}\hat{P}_0+\ldots+\operatorname{supp}\hat{P}_m).$$

Thus μ is a kind of Riesz product (cf. [4]).

In order to prove that μ^n is in M_s, take any finite set K in Γ. Then $\{j\theta_m\colon j = \pm1, \ldots, \pm r_m\}$ is disjoint from K whenever m is large enough. Moreover, (3), (6)' and (5) yield

$$\int_G Q_m\,d\mu^n = \sum_{\gamma\in\Gamma} \hat{Q}_m(-\gamma)\hat{\mu}^n(\gamma) = \int_G Q_m P_m^{(n)}\,d\lambda_G > 1-2^{-m}$$

for all m. It follows from (3), (4) and Doss' theorem [2] that μ^n is a singular measure.

To confirm that $\mu*\tilde{\mu}$ is in M_a, notice that

$$R_m*\tilde{R}_m = (P_0*\tilde{P}_0)\ldots(P_m*\tilde{P}_m) \quad (m \geq 0)$$

by (1) and (3). It follows from (2) that

$$\|R_{m-1}*\tilde{R}_{m-1}-R_m*\tilde{R}_m\|$$
$$\leq \|1-P_m*\tilde{P}_m\|\cdot\|R_{m-1}*\tilde{R}_{m-1}\|_\infty < 2^{-m} \quad (m \geq 1).$$

Therefore the sequence $(R_m*\tilde{R}_m)$ converges in norm to some $f \in L^1(G)$. Since

$$\hat{f}(\gamma) = \lim_{m\to\infty} |\hat{R}_m(\gamma)|^2 = |\hat{\mu}(\gamma)|^2 \quad (\gamma \in \Gamma),$$

we conclude that $\mu*\tilde{\mu}$ is absolutely continuous. This completes the proof.

Proofs of Theorems 1 and 2. Let G be a nondiscrete LCA group. To prove the required results, we may replace G by any open subgroup of G. Thus, by the well-known structure theorem [6], we may also assume that G has the form $\boldsymbol{R}^m \times K$ for some integer $m \geq 0$ and some compact abelian group K. If $m \geq 1$, then our results follow easily from the results in [1]. So assume that G is compact. If G is also an I-group, then G must be of unbounded order. In the last case, Lemma 5 with $n = 2$ yields both Theorems 1 and 2. Consequently it will suffice to prove that the conclusion of Theorem 2 holds for every infinite compact abelian group of bounded order. Let G be a such group. Then G has the form $G = \prod_{n=1}^{\infty} G_n$, where every G_n is an infinite compact abelian group (see (A.25) of [3]).

By Theorem 2.4 of [8] and its proof, there exist two symmetric probability measures μ_n, ν_n in $M(G_n)$ such that $\mu_n*\nu_n \in M_s$, $\mu_n^2 \in M_a$, $\nu_n^2 \in M_a$, and

$$\text{(1)}\qquad \|1-\mu_n^2\|+\|1-\nu_n^2\| < 2^{-n}.$$

Setting $\lambda_n = (1+i)^{-1}(\mu_n + i\nu_n)$, we then have

$$\lambda_n^2 = \mu_n * \nu_n + (2i)^{-1}(\mu_n^2 - \nu_n^2), \tag{2}$$

$$\lambda_n * \tilde{\lambda}_n = 2^{-1}(\mu_n^2 + \nu_n^2). \tag{3}$$

Now define $\varrho_n \in M(G)$ by setting

$$\varrho_n = \lambda_1^2 \times \ldots \times \lambda_n^2 \times \prod_{k=n+1}^{\infty} (\mu_k * \nu_k) \quad (n = 1, 2, \ldots). \tag{4}$$

It follows from (1) and (2) that the sequence (ϱ_n) converges in norm to a complex measure $\nu \in M(G)$. Since every ϱ_n is a singular measure, so is ν. Moreover, (3) and (4) imply that

$$|\hat{\nu}| = \Big[\prod_{n=1}^{\infty} 2^{-1}(\mu_n^2 + \nu_n^2)\Big]^{\wedge}.$$

It follows at once that $|\hat{\nu}|$ is in $(M_a)^{\wedge}$. This completes the proof.

Remarks. (a) Applying the method in Section 6 of [8], we can generalize Theorem 1 as follows. Let G be an I-group and let r be a natural number. Then there exist r probability measures $\mu_1, \ldots, \mu_r$ in $M(G)$ such that

$$\mu_1 * \tilde{\mu}_1 * \ldots * \mu_r * \tilde{\mu}_r \in M_a(G),$$

and such that

$$\mu_1^{n_1} * \ldots * \mu_r^{n_r} \in M_s(G)$$

for every r-tuple $(n_1, \ldots, n_r)$ of non-negative integers with $\min\{n_j\colon 1 \leqslant j \leqslant r\} \leqslant 2$.

(b) Suppose that G contains an open subgroup isomorphic to $\boldsymbol{Z}(2)^{\omega}$ for some infinite cardinal ω. Then it is easy to show that there exists no probability measure in $M(G)$ as in the conclusion of Theorem 1.

(c) Let Σ_G denote the set of all symmetric complex homomorphisms of $M(G)$. Then $\mu \in M(G)$ with $\mu * \tilde{\mu} \in M_a$ has Gelfand transform vanishing on $\Sigma_G \setminus \Gamma$. Moreover, it is not difficult to prove that if G is a group as in (b) and if $\mu \in M(G)$ has Gelfand transform vanishing on $\Sigma_G \setminus \Gamma$, then $\|\mu^n + M_a\|^{1/n} = o(1)$ as $n \to \infty$ (cf. Problem (ii) on p. 253 of [7]). Consequently, for any group G as in (b), there exists no nonzero measure μ in $M(G)$ such that $\mu * \tilde{\mu} \in M_a$ and $\mu^n \in M_s$ for all $n \geqslant 1$.

(d) Does there exist a sequence (P_n) of non-negative trigonometric polynomials on $\boldsymbol{T}$ such that $\hat{P}_n(0) = 1$ for all n,

$$\sum_{n=1}^{\infty} \|P_n * \tilde{P}_n - 1\| < \infty \quad \text{and} \quad \limsup_{n\to\infty} \|P_n^{(n)} - 1\|^{1/n} > 0\,?$$

If such a sequence exists, we can prove that every I-group has a probability measure μ such that $\mu * \tilde{\mu} \in M_a$ and $\mu^n \in M_s$ for all natural numbers n. In view of the fact stated in (c), we feel that such a μ can not exist, and therefore that the conclusion of Lemma 5 must fail to hold for some $n > 0$.

References

[1] D. M. Connolly and J. H. Williamson, *An application of a theorem of Singer*, Proc. Edinburgh Math. Soc. 19 (1974), pp. 119–123.

[2] R. Doss, *On the transform of a singular or an absolutely continuous measure*, Proc. Amer. Math. Soc. 19 (1968), pp. 361–363.

[3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I, Springer-Verlag, Berlin 1963.

[4] E. Hewitt and H. S. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Cambridge Philos. Soc. 62 (1966), pp. 399–420.

[5] P. M. Ludvik, *On a singular measure of D. M. Connolly and J. H. Williamson*, Proc. Edinburgh Math. Soc. 20 (1976), pp. 59–61.

[6] W. Rudin, *Fourier analysis on groups*, Interscience, New York 1962.

[7] S. Saeki, *Asymmetric maximal ideals in $M(G)$*, Trans. Amer. Math. Soc. 222 (1976), pp. 241–254.

[8] — *Singular measures having absolutely continuous convolution powers*, Illinois J. Math. 21(1977), pp. 395–412.

[9] — *On infinite convolution products of discrete probability measures*, J. London Math. Soc. (2) 16 (1977), pp. 172–176.

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