A. Erdmann [12] has proved in this thesis that the exponent $\lambda = |1/2 - 1/p|$ in the inequality of Corollary 1 is also the best possible in the case $1 < p < \infty$.

COROLLARY 2. *Let E_n be a subspace of E. Then there is a projection P from E onto E_n such that*

$$\iota_p(P) \leqslant n^{\max(1/p,1/2)} \quad \textit{for} \quad 1 \leqslant p \leqslant \infty.$$

Proof. For $1 \leqslant p \leqslant 2$ the statement follows from the well-known result of S. Kwapień (cf. [10], Chapter 29) that there is a projection P with $\pi_2(P) = n^{1/2}$ and Proposition 4. For $2 \leqslant p \leqslant \infty$ the statement is implied by the fact that $\iota_p \leqslant \pi_2$.

References

[1] G. Bennett, *Schur multipliers*, Duke Math. J. 44 (1977), pp. 603–639.
[2] G. Bennett, V. Goodman, C. M. Newman, *Norms of random matrices*, Pacific J. Math. 59 (1975), 359–365.
[3] B. Carl, *A remark on p-integral and p-absolutely summing operators from l_u into l_v*, Studia Math. 57 (1976), pp. 257–262.
[4] B. Carl, B. Maurey, J. Puhl, *Grenzordnungen von absolut-(r, p)-summierenden Operatoren*, Math. Nachr. 82 (1978), pp. 205–218.
[5] A. Erdmann, Diplomarbeit, Jena 1974.
[6] D. J. H. Garling, Y. Gordon, *Relations between some constants associated with finite dimensional Banach spaces*, Israel J. Math. 9 (1971), pp. 346–361.
[7] E. D. Gluskin, A. Pietsch, J. Puhl, *A Generalization of Khintchine's inequality and its application in the theory of operator ideals*, Studia Math. 67 (1980), pp. 149–155.
[8] D. R. Lewis, *Finite dimensional subspaces of L_p*, ibid., to appear.
[9] A. Persson, A. Pietsch, *p-nukleare und p-integrale Operatoren in Banachräumen*, Studia Math. 33 (1969), pp. 21–62.
[10] A. Pietsch, *Operator ideals*, Berlin 1978.
[11] J. Puhl, *Quotienten von Operatorenidealen*, Math. Nachr. 79 (1977), pp. 131–144.
[12] F. Saphar, *Applications p décomposantes et p absolument sommantes*, Israel J. Math. 11 (1972), pp. 164–179.

Received June 20, 1978 (1439)

STUDIA MATHEMATICA, LXIX. (1980)

Maximal seminorms on Weak L^1

by

MICHAEL CWIKEL (Haifa) and CHARLES FEFFERMAN (Princeton, N.J.)

Abstract. For each $f \in \mathrm{Weak}\,L^1$ let $q_1(f) = \sup_{a>0} a\mu(\{x|\ |f(x)| > a\})$ and let $q(f)$ be the seminorm

$$q(f) = \inf_{f=f_1+f_2+\ldots+f_n} \sum_{j=1}^{n} q_1(f_j).$$

We obtain two equivalent expressions for $q(f)$ and make some remarks about the dual of $\mathrm{Weak}\,L^1$.

Introduction. For $0 < p < \infty$ the space $\mathrm{Weak}\,L^p$ taken over the measure space (X, Σ, μ) consists of all (equivalence classes of) measurable functions f for which the quasinorm

$$q_p(f) = \sup_{a>0} a\left[\mu(\{x|\ |f(x)| > a\})\right]^{1/p}$$

is finite.

In [4] and [3] it was shown (independently) that $\mathrm{Weak}\,L^1$ has a non trivial dual space. This is at first sight a surprising result since one can readily show that any continuous linear functional on $\mathrm{Weak}\,L^1$ necessarily vanishes on all simple functions if μ is non atomic. Thus the functionals and seminorms on $\mathrm{Weak}\,L^1$ measure only the "behavior at infinity" (in a sense to be indicated below and in [3]) of the functions on which they act.

Any continuous linear functional on $\mathrm{Weak}\,L^1$ is dominated by a constant multiple of the quasinorm q_1 and therefore also by a multiple of the seminorm q defined by

$$q(f) = \inf_{f=f_1+f_2+\ldots+f_n} \sum_{j=1}^{n} q_1(f_j)$$

where the infimum is taken over all finite decompositions $f = f_1+f_2+\ldots+f_n$ of f in $\mathrm{Weak}\,L^1$. The quotient space of $\mathrm{Weak}\,L^1$ modulo the subspace of elements f satisfying $q(f) = 0$ is a Banach space W normed by q whose dual coincides with the dual of $\mathrm{Weak}\,L^1$.

We shall show here that when the measure μ is non atomic, q is equivalent to each of the following seminorms:

$$I(f) = \lim_{n\to\infty}\Big(\sup_{b/a>n}(\log(b/a))^{-1}\int_{\{x|\ a\leqslant|f(x)|\leqslant b\}}|f(x)|\,d\mu\Big),$$

$$J(f) = \lim_{h\to 1}\Big(\sup_{\substack{1>r>h\\ t>0}}(1-r)t^{r-1}\int_0^t f^*(s)^r\,ds\Big)$$

where f^* is the non increasing rearrangement of f. One might hope that with the help of I or J it might be possible to give some sort of description of $W^* = (\mathrm{Weak}\,L^1)^*$ but in fact the remarks in the last part of this note suggest that there is little hope of obtaining a reasonable characterisation of the elements of the dual space.

Equivalence of the seminorms q, I and J. To verify the triangle inequality for I and for J, let f and g be in $\mathrm{Weak}\,L^1$. It is immediate that $J(f+g)\leqslant J(f)+J(g)$ since

$$\int_0^t f^*(s)^r\,ds = \sup_{\substack{E\in\Sigma\\ \mu(E)=t}}\int_E |f(x)|^r\,d\mu$$

and

$$|f(x)+g(x)|^r \leqslant |f(x)|^r + |g(x)|^r$$

for each $r\leqslant 1$. For the case of the seminorm I let us fix numbers a and b, $0<a<b$ and let $E_0 = \{x|\ a\leqslant|f(x)+g(x)|\leqslant b\}$. Then $E_0\subset E_1\cup E_2\cup E_3\cup\cup E_4$ where

$$E_1 = \{x|\ |f(x)|>b/2\}\cup\{x|\ |g(x)|>b/2\},$$

$$E_2 = \{x|\ a/2\leqslant|f(x)|\leqslant b/2;\ a/2\leqslant|g(x)|\leqslant b/2\},$$

$$E_3 = \{x|\ |f(x)|<a/2,\ a/2\leqslant|g(x)|\leqslant b/2\},$$

and

$$E_4 = \{x|\ |g(x)|<a/2,\ a/2\leqslant|f(x)|\leqslant b/2\}.$$

$\int_{E_0\cap E_1}|f+g|\,d\mu\leqslant b\mu(E_1)\leqslant 2\big(q_1(f)+q_1(g)\big)$ and similarly $\int_{E_3}|f|\,d\mu\leqslant q_1(g)$ and $\int_{E_4}|g|\,d\mu\leqslant q_1(f)$. Consequently,

$$\int_{E_0}|f+g|\,d\mu\leqslant 3\big(q_1(f)+q_1(g)\big)+\int_{E_2\cup E_4}|f|\,d\mu+\int_{E_2\cup E_3}|g|\,d\mu$$

$$\leqslant 3\big(q_1(f)+q_1(g)\big)+\int_{\{x|\ a/2\leqslant|f(x)|\leqslant b/2\}}|f|\,d\mu+\int_{\{x|\ a/2\leqslant|g(x)|\leqslant b/2\}}|g|\,d\mu.$$

Upon dividing by $\log(b/a)$ and taking the lim sup as b/a tends to infinity we obtain that $I(f+g)\leqslant I(f)+I(g)$.

The rest of the verification that I and J are seminorms is trivial and it is also easy to see that $I(f)\leqslant q_1(f)$, $J(f)\leqslant q_1(f)$ for each $f\in\mathrm{Weak}\,L^1$. It follows that $I(f)\leqslant q(f)$ and $J(f)\leqslant q(f)$, and incidentally these inequalities imply that $I(h)=J(h)=q(h)=1$ for any function h with $h^*(t)=1/t$.

The equivalence of the three seminorms now follows from the following theorem.

THEOREM. *Let (X,Σ,μ) be a non atomic measure space. Then for each $f\in\mathrm{Weak}\,L^1(\mu)$:*

(i) $q(f)\leqslant 2I(f)$;

(ii) $I(f)\leqslant eJ(f)$.

Proof of (i). Clearly it suffices to prove the inequality for all functions of the form $f=\sum_{k=-\infty}^{\infty}\lambda^k\chi_{E_k}$ where the sets E_k are disjoint and λ is any fixed positive number as close as we please to 1. In fact we may take the measure space to be the half line $(0,\infty)$ equipped with Lebesgue measure and the sets E_k to be abutting intervals such that f is non increasing. The extension of the proof to the case of a general non atomic measure space is then straightforward.

Specifically then let $f=\sum_{k=-\infty}^{\infty}\lambda^k\chi_{I_k}$ where $\lambda>1$, $I_k=[h_{k+1},h_k)$ and $(h_k)_{k=-\infty}^{\infty}$ is a non increasing sequence of non negative numbers with $\lim_{k\to+\infty}h_k=0$. Fix an integer N and for $j=1,2,\ldots,N$ and $k=0,\pm1,\pm2,\ldots$ let $t_{kj}=h_{k+1}+j|I_k|/N$, where $|I_k|=\mu(I_k)=h_k-h_{k+1}$. Define

$$g_j(t) = \sum_{k=-\infty}^{\infty}\frac{\lambda^k|I_k|}{|t-t_{kj}|+|I_k|/N}\chi_{I_k}(t).$$

For $t\in I_k$,

$$\sum_{j=1}^N g_j(t)\geqslant\sum_{j=1}^N g_j(h_{k+1}) = f(t)N\Big(\sum_{j=1}^N\frac{1}{j+1}\Big).$$

Consequently, for all $t\in(0,\infty)$

$$f(t)\leqslant N^{-1}\Big(\sum_{j=1}^N\frac{1}{j+1}\Big)^{-1}\sum_{j=1}^N g_j(t)$$

and

$$q(f)\leqslant N^{-}\Big(\sum_{j=1}^N\frac{1}{(j+1)}\Big)^{-1}\sum_{j=1}^N q_1(g_j).$$

For any $a > 0$ the set $E_{kj} = I_k \cap \{t \mid g_j(t) > a\}$ coincides with I_k if $a < \lambda^k(1+1/N)^{-1}$ and is empty if $a \geqslant \lambda^k N$. If $\lambda^k(1+1/N)^{-1} \leqslant a < \lambda^k N$, then $\mu(E_{kj}) \leqslant 2|I_k|\left(\frac{\lambda^k}{a} - \frac{1}{N}\right)$. So

$$a\mu(\{t \mid g_j(t) > a\}) \leqslant a \sum_{a<\lambda^k(1+1/N)^{-1}} |I_k| + a \sum_{\lambda^k(1+1/N)^{-1} \leqslant a \leqslant \lambda^k N} 2|I_k|\lambda^k/a$$
$$\leqslant a\mu(\{t \mid f(t) > a(1+1/N)\}) + 2 \int_{\{t \mid a/N \leqslant f(t) \leqslant a(1+1/N)\}} f(t)\,dt.$$

It follows that

$$q_1(g_j) \leqslant q_1(f) + 2 \sup_{b/a=N+1} \int_{\{t \mid a \leqslant |f(t)| \leqslant b\}} |f(t)|\,dt$$

and this in turn implies that

$$q(f) \leqslant \left(\sum_{j=1}^{N} \frac{1}{j+1}\right)^{-1} \left(q_1(f) + 2 \sup_{b/a \geqslant N+1} \int_{\{t \mid a < |f(t)| < b\}} |f(t)|\,dt\right).$$

Now letting N tend to infinity we obtain $q(f) \leqslant 2I(f)$.

Proof of (ii). Let us fix f in WeakL^1. We may assume without loss of generality that $q_1(f) = 1$. There exist sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty} b_n/a_n = \infty$ and

$$\lim_{n\to\infty}\left[(\log(b_n/a_n))^{-1} \int_{\{x \mid a_n \leqslant |f(x)| \leqslant b_n\}} |f(x)|\,d\mu\right] = I(f).$$

But since $q_1(f) = 1$,

$$(\log(b_n/a_n))^{-1} \int_{\{t \mid a_n \leqslant f^*(t) \leqslant b_n\}} f^*(t)\,dt \leqslant (\log(b_n/a_n))^{-1} \int_0^{1/a_n} \min(b_n, f^*(t))\,dt$$
$$\leqslant (1-r_n)(1/a_n)^{r_n-1} \int_0^{1/a_n} f^*(t)^{r_n}\,dt\,(b_n/a_n)^{1-r_n}[(1-r_n)\log(b_n/a_n)]^{-1}$$

for any choice of $r_n \in (0,1)$. Here we shall choose $r_n = 1 - (\log(b_n/a_n))^{-1}$ so that $(b_n/a_n)^{1-r_n} = e$ and $\lim_{n\to\infty} r_n = 1$. (This also minimizes the expression $(b_n/a_n)^{1-r_n}[(1-r_n)\log(b_n/a_n)]^{-1}$.) In the limit as n tends to infinity the above inequalities imply that $I(f) \leqslant eJ(f)$.

Remarks about the dual and predual of W. For $p < 1$ the dual space of WeakL^p is trivial in the non atomic case ([4], [1]). (Weak $L^p)^*$ for $p > 1$ was studied in [2] where it was seen to be the direct sum of the Lorentz space $L^{p',1}$ and a space S of "singular" functionals which, like those of (Weak$L^1)^*$ vanish on simple functions. There is a representation, though not a canonical one, for elements of S, and, in the case of a finite measure space, each element of S is supported on a set whose measure may be chosen to be arbitrarily small. Furthermore, WeakL^p has a perfectly civilized predual, namely $L^{p',1}$.

We show here that anologous results for the case $p = 1$ are either untrue or unlikely. Let us consider a finite measure space (X, Σ, μ) with $\mu(X) = 1$. We first observe that there exists a continuous linear functional l on Weak$L^1(X, \mu)$ which cannot be supported on any set of measure smaller than 1. Let Γ be the class of non negative functions f in WeakL^1 for which $\lim_{r\to 1}\left((1-r)\int_X f(x)^r\,d\mu\right)^{1/r}$ exists. For each $f \in \Gamma$ define $l(f)$ to equal this limit. In view of the inequalities

$$\left(\int f^r\,d\mu\right)^{1/r} + \left(\int g^r\,d\mu\right)^{1/r} \leqslant \left(\int (f+g)^r\,d\mu\right)^{1/r} \leqslant 2^{1/r-1}\left[\left(\int f^r\,d\mu\right)^{1/r} + \left(\int g^r\,d\mu\right)^{1/r}\right]$$

which hold for each positive $r \leqslant 1$ and positive functions f and g we can immediately see that $f \in \Gamma$, $g \in \Gamma$ implies that $f+g \in \Gamma$ with $l(f+g) = l(f) + l(g)$, and that l has a unique extension to a bounded linear functional on the subspace $\Gamma - \Gamma$ of WeakL^1 and thus extends to all of WeakL^1. Clearly $l(f) = 1$ for every non negative function f with $f^*(t) = 1/t$ on $(0, \varepsilon)$ where ε may be arbitrarily small.

Next we specifically choose X to be $(0,1)$ with Lebesgue measure. Let $g_a(x) = 1/2|x-a|$ for each $a \in (0,1)$ and let Y be the subspace of WeakL^1 spanned by these functions. Consider any $f = \sum_{j=1}^{N} \lambda_j g_{a_j}$ in Y where the a_j's are distinct. Choose disjoint open intervals I_j such that $a_j \in I_j$. Let $\tilde{f} = \sum \lambda_j g_{a_j} \chi_{I_j}$. Then $f - \tilde{f}$ is bounded, $q(f-\tilde{f}) = 0$ and so $q(f) = q(\tilde{f})$. For all a sufficiently large $\mu\{x \mid \tilde{f}(x) > a\} = \sum_{j=1}^{N} |\lambda_j|/a$. It follows that $q(f) = \sum_{j=1}^{N} |\lambda_j|$. Now define the linear functional γ on Y by $\gamma\left(\sum_{j=1}^{N} \lambda_j g_{a_j}\right) = \sum_{j=1}^{N} \lambda_j m(a_j)$ where m is a bounded function on $(0,1)$. γ can be extended to a linear functional on all WeakL^1 with bound equal to $\sup_{0<x<1} |m(x)|$. Any representation which one might hope to obtain for such functionals would also have to be meaningful in the case where m is not measurable.

This last example seems to rather dim the prospects of finding a reasonable representation for elements of (Weak$L^1)^*$. One is thus led to ask whether WeakL^1, or rather the corresponding Banach space W has a predual which might be easier to describe and which might perform some of the functions of the inaccessible dual. We conclude these remarks by showing that W cannot have a "natural" predual; any seminorm induced on WeakL^1 by a predual of W necessarily has properties which are incompatible with the basic structures of WeakL^1. One would expect any reasonable seminorm $w(f)$ on WeakL^1 to be rearrangement invariant and, more pertinently for us, it ought to be invariant, as are I, J, and q with respect to the natural rescalling of WeakL^1 obtained when the measure

μ is replaced by 2μ. Specifically we expect to have that

$$(*) \qquad f^*(t) = 2g^*(2t) \quad \text{implies that} \quad w(f) = w(g).$$

But it follows from $(*)$ that the unit ball of W with respect to the dual norm has no extreme points, which is a contradiction. To see this we first note that, if $f \in \mathrm{Weak}\,L^1$ has the form $f = \sum_{k=1}^{\infty} \lambda_k \chi_{E_k}$ where the sets E_k are disjoint, then f cannot be an extreme point of the dual norm unit ball of W, since $f = f_0 + f_1$ where $w(f_0) = w(f_1) = \frac{1}{2}w(f)$ and $f_0 \neq f_1$. To construct f_0 and f_1 divide each set E_k into two disjoint sets of equal measure $E_k = E_{k0} \cup E_{k1}$ and let $f_j = \sum_{k=1}^{\infty} \lambda_k \chi_{E_{kj}}$, $j = 0, 1$. Clearly $f_j^*(t) = f^*(2t)$, so that $w(f_j) = \frac{1}{2}w(f)$. Finally it suffices to observe that for any function $g \in \mathrm{Weak}\,L^1$ there exists a function $f = \sum_{k=1} \lambda_k \chi_{E_k}$ of the above form which represents the same element of W. To construct f we let $G_n = \{x|\ 2^n \leqslant |g(x)| < 2^{n+1}\}$ for each integer n and let f restricted to G_n be a simple function such that $\|(f-g)\chi_{G_n}\|_{L^1} \leqslant 2^{-|n|}$. Clearly f has the required form. For each positive integer N let $V_N = \bigcup_{|n| \leqslant N} G_n$. Then $I\big((f-g)\chi_{V_N}\big) = 0$ for each N and

$$I(f-g) = I\big((f-g)\chi_{X \setminus V_N}\big) \leqslant \|(f-g)\chi_{X \setminus V_N}\|_{L^1} \leqslant \sum_{|n|>N} 2^{-|n|}.$$

Since N may be arbitrarily large, we deduce that f and g are equivalent.

Acknowledgement. We thank Dr. Yoav Benyamini for a helpful remark.

References

[1] M. Cwikel, *On the conjugates of some function spaces*, Studia Math. 45 (1973), pp. 49–55.

[2] — *The dual of* WeakL^p, Ann. Inst. Fourier, Grenoble 25, 2 (1975), pp. 81–126. (Correction note to appear.)

[3] M. Cwikel and Y. Sagher, $L(p, \infty)^*$, Indiana Univ. Math. J. 21 (1972), pp. 781–786.

[4] A. Haaker, *On the conjugate space of Lorentz space*, Technical report, Lund 1970.

DEPARTMENT OF MATHEMATICS
TECHNION-I.I.T., HAIFA
and
DEPARTMENT OF MATHEMATICS
PRINCETON UNIVERSITY, PRINCETON, N.J.

Received June 20, 1978 (1440)

STUDIA MATHEMATICA, T. LXIX. (1980)

On the strong maximal function and Zygmund's class $L(\log^+ L)^n$

by

N. A. FAVA, E. A. GATTO and C. GUTIÉRREZ (Buenos Aires)

Abstract. We show that the basic inequality of a weak type estimate may in a sense be reversed, thus allowing us to give a characterization of the class $L(\log^+ L)^n$ in terms of the integrability of certain functions associated with the strong differentiability of integrals.

1. Introduction and statement of results. In this paper m is used to denote Lebesgue measure in any number of dimensions; this number being clear in each case from the context.

Let $f(x) = f(x_1, \ldots, x_n)$ be an integrable function with support in the unit cube of $\boldsymbol{R}^n$, defined by the inequalities $0 \leqslant x_i \leqslant 1$ $(i = 1, \ldots, n)$. We consider the *partial* maximal operators M_i defined by

$$(1) \qquad M_i f(x) = \sup_{a \leqslant x_i \leqslant b} \frac{1}{b-a} \int_a^b |f(x_1, \ldots, x_{i-1}, u, x_{i+1} \ldots, x_n)|\, du$$

at each point x in $\boldsymbol{R}^n$. We also consider the *strong maximal function*

$$(2) \qquad f^*(x) = \sup_{I \ni x} \frac{1}{m(I)} \int_I |f(y)|\, dy,$$

where the supremum is taken over the set of all intervals I (cells with sides parallel to the axes) containing the point x. As usual, we denote by $L(\log^+ L)^k$ the class of all functions f such that the integral

$$\int |f(x)| \big(\log^+ |f(x)|\big)^k dx$$

is finite.

One of the authors proved in [1] the following theorem:

THEOREM A. *The operation $M_n \ldots M_1 f$ is well defined for any function f in $L(\log^+ L)^{n-1}$ and there exists a constant C depending only on n, such that for each $t > 0$,*

$$(3) \qquad m\{x\colon M_n \ldots M_1 f(x) > 4t\} \leqslant C \int \frac{|f|}{t} \left(\log^+ \frac{|f|}{t}\right)^{n-1} dx.$$