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Inequalities between absolutely (p, q)-summing norms

by

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Abstract. Let E_n be an n-dimensional Banach space and let F be an arbitrary Banach space. By $\pi_{(p,q)}$ we denote the absolutely (p, q)-summing norm. Then

$$\pi_{(p,q)}(S\colon E_n \to F) \leqslant n^{\max(q/2,1)(1/q-1/s)}\pi_{(r,s)}(S\colon E_n\to F)$$

for $1 \leqslant q \leqslant s \leqslant \infty$ and $1/q-1/p = 1/s-1/r$.

Similar inequalities between p-integral and absolutely r-summing norms are established. We also obtain that there is a projection P from a Banach space E onto an n-dimensional subspace E_n such that for the p-integral norm the following estimate holds:

$$\iota_p(P\colon E\to E_n) \leqslant n^{\max(1/p,1/2)}, \qquad 1 \leqslant p \leqslant \infty.$$

The results obtained here are related to those in [8] and [6]. We recall some basic definitions. The Banach space of all (bounded linear) operators from a Banach space E into a Banach space F is denoted by $\mathscr{L}(E, F)$. The identity operator on an n-dimensional Banach space E_n is denoted by I_n. We refer to [10] for definitions and fundamental properties of the operator ideals $[\mathscr{P}_{(p,q)};\ \pi_{(p,q)}]$ and $[\mathscr{I}_r;\ \iota_r]$ of absolutely (p, q)-summing and r-integral operators, respectively. For $p = q$ we have the operator ideals $[\mathscr{P}_p;\ \pi_p]$ of absolutely p-summing operators.

1. The operator ideals $[\mathfrak{M}_{(s,q)};\ \mu_{(s,q)}]$. Let us recall that an operator $S \in \mathscr{L}(E, F)$ is called (s, q)-*mixing*, $1 \leqslant q \leqslant s \leqslant \infty$, if there is a constant $\sigma \geqslant 0$ such that

$$\Big(\sum_{i=1}^{m}\Big(\sum_{k=1}^{n} |\langle Sx_i,\ b_k\rangle|^s\Big)^{q/s}\Big)^{1/q} \leqslant \sigma \sup_{\|a\|\leqslant 1}\Big(\sum_{i=1}^{m}|\langle x_i,\ a\rangle|^s\Big)^{1/s}\Big(\sum_{k=1}^{n}\|b_k\|^q\Big)^{1/q}$$

for all finite families of elements $x_1, \ldots, x_m \in E$ and functionals $b_1, \ldots, b_n \in F'$.

The phrase "(s, q)-mixing" is derived from the fact that these operators are characterized by the following property: Every weakly q-summable sequence (x_n) is mapped into a sequence (y_n) which can be written as a product of an r-summable scalare sequence (β_n) and a weakly s-summable sequence (y_n^0). Here $1/r+1/s = 1/q$. Putting

$$\mu_{(s,q)}(S) := \inf \sigma,$$

then the class of all (s, q)-mixing operators forms a normed operator ideal denoted by $[\mathfrak{M}_{(s,q)}; \mu_{(s,q)}]$. These operator ideals were extensively investigated by A. Pietsch in [10]. There one can also find the following properties of these operator ideals:

Let $1 \leqslant q \leqslant s \leqslant \infty$ *and* $1/q - 1/p = 1/s - 1/r$. *Then*

(1) $$[\mathscr{P}_{(r,s)}; \pi_{(r,s)}] \circ [\mathfrak{M}_{(s,q)}; \mu_{(s,q)}] \subseteq [\mathscr{P}_{(p,q)}; \pi_{(p,q)}],$$

(2) $$[\mathfrak{M}_{(\infty,q)}; \mu_{(\infty,q)}] = [\mathscr{P}_q; \pi_q], \qquad [\mathfrak{M}_{(q,q)}; \mu_{(q,q)}] = [\mathscr{L}; \|\cdot\|].$$

Let $1 \leqslant s_1 \leqslant s_0 \leqslant \infty$. *Then*

(3) $$[\mathfrak{M}_{(s_0,q)}; \mu_{(s_0,q)}] \subseteq [\mathfrak{M}_{(s_1,q)}; \mu_{(s_1,q)}].$$

Let $1 \leqslant s_1 \leqslant s \leqslant s_0 \leqslant \infty$ *and* $1/s = (1-\theta)/s_0 + \theta/s_1$, $0 < \theta < 1$. *Then*

(4) $$\mu_{(s,q)}(S) \leqslant \mu^{1-\theta}_{(s_0,q)}(S)\, \mu^{\theta}_{(s_1,q)}(S) \text{ for } S \in \mathfrak{M}_{(s_0,q)}(E, F).$$

2. Inequalities between absolutely (p, q)-summing norms. In order to prove our main result we need the following elementary

LEMMA 1. *Let* $1 \leqslant q \leqslant s \leqslant \infty$ *and* $n = 1, 2, \ldots$ *Then*

$$\mu_{(s,q)}(I_n) \leqslant [\pi_q(I_n)]^{1-q/s}.$$

Proof. Using (4) and (2) we get

$$\mu_{(s,q)}(I_n) \leqslant \mu^{1-\theta}_{(\infty,q)}(I_n)\, \mu^{\theta}_{(q,q)}(I_n) \leqslant [\pi_q(I_n)]^{1-\theta} \|I_n\|^{\theta} \leqslant [\pi_q(I_n)]^{1-\theta}$$

for $1/s = (1-\theta)/\infty + \theta/q$, $0 < \theta < 1$. This implies the assertion. First we treat a special case; see also [8].

PROPOSITION 2. *Let* $2 \leqslant s \leqslant \infty$ *and* $n = 1, 2, \ldots$ *Then*

$$\pi_2(S\colon E_n \to F) \leqslant n^{1/2-1/s} \pi_s(S\colon E_n \to F).$$

Proof. From the well-known equality $\pi_2(I_n) = n^{1/2}$ (cf. [10], Chapter 29) and the above Lemma 1 it follows that

$$\mu_{(s,2)}(I_n) \leqslant [\pi_2(I_n)]^{1-2/s} = n^{1/2-1/s}.$$

Now formula (1) yields

$$\pi_2(S\colon E_n \to F) \leqslant \mu_{(s,2)}(I_n)\, \pi_s(S\colon E_n \to F) \leqslant n^{1/2-1/s} \pi_s(S\colon E_n \to F).$$

As a consequence of the preceding result we get an estimate for the q-integral norms of the identity operator on n-dimensional Banach spaces which improves the result obtained by D. J. H. Garling and Y. Gordon [6] for complex Banach spaces.

PROPOSITION 2. *Let* $1 \leqslant q \leqslant \infty$ *and* $n = 1, 2, \ldots$ *Then*

$$\iota_q(I_n) \leqslant n^{\max(1/q, 1/2)}.$$

Proof. Denote by q' the conjugate index of q. Proposition 1 yields

$$\pi_2(T\colon E_n \to F) \leqslant n^{1/q-1/2} \pi_{q'}(T\colon E_n \to F) \quad \text{for} \quad 2 \leqslant q' \leqslant \infty.$$

Now a duality argument [9], [10] implies

$$\iota_q(S\colon F \to E_n) \leqslant n^{1/q-1/2} \iota_2(S\colon F \to E_n) \quad \text{for} \quad 1 \leqslant q \leqslant 2.$$

Since $\iota_2 = \pi_2$ and $\pi_2(I_n) = n^{1/2}$, we get

$$\iota_q(I_n) \leqslant n^{1/q-1/2} \pi_2(I_n) \leqslant n^{1/q} \quad \text{for} \quad 1 \leqslant q \leqslant 2.$$

The case $2 \leqslant q \leqslant \infty$ can be checked by using the fact that $\iota_q \leqslant \iota_2$.

LEMMA 2. *Let* $1 \leqslant q \leqslant s \leqslant \infty$ *and* $n = 1, 2, \ldots$ *Then*

$$\mu_{(s,q)}(I_n) \leqslant n^{\max(q/2,1)(1/q-1/s)}.$$

Proof. Since $\pi_q \leqslant \iota_q$, $1 \leqslant q \leqslant \infty$, the estimate follows from Lemma 1 and Proposition 2.

Now we are able to prove the main result of this paper.

THEOREM. *Let* $1 \leqslant q \leqslant s \leqslant \infty$, $1/q - 1/p = 1/s - 1/r$ *and* $n = 1, 2, \ldots$ *Then*

$$\pi_{(p,q)}(S\colon E_n \to F) \leqslant n^{\max(q/2,1)(1/q-1/s)} \pi_{(r,s)}(S\colon E_n \to F).$$

The inequality is the best possible one in the sense that, if the above conditions are satisfied, then the exponent in the inequality can not be improved.

Proof. Using (1) and Lemma 2 we get the required inequality.

To see that the exponent $\max(q/2, 1)(1/q - 1/s)$ is the best possible one we have to show that, if

$$\pi_{(p,q)}(S\colon E_n \to F) \leqslant n^{\lambda} \pi_{(r,s)}(S\colon E_n \to F)$$

holds for all operators S and $n = 1, 2, \ldots$, then

$$\lambda \geqslant \max(q/2, 1)(1/q - 1/s).$$

First we suppose $1 \leqslant q \leqslant 2$. Taking the identity operators $I_n\colon l^n_\infty \to l^n_\infty$ we have $\pi_{(r,s)}(I_n) = n^{1/r}$ for $1 \leqslant s \leqslant r \leqslant \infty$. If the inequality

$$n^{1/p} = \pi_{(p,q)}(I_n) \leqslant n^{\lambda} \pi_{(r,s)}(I_n) \leqslant n^{\lambda} n^{1/r}$$

is true for every positive integer n, then we obtain

$$\lambda \geqslant 1/p - 1/r = 1/q - 1/s,$$

as desired.

Now, let $2 \leqslant q \leqslant \infty$ and put $m := [n^{q/2}]$. We show for each $n = 1, 2, \ldots$, that there is a (m, n)-matrix A_n whose entries are ± 1 such that

(*) $$\pi_{(p,q)}(A_n\colon l_1^n \to l_r^m) \geqslant c_{pq} n^{q/2p}$$

and

(**) $$\pi_{(r,s)}(A_n\colon l_1^n \to l_r^m) \leqslant \begin{cases} c_{rs} n^{q/2r} & \text{for } r > s \geqslant 2, \\ c_{r\varepsilon} n^{q/2r+\varepsilon} & \text{for } r = s \geqslant 2 \text{ and all } \varepsilon > 0. \end{cases}$$

Here $c_{pq}, \ldots$ denote certain positive constants not depending on n.

First we show (*). By [1], [2] and [4] there is an (n, m)-matrix B_n having entries ± 1 and satisfying

$$\|B_n\colon l_{q'}^m \to l_1^n\| \leqslant c_q^{-1} n.$$

If $A_n := B_n'$, where B_n' denotes the transposed matrix of B_n, we have

$$\|A_n B_n e_i\|_r \geqslant n \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Here (e_i) is the canonical basis of $l_{q'}^m$. By the definition of the absolutely (p, q)-summing operators it follows that

$$\begin{aligned}[n^{q/2}]^{1/p} n &\leqslant \Big\{\sum_1^m \|A_n B_n e_i\|_r^p\Big\}^{1/p} \\ &\leqslant \pi_{(p,q)}(A_n\colon l_1^n \to l_r^m)\,\|B_n\colon l_{q'}^m \to l_1^n\| \\ &\leqslant \pi_{(p,q)}(A_n\colon l_1^n \to l_r^m)\; c_q^{-1} n.\end{aligned}$$

This implies (*).

Since $\|A_n\colon l_1^n \to l_r^m\| = [n^{q/2}]^{1/r}$, the estimate (**) can be checked by using the following results of G. Bennett [1] and P. Saphar [12]:

$$\mathcal{L}(l_1, l_r) = \mathcal{P}_{(r,s)}(l_1, l_r) \quad \text{for} \quad r > s \geqslant 2$$

and

$$\mathcal{L}(l_1, l_r) = \mathcal{P}_{(r,\varepsilon)}(l_1, l_r) \quad \text{for} \quad r > 2 \text{ and all } \varepsilon > 0.$$

Now, if the inequality

$$\pi_{(p,q)}(A_n) \leqslant n^\lambda \pi_{(r,s)}(A_n)$$

is true for every n, then we get $\lambda \geqslant q/2(1/p - 1/r) = q/2(1/q - 1/s)$.

As a consequence of our Theorem we have

Proposition 3. *Let $1 \leqslant p \leqslant r \leqslant \infty$ and $n = 1, 2, \ldots$ Then*

$$\pi_p(S\colon E_n \to F) \leqslant n^{\max(p/2,1)(1/p-1/r)} \pi_r(S\colon E_n \to F).$$

Using a duality argument as before from Proposition 3 we get

Proposition 3'. *Let $1 \leqslant p \leqslant r \leqslant \infty$ and $n = 1, 2, \ldots$ Then*

$$\iota_p(S\colon E \to F_n) \leqslant n^{\max(r'/2,1)(1/p-1/r)} \iota_r(S\colon E \to F_n).$$

Proposition 4. *Let one of E, F be n-dimensional. Then*

$$\iota_p(S\colon E \to F) \leqslant n^{1/p-1/r} \pi_r(S\colon E \to F) \quad \textit{for} \quad 1 \leqslant p \leqslant 2 \leqslant r \leqslant \infty.$$

Proof. At first we show that if $1 \leqslant p \leqslant 2$, then

$$\iota_p(S\colon E_n \to F) \leqslant n^{1/p-1/2} \pi_2(S\colon E_n \to F).$$

The operator S can be factorized through the natural injection i as

$$S\colon E_n \xrightarrow{S_0} S(E_n) \xrightarrow{i} F.$$

Since $\iota_2 = \pi_2$, from Proposition 3' and the injectivity of the absolutely p-summing norms it follows that

$$\begin{aligned}\iota_p(S\colon E_n \to F) \leqslant \iota_p(S_0\colon E_n \to S(E_n)) &\leqslant n^{1/p-1/2} \pi_2(S_0\colon E_n \to S(E_n)) \\ &\leqslant n^{1/p-1/2} \pi_2(S\colon E_n \to F).\end{aligned}$$

From this inequality we obtain again by the duality argument that

$$\pi_2(S\colon E \to F_n) \leqslant n^{1/2-1/r} \pi_r(S\colon E \to F_n) \quad \text{for} \quad 2 \leqslant r \leqslant \infty.$$

Our assertion is now a combination of both these inequalities with Proposition 3 and 3'.

Remark. Proposition 4 has also been proved by D. R. Lewis [8] up to the constant $2(\pi/2)^{1/2}$ (in the complex case) on the right-hand side of the inequality.

Corollary 1. *Let one of E, F be n-dimensional. Then*

$$\iota_p(S\colon E \to F) \leqslant n^{|1/2-1/p|} \pi_p(S\colon E \to F) \quad \textit{for} \quad 1 \leqslant p \leqslant \infty.$$

Proof. Let $1 \leqslant p \leqslant 2$. Then from Proposition 4 we have

$$\iota_p(S) \leqslant n^{1/p-1/2} \pi_2(S) \leqslant n^{1/p-1/2} \pi_p(S).$$

Analogously, for $2 \leqslant p \leqslant \infty$ we get

$$\iota_p(S) \leqslant \iota_2(S) \leqslant n^{1/2-1/p} \pi_p(S).$$

Remark. In the cases $p = 1$ and $p = \infty$ the exponent in the inequality of Corollary 1 can not be improved (in the sense of our Theorem). Comparing the "limit order diagrams" of $\mathcal{P}_p$ and $\mathcal{I}_p$ for $1 < p < \infty$ in [3] or [7] we see that the inequality

$$\iota_p(S) \leqslant n^\lambda \pi_p(S) \quad \text{for} \quad n = 1, 2, \ldots$$

implies $\lambda \geqslant 1/4\,|1/2 - 1/p|$.

A. Erdmann [12] has proved in this thesis that the exponent $\lambda = |1/2 - 1/p|$ in the inequality of Corollary 1 is also the best possible in the case $1 < p < \infty$.

Corollary 2. *Let E_n be a subspace of E. Then there is a projection P from E onto E_n such that*

$$\iota_p(P) \leqslant n^{\max(1/p,1/2)} \quad \textit{for} \quad 1 \leqslant p \leqslant \infty.$$

Proof. For $1 \leqslant p \leqslant 2$ the statement follows from the well-known result of S. Kwapień (cf. [10], Chapter 29) that there is a projection P with $\pi_2(P) = n^{1/2}$ and Proposition 4. For $2 \leqslant p \leqslant \infty$ the statement is implied by the fact that $\iota_p \leqslant \pi_2$.

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Maximal seminorms on Weak L^1

by

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Abstract. For each $f \in \text{Weak}\,L^1$ let $q_1(f) = \sup_{a>0} a\mu(\{x \mid |f(x)| > a\})$ and let $q(f)$ be the seminorm

$$q(f) = \inf_{f=f_1+f_2+\ldots+f_n} \sum_{j=1}^{n} q_1(f_j).$$

We obtain two equivalent expressions for $q(f)$ and make some remarks about the dual of Weak L^1.

Introduction. For $0 < p < \infty$ the space Weak L^p taken over the measure space (X, Σ, μ) consists of all (equivalence classes of) measurable functions f for which the quasinorm

$$q_p(f) = \sup_{a>0} a\left[\mu(\{x \mid |f(x)| > a\})\right]^{1/p}$$

is finite.

In [4] and [3] it was shown (independently) that Weak L^1 has a non trivial dual space. This is at first sight a surprising result since one can readily show that any continuous linear functional on Weak L^1 necessarily vanishes on all simple functions if μ is non atomic. Thus the functionals and seminorms on Weak L^1 measure only the "behavior at infinity" (in a sense to be indicated below and in [3]) of the functions on which they act.

Any continuous linear functional on Weak L^1 is dominated by a constant multiple of the quasinorm q_1 and therefore also by a multiple of the seminorm q defined by

$$q(f) = \inf_{f=f_1+f_2+\ldots+f_n} \sum_{j=1}^{n} q_1(f_j)$$

where the infimum is taken over all finite decompositions $f = f_1+f_2+\ldots+f_n$ of f in Weak L^1. The quotient space of Weak L^1 modulo the subspace of elements f satisfying $q(f) = 0$ is a Banach space W normed by q whose dual coincides with the dual of Weak L^1.