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Aronszajn–Kolmogorov type theorems for positive definite kernels in locally convex spaces

by

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Abstract. It is shown that Aronszajn–Kolmogorov type kernel theorems characterize some new classes of locally convex spaces. These spaces have the factorization property and the strong factorization property. First class includes all pseudo-barrelled spaces and second one includes all barrelled spaces. Some applications to the dilation theory are discussed.

1. Introduction. The study of dilations of operator functions in non-Hilbert spaces has been initiated by the probability theory on Banach spaces. Recent works has revealed a close nexus between the dilation theory and the theory of reproducing kernel Hilbert spaces, see [9], [16]. Following [6] and [3] we will study here X-to-X^* operator valued functions, where X is a locally convex space. The motivation for this general approach is given by the following three facts:

(i) the correlation function of a generalized random field on $\boldsymbol{R}^N$ is a positive definite kernel $K(\cdot,\cdot)$ with values in a set of all antilinear X-to-X^* operators, where X is a space of test functions on $\boldsymbol{R}^N$ ([2]);

(ii) the covariance operator of weak second order probability measures on locally convex spaces is a positive X^*-to-X^{**} valued operator ([14]);

(iii) the theory of Abstract Wiener Spaces, which purpose is to describe Gaussian processes, has been developed recently for locally convex Hausdorff spaces ([8]).

The importance of positive definite kernels is well known culminating from the basic results of Kolmogorov [7] and Aronszajn [1]. The purpose of this paper is to obtain some generalizations of the Aronszajn–Kolmogorov kernel theorem for the operator valued kernels in locally convex spaces. In Section 2 we will show that the complete analogue of the Aronszajn–Kolmogorov theorem does not hold in all locally convex spaces. We will give some modification of this theorem which holds in all locally convex spaces (Th. (2.7)). Theorem (2.12) and (2.16) present two versions of the Aronszajn–Kolmogorov type theorem. The characterization of those locally convex spaces for which and only for which these theorems hold are next obtained. By the way we give an answer to the question of P. Masani (Remark (2.17)). The above characterizations lead us to consider in Section 3 spaces with the factorization property, which first appeared in [3]. The last Section 4 is devoted to deduce some dilation theorems as applications of the kernel theorems.

2. Kernel theorems. Let X be a complex locally convex space and X^* its topological dual. By $\bar{L}(X, X^*)$ ($C\bar{L}(X, X^*)$) we denote the space of all antilinear operators from X into X^* (continuous antilinear operators from X into X^*, where X^* has the strong topology $\beta(X^*, X)$, i.e., the topology of uniform convergence on bounded subsets of X). $L(X, H)$ ($CL(X, H)$) denotes the space of all linear operators (continuous linear operators) from X into a Hilbert space H.

For $A \in L(X, H)$, where in H is given an inner product $(\cdot, \cdot)$, we define the adjoint operator $A^* \in \bar{L}(H, X^*)$ by the formula

$$(A^*f)(x) = (Ax, f), \quad f \in H, x \in X. \tag{2.1}$$

(2.2) Definition. Let Z be a set. An $\bar{L}(X, X^*)$-valued function K on $Z \times Z$ is called *positive definite kernel* if for each $n, z_1, \ldots, z_n \in Z$ and $x_1, \ldots, x_n \in X$ we have

$$\sum_{i,k=1}^{n} \big(K(z_i, z_k)x_i\big)(x_k) \geqslant 0.$$

If X is a Banach space, then the next generalization of the Aronszajn–Kolmogorov theorem was obtained in [9], Th. (2.9) and in [15], Th. (4.6).

(2.3) Proposition. *Let X be a Banach space and suppose that the kernel $K(\cdot, \cdot)\colon Z \times Z \to C\bar{L}(X, X^*)$ is positive definite. Then there is a Hilbert space H and an operator function $T(\cdot)\colon Z \to CL(X, H)$ such that*

$$K(u, v) = T^*(v)T(u) \quad \textit{for} \quad u, v \in Z. \tag{2.4}$$

If H is minimal, then H and $T(\cdot)$ are unique up to unitary equivalence.

This result plays an important role in the theory of the second order stochastic processes and in the dilation theory (cf. [9], [10], [12], [15] and [16]).

Following [6], where positive definite operator valued functions in linear spaces were investigated, it is interesting to ask if this proposition is also true for locally convex spaces. The example which we present below shows that in general the answer is not.

(2.5) Example. Let s_0 be a space of all complex sequences having only a finite number of coordinates different from zero. Let us introduce in s_0 a topology τ (cf. [4]) by the family of seminorms

$$p_{N_0,\{M_n\}}(x) = \sum_{n \in N_0} M_n|x_n|, \quad x = (x_1, x_2, \ldots) \in s_0, \tag{2.6}$$

where N_0 is an arbitrary subset of natural numbers with the density equal to zero and $\{M_n\}$ is an arbitrary non-negative sequence of real numbers. Note that a subset $B \subset s_0$ is τ-bounded iff there exist a natural number n_0 and constant $C > 0$ such that the condition $x = \{x_n\} \in B$ implies $x_n = 0$ for $n > n_0$ and $|x_n| < C$.

Let Z be a singleton $\{z\}$ and put

$$\big(K(z, z)x\big)(y) = \sum_{k=1}^{\infty} \bar{x}_k y_k, \quad x, y \in s_0.$$

Then

$$K(\cdot, \cdot)\colon Z \times Z \to C\bar{L}\big((s_0, \tau), (s_0^*, \beta(s_0^*, s_0))\big)$$

is a positive definite kernel ($K(z, z)$ is a positive continuous operator on (s_0, τ) to $(s_0^*, \beta(s_0^*, s_0))$).

If we suppose that similarly to Proposition (2.3) there exist a Hilbert space H and an operator $T_z \in CL(s_0, H)$ such that

$$K(z, z) = T_z^* T_z,$$

then the continuity of T_z implies that the function

$$f(x) = \big[\big(K(z, z)x\big)(x)\big]^{1/2} = \Big(\sum_{k=1}^{\infty} |x_k|^2\Big)^{1/2}$$

is continuous on (s_0, τ). But following the definition of the topology τ, $f(x)$ is not continuous because for each $\varepsilon > 0$ and for each seminorm p from the family (2.6) there exists $x \in X$ such that $f(x) > \varepsilon p(x)$.

This contradiction establishes the fact that the complete analogue of the Aronszajn–Kolmogorov theorem (as Proposition (2.3)) is not true in all locally convex spaces.

Moreover, we give a characterization of those locally convex spaces for which and only for which this theorem is true. But it is convenient to start from some modification of the Aronszajn–Kolmogorov theorem which is true for all locally convex spaces.

(2.7) Theorem. (On factorization.) *Let Z be a set and let X be a complex locally convex space. If a positive definite kernel $K(\cdot, \cdot)\colon Z \times Z \to \bar{L}(X, X^*)$ satisfies the condition*

$$K(z, z) \in C\bar{L}(X, X^*) \quad \textit{for all } z \in Z, \tag{α}$$

then there exist a Hilbert space H and an operator function $T(\cdot)\colon Z \to L(X, H)$, where $T(z)$ for each $z \in Z$ is a continuous operator on bounded *subsets of* X *such that*

$$K(u, v) = T^*(v)\,T(u), \quad u, v \in Z. \tag{2.8}$$

If H *is minimal, i.e.,* $H = \bigvee_{z\in Z} T(z)X$, *then* H *and* $T(\cdot)$ *are unique up to unitary equivalence. This means that if* H_1, H_2 *are Hilbert spaces and* $T_i(\cdot)\colon X \to H_i$ *as well* $H_i = \bigvee_{z\in Z} T_i(z)X$, $i = 1, 2$, *and* $T_1^*(v)T_1(u) = T_2^*(v)T_2(u)$ *for all* $u, v \in Z$, *then there is an unitary operator* $U\colon H_1 \to H_2$ *such that* $UT_1(u) = T_2(u)$ *for* $u \in Z$.

Proof. It is possible to present two different proofs. The first one uses the original idea of A. N. Kolmogorov and the second the idea of N. Aronszajn (cf. the proof in Banach space case in [15] and [9], respectively). We use the first possibility.

Let us consider the scalar valued kernel $R(h, g)$ of two variables $h = (y, v)$ and $g = (x, u)$ from $X\times Z$ given by the following relation

$$R(h, g) = \big(K(u, v)x\big)(y). \tag{2.9}$$

Since $K(\cdot,\cdot)$ is an operator valued positive definite kernel, then for each $x \in X$ and $u, v \in Z$, $K(u, v)x$ is an antilinear operator and $\big(K(u, v)x\big)(\cdot)$ is a linear functional. Consequently, for each n and every n-tuples $g_i = (x_i, z_i) \in X\times Z$, $c_i \in C$ we have

$$\sum_{i,k=1}^{n} R(g_i, g_k)c_i\bar{c}_k = \sum_{i,k=1}^{n} \big(K(z_k, z_i)x_k\big)(x_i)\,c_i\bar{c}_k$$
$$= \sum_{i,k=1}^{n} \big(K(z_k, z_i)c_k x_k\big)(c_i x_i) \geqslant 0$$

and thus the scalar valued kernel $R(h, g)$ is positive definite. If $g_1, g_2, \ldots, g_n$ are n points in $X\times Z$ then complex matrix $[R(g_j, g_k)]$ is positive definite. Hence there exists an n-dimensional complex Gaussian probability distribution $\mu^{g_1\ldots g_n}$ with mean vector zero and covariance matrix $[R(g_j, g_k)]$. The family of probability measures $\mu^{g_1\ldots g_n}$ is obviously consistent. Consider the Borel space Ω of all complex valued functions on $X\times Z$ with the smallest σ-field relative to which every projection map $\pi_g\colon f \to f(g)$ from Ω into the complex plane is measurable. By Kolmogorov's theorem there exists a measure μ on Ω such that the joint distribution of $\big(f(g_1), \ldots f(g_n)\big)$, $f \in \Omega$, is $\mu^{g_1\ldots g_n}$ for every $g_1, \ldots, g_n \in X\times Z$. If we consider the Hilbert space $L^2(\mu)$ and define $\xi(g) = f(g)$, $f \in \Omega$, then $\xi(g)$ is an element of $L^2(\mu)$ (ξ is a Gaussian stochastic process on $X\times Z$) and

$$\big(\xi(g), \xi(h)\big) = \int_{\Omega} f(g)\overline{f(h)}\,d\mu(f) = R(g, h). \tag{2.10}$$

Now we define H as the subspace of $L^2(\mu)$ spanned by the vectors $\xi(g)$, $g \in X\times Z$. Thus by (2.9) and (2.10) we have

$$\big(K(u, v)x\big)(y) = R(h, g) = \big(\xi(h), \xi(g)\big) = \big(\xi(y, v), \xi(x, u)\big),$$

where $\xi(x, u)$ is an H-valued function of two variables $x \in X$ and $u \in Z$.

Define an operator function $T(\cdot)\colon X \to H$ on Z by the formula

$$T(u)(x) = \xi(x, u), \quad x \in X,\ u \in Z.$$

For each $u \in Z$, $T(u)$ is a linear operator according to the fact that the function $\xi(x, u)$ is linear on the first variable when the second is fixed. By (2.10) and (2.1) we have for all $x, y \in X$ and $u, v \in Z$

$$\big(K(u, v)x\big)(y) = \big(T(v)y, T(u)x\big) = \big(T^*(v)T(u)x\big)(y),$$

which shows that (2.8) holds. Since the kernel $K(\cdot,\cdot)$ satisfies condition (α) and X^* has the strong topology, we obtain that for each bounded net $x_a \to 0$, $x_a \in X$,

$$\|T(u)x_a\|^2 = \big(T(u)x_a, T(u)x_a\big) = \big(\xi(x_a, u), \xi(x_a, u)\big)$$
$$= \big(K(u, u)x_a\big)(x_a) \to 0.$$

Hence, for each $u \in Z$, $T(u)$ is a continuous operator on bounded subsets of X.

Since H is minimal, only uniqueness requires a proof. Thus suppose that there exist Hilbert spaces H_1, H_2 and $T_i(\cdot)\colon X \to H_i$, $H_i = \bigvee_{z\in Z} T_i(z)X$ for $i = 1, 2$ such that

$$T_1^*(v)T_1(u) = K(u, v) = T_2^*(v)T_2(u).$$

Hence the map

$$\sum_{k=1}^{N} T_1(u_k)x_k \to \sum_{k=1}^{N} T_2(u_k)x_k$$

extends to the unitary operator $U\colon H_1 \to H_2$ such that $UT_1(u) = T_2(u)$ for $u \in Z$. ■

Remark. In the factorization (2.8) the operator function $T(.)$ may not belong to $CL(X, H)$ (see (2.5)). Let us note that for a complex linear space X in a similar way as in [6], Th. 1, the above proof gives a theorem an algebraic factorization of $\bar{L}(X, X')$-valued kernels.

(2.11) Definition. A locally convex space X has the *factorization property* (the *strong factorization property*) if for each positive operator $R \in C\bar{L}(X, X^*)$ ($R \in \bar{L}(X, X^*)$) the function $x \to (Rx)(x)$ is continuous.

(2.12) Theorem (On continuous factorization I.) *Let* Z *be a set and* X *a complex locally convex space with the factorization property. If a positive definite kernel* $K(\cdot,\cdot)\colon Z\times Z \to \bar{L}(X, X^*)$ *satisfies the condition*

$$K(z, z) \in C\bar{L}(X, X^*) \quad \textit{for all } z \in Z, \tag{α}$$

then there exist a Hilbert space H and a $CL(X, H)$-valued function $T(\cdot)$ on Z such that

$$K(u, v) = T^*(v)T(u), \quad u, v \in Z.$$

Moreover, if H is minimal, then H and $T(\cdot)$ are unique up to unitary equivalence.

Proof. By Theorem (2.7) there exist a Hilbert space H and a $L(X, H)$-valued function $T(\cdot)$ on Z such that $K(u, v) = T^*(v)T(u)$. We must to prove that the assumption "X has the factorization property" yields that $T(u) \in CL(X, H)$ for each $u \in Z$. But

$$\|T(u)x\|^2 = \big(K(u, u)x\big)(x) \quad \text{for } x \in X, u \in Z.$$

If we put $R = K(u, u)$, $u \in Z$, then R is positive by (α) $R \in C\bar{L}(X, X^*)$ and its square root $T(u)$ is continuous according to (2.11). ∎

(2.13) Proposition. *Let X be a complex locally convex space and X^* its topological dual with the strong topology. Then the following conditions are equivalent:*

(i) *For each inner product $(\cdot,\cdot)$ defined on X, for which*

(2.14) $\quad \sup_{y \in B} |(x, y)| = p_B(x) < \infty$ *(for each bounded subset $B \subset X$) and the seminorm p_B is continuous,*

the function $\|x\| \overset{\text{df}}{=} (x, x)^{1/2}$ is continuous.

(ii) *X has the factorization property.*

(iii) *For each positive definite kernel the analogue of the Aronszajn–Kolmogorov theorem* (2.12) *holds true.*

(iv) *For each positive operator $R \in C\bar{L}(X, X^*)$ there exist a Hilbert space H and a square root $T \in CL(X, H)$, i.e., $R = T^*T$. Moreover, if H is minimal, then H and T are unique up to unitary equivalence.* [1]

Proof. (i) ⇒ (ii). Let us define an inner product in X by the formula $(y, x) = (Rx)(y)$, $x, y \in X$, where $R \in C\bar{L}(X, X^*)$ is positive. This inner product satisfies (2.14). Indeed, if $B \subset X$ is a bounded subset, then for each $x \in X$, $p_B(x) = \sup_{y \in B} |(Rx)(y)| < \infty$ since Rx is bounded on B. Moreover, p_B is continuous which follows from the assumption $R \in C\bar{L}(X, X^*)$. Consequently, by (i) the function $x \to \|x\|^2 = (Rx)(x)$ is continuous which by (2.11) shows that X has the factorization property.

(ii) ⇒ (iii). Theorem (2.12).

(iii) ⇒ (iv). We put Z is singleton in (2.12).

(iv) ⇒ (i). By (iv) for each $x \in X$ $(Rx)(x) = \|Tx\|^2$ and consequently

[1] For the Banach space X this result reduces to Vakhania's lemma on factorization [13]. Cf. also [14].

the function $x \to (Rx)(x)$ is continuous, since $T \in CL(X, H)$. Let a contrario $(\cdot,\cdot)$ be an inner product in X for which (2.14) holds but the function $\|x\| = (x, x)^{1/2}$ is not continuous. Then the operator R given by the formula

$$(Rx)(y) = (y, x), \quad x, y \in X,$$

is well defined, positive and $R \in C\bar{L}(X, X^*)$, but the function $x \to (Rx)(x)$ is not continuous. This contradiction completes the proof. ∎

(2.15) Corollary. *Let X and $K(\cdot,\cdot)$ be as in* (2.12); *then for each $u, v \in Z$, $K(u, v) \in C\bar{L}(X, X^*)$.*

Proof. By Theorem (2.12) there exist the Hilbert space H and the operator function $T(\cdot)$: $Z \to CL(X, H)$ such that $K(u, v) = T^*(v)T(u)$. But $T(u) \in CL(X, H)$ and $T^*(v) \in C\bar{L}(H, X^*)$. Thus $K(u, v) \in C\bar{L}(X, X^*)$.

(2.16) Theorem. (On continuous factorization II.) *Let Z be a set and X a complex locally convex space with strong factorization property. For each positive definite kernel $K(\cdot,\cdot)$: $Z \times Z \to \bar{L}(X, X^*)$ there exist a Hilbert space H and an $CL(X, H)$-valued function $T(\cdot)$ on Z such that $K(u, v) = T^*(v)T(u)$. Moreover, if H is minimal, then H and $T(\cdot)$ are unique up to unitary equivalence.*

Proof. We use Theorem (2.7). It is sufficient to check the continuity of $T(z)$, $z \in Z$, on X. For each $z \in Z$, $K(z, z)$ is a positive $\bar{L}(X, X)^*$-valued operator and by definition (2.11) the function $x \to \big(K(z, z)x\big)(x)$ is continuous. But according to the factorization (2.8), where $T(z) \in L(X, H)$, we have that the function $x \to \big(K(z, z)x\big)(x) = \|T(z)x\|^2$ is continuous. Hence $T(z) \in CL(X, H)$, $z \in Z$. ∎

(2.17) Remark. P. Masani ([10], (7.1)) asked in what form does Proposition (2.3) survive for (not necessarily continuous) $L(X, X^*)$-valued positive definite kernels on $Z \times Z$? The above theorem gives an answer to this question in a more general setting of locally convex spaces. Because Banach spaces have the strong factorization property (see Section 3) Theorem (2.16) shows that Proposition (2.3) is true without any change for non-continuous operators.

The next result is similar to (2.13). Its proof is analogous to the proof of (2.13) and will be omitted.

(2.18) Proposition. *Let X be a complex locally convex space and X^* its topological dual. Then the following conditions are equivalent:*

(i) *For each inner product $(\cdot,\cdot)$ defined on X, which is coordinatewise continuous the function $\|x\| \overset{\text{df}}{=} (x, x)^{1/2}$ is continuous.*

(ii) *X has the strong factorization property.*

(iii) *For each positive definite kernel the analogue of the Aronszajn–Kolmogorov theorem* (2.16) *is true.*

(iv) *For each positive operator* $R \in \bar{L}(X, X^*)$ *there exist a Hilbert space* H *and a square root* $T \in CL(X, H)$. *Moreover, if* H *is minimal, then* H *and* T *are unique up to unitary equivalence.*

3. Spaces with factorization property. These spaces were introduced in [3]. Proposition (2.13) and Example (2.5) show that there exist locally convex spaces without the factorization property. However, the class of spaces with the factorization property is large and contains, for example, barrelled spaces, bornological spaces, quasi-barrelled spaces, and DF-spaces. Consequently, it contains Banach spaces.

Let us denote

$\mathscr{F}$ — class of spaces with the factorization property,
$\mathscr{SF}$ — class of spaces with the strong factorization property,
$\mathscr{PB}$ — class of pseudo-barrelled spaces ($X \in \mathscr{PB}$ if each lower semi-continuous and continuous on bounded subsets of X seminorm p on X is continuous),
$\mathscr{B}$ — class of barrelled spaces.

From the definition immediately follows that the barrelled spaces, bornological spaces and quasi-barrelled spaces belongs to the class $\mathscr{PB}$.

(3.1) PROPOSITION. *Each pseudo-barrelled space has the factorization property.*

Proof. Let X be a pseudo-barrelled space and let R be a positive operator from $C\bar{L}(X, X^*)$. We have to prove that the function $x \to (Rx)(x)$ s continuous.

Consider the operator R as a positive definite kernel on a singleton. By Theorem (2.7) there exist a minimal Hilbert space H and an operator $T \in L(X, H)$ which is continuous on bounded subsets of X and such that

$$R = T^*T.$$

An analogous factorization of R may be obtained from the earlier algebraic version of the dilation theorem ([6], Th. 1)

$$R = A^*A, \quad A \in L(X, F).$$

In view of the uniqueness property, H is unitary equivalent by U to the completion of the pre-Hilbert space F with the inner product $(\cdot, \cdot)$ given by formula (2) in [6] and $A = UT$.

Let us define on X a family of seminorms $(p_y)_{y \in F, \|y\| \leqslant 1}$ by the equality

$$p_y(x) = |(y, Ax)|. \tag{3.2}$$

Note that each $p_y(\cdot)$ is continuous on X. Indeed, using the form of the inner product and the operator A, we have for a net $x_a \to x$

$$p_y(x_a) = |(y, Ax_a)| = |(Rx_0)(x_a)| \xrightarrow{a} |(Rx_0)(x)| = |(y, Ax)| = p_y(x)$$

where $x_0 \in X$ generates $y \in F$ in the sense of formula (1) in [6]. Hence the new seminorm

$$p(x) = \|Ax\| = \sup_{\substack{y \in F \\ \|y\| \leqslant 1}} |(y, Ax)| = \sup_{\substack{y \in F \\ \|y\| \leqslant 1}} p_y(x) \tag{3.3}$$

as the supremum of continuous seminorm is lower semicontinuous. Since

$$p(x) = \|Ax\| = \|UTx\| = \|Tx\|, \tag{3.4}$$

the seminorm $p(x)$ is continuous on bounded subsets of X. Hence, by the definition of pseudo-barrelled spaces, the seminorm $p(x)$ is continuous. Thus the function

$$x \to (Rx)(x) = (T^*Tx)(x) = \|Tx\|^2 = p^2(x)$$

is continuous and according to definition (2.11) X has the factorization property. ■

(3.5) PROPOSITION. *Each barrelled space has the strong factorization property.*

Proof. The proof is similar to the above one.

Let X be a barrelled space and R a positive operator from $\bar{L}(X, X^*)$. We prove that the function $x \to (Rx)(x)$ is continuous. Choose the space H, operator A, seminorms $p_y(x)$ and $p(x)$ as in the proof of (3.1). The seminorms $p_y(\cdot)$ defined by (3.2) are continuous and its supremum $p(\cdot)$ (cf. (3.3) and (3.4)) is lower semicontinuous. Since X is barrelled space, we have that $p(x)$ is continuous what in view of definition (2.11) shows that X has the strong factorization property. ■

Examples given in [3] show that:

(3.6) PROPOSITION ([3]).

(1) $\mathscr{SF} \subsetneq \mathscr{F}$.
(2) $\mathscr{PB} \subsetneq \mathscr{F}$.
(3) $\mathscr{PB} \cap \mathscr{SF} \supsetneq \mathscr{B}$.
(4) $\mathscr{PB} \neq \mathscr{PB} \cap \mathscr{SF} \neq \mathscr{SF}$.

4. Dilation theorems. Here we give some applications of kernel theorems to the dilation theory. For the results of this type for the Hilbert space case we refer to [11] and [9]. The Banach space case was studied in [12] and [16].

Let S be a multiplicative semi-group and H a Hilbert space. The operator valued function $\pi(\cdot)$: $S \to CL(H, H) = CL(H)$ is called a *representation* if $\pi(uv) = \pi(u)\pi(v)$ for $u, v \in S$. If S has a unit e, then π is called *unital* if $\pi(e) = I_H$ — the identity operator in H.

(4.1) DEFINITION. Let X be a complex locally convex space, $R \in CL(X, H)$ and let $\pi(\cdot)\colon S \to CL(H)$ be a representation of semi-group S in H. We say that $\pi(\cdot)$ is an *R-dilation* of kernel $K(\cdot,\cdot)\colon S\times S \to \bar{L}(X, X^*)$ if

$$K(u, v) = R^*\pi(v)^*\pi(u)R, \qquad u, v \in S. \tag{4.2}$$

Let us note that (4.2) implies that kernel $K(\cdot,\cdot)$ is positive definite and satisfies the *boundedness condition*, i.e., there is a finite function $\varrho\colon S \to \boldsymbol{R}^+$ such that

$$\sum_{i,k=1}^{n} \big(K(su_i, su_k)x_i\big)(x_k) \leqslant \varrho(s) \sum_{i,k=1}^{n} \big(K(u_i, u_k)x_i\big)(x_k) \tag{4.3}$$

for all n, $u_1, \ldots, u_n \in S$ and $x_1, \ldots, x_n \in X$.

Now we generalize to the non-Banach space case the dilation theorem given in [9], [12].

(4.4) THEOREM. (General dilation theorem.) *Let S be a unital semi-group, X a complex locally convex space with the factorization property, and $K(\cdot,\cdot)\colon S\times S \to \bar{L}(X, X^*)$ a positive definite kernel which satisfies condition (α),*

$$K(s, s) \in C\bar{L}(X, X^*) \qquad \text{for each } s \in S.$$

If the kernel $K(\cdot,\cdot)$ satisfies the boundedness condition (4.3), *then $K(\cdot,\cdot)$ has an R-dilation which is a unital representation π of S. The minimality condition $H = \bigvee_{s\in S} \pi(s)R(X)$ determines H and $\pi(\cdot)$ up to unitary equivalence.*

Proof. By Theorem (2.12) there exist a Hilbert space H and an operator function $T(\cdot)\colon S \to CL(X, H)$ such that

$$K(u, v) = T^*(v)T(u), \qquad u, v \in S.$$

Following [9], [12] we define for $s \in S$ the operator

$$\pi(s)\sum_{j=1}^{n} T(u_j)x_j = \sum_{j=1}^{n} T(su_j)x_j$$

on a dense linear manifold of $H = \bigvee_{s\in S} T(s)X$. Condition (4.3) implies that $\pi(s)$ extends in a unique way to $\pi(s) \in CL(H)$. Since

$$\pi(st)T(u)x = T(stu)x = \pi(s)\big(T(tu)x\big) = \pi(s)\big(\pi(t)T(u)x\big)$$

for $s, t, u \in S$ and $x \in X$ and

$$\pi(e)T(u)x = T(eu)x = T(u)x = I_H,$$

$\pi(\cdot)$ is a unital representation of S in H. Moreover,

$$K(us, vt) = T^*(vt)T(us) = T^*(t)\pi(v)^*\pi(u)T(s), \qquad u, v, t, s \in S.$$

If we put $t = s = e$ and $R = T(e)$, then we have

$$K(u, v) = R^*\pi(v)^*\pi(u)R.$$

Since the proof of uniqueness is standard, we omit them. ∎

If S is a *-semi-group (i.e., $(st)^* = t^*s^*$, $s^{**} = s$ and $e^* = e$) and $K(u, v) = B(v^*u)$, then Theorem (4.4) reduces to the extention of the Sz.-Nagy theorem proved in another way in [5].

(4.5) COROLLARY. (Sz.-Nagy dilation theorem.) *Let S be a *-semi-group, X a complex locally convex space with factorization property and $B(\cdot)\colon S \to \bar{L}(X, X^*)$ positive definite function for which $B(s^*s) \in C\bar{L}(X, X^*)$. If $B(\cdot)$ satisfies condition* (4.3), *then $B(\cdot)$ has an R-dilation which is a unital *-representation π of S. The minimality condition determines H and π up to unitary equivalence.*

(4.6) COROLLARY. (Naĭmark dilation theorem.) *Let X be a complex locally convex space with the factorization property. If F is a positive $C\bar{L}(X, X^*)$-valued measure on a measurable space (S, Σ), then $F(\cdot)$ has an R-dilation which is a spectral measure $E(\cdot)$. The minimality condition determines the Hilbert space H and spectral measure $E(\cdot)$ up to unitary equivalence.*

Proof. Similar to that of Proposition 3 from [17], but we should use Corollary (4.5).

(4.7) PROPOSITION. *Let X be a complex locally convex space and X^* its topological dual with the strong topology. Then each of conditions* (i)–(iv) *from Proposition* (2.13) *is equivalent to the following ones:*

(v) *For each positive definite kernel the general dilation theorem* (4.4) *holds.*

(vi) *For each positive definite function on a *-semi-group Sz.-Nagy dilation theorem* (4.5) *holds.*

(vii) *For each positive measure the Naĭmark dilation theorem* (4.6) *holds.*

Proof. It is easy to observe that (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (iv).

(4.8) Remark. In a similar way by Proposition (2.16) one may obtain variants of the above results for locally convex spaces X with the strong factorization property.

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Convexity, type and the three space problem

by

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Abstract. A twisted sum of two quasi-Banach spaces X and Y is a quasi-Banach space Z with a closed subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$.

We show that if X is p-convex and Y is q-convex where $p \neq q$, then Z is $\min(p, q)$ convex. Similarly, if X is a type p Banach space and Y is a type q Banach space where $p \neq q$ then Z is type $\min(p, q)$.

If X and Y are Banach spaces, we show that Z is *log convex*, i.e., for some $C < \infty$

$$\|z_1+\ldots+z_n\| \leqslant C\left(\sum_{k=1}^{n}\|z_k\|\left(1+\log\frac{1}{\|z_k\|}\right)\right)$$

where $\|z_1\|+\ldots+\|z_n\| = 1$. Conversely, every log convex space is the quotient of a subspace of a twisted sum of two Banach spaces.

If X and Y are type p Banach spaces $(1 < p \leqslant 2)$ and one is the quotient of a subspace of some L_p-space, then Z is *log type* p, i.e.,

$$\left\{\int_0^1 \|\varepsilon_1(t)z_1+\ldots+\varepsilon_n(t)z_n\|^p\,dt\right\}^{1/p} \leqslant c\left\{\sum \|z_k\|^p\left(1+\left(\log\frac{1}{\|z_k\|}\right)^p\right)\right\}^{1/p}$$

where $\|z_1\|^p+\ldots+\|z_n\|^p = 1$. This result is best possible in a certain sense.

We also show that if $p < 1$ type p implies p-convexity, but if $p = 1$ a type 1 space need not be convex.

We investigate which Orlicz sequence spaces and Köthe sequence spaces are $\mathscr{K}$-spaces, i.e., such that every twisted sum with $\boldsymbol{R}$ is a direct sum.

1. Introduction. A quasi-Banach space Z is a twisted sum of X and Y if it has a subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$. The so-called *three space problem* is to study the properties of Z in terms of those of X and Y.

In [1], Enflo, Lindenstrauss and Pisier showed that a Banach space which is a twisted sum of two Hilbert spaces need not be a Hilbert space. Independently, the author [6], Ribe [15] and Roberts [16] showed that a twisted sum of a line and a Banach space need not be locally convex. In [9] the author and Peck showed that these results are related by describing a general construction which shows that for every p, $0 < p < \infty$, there is a twisted sum of l_p with l_p which is not a direct sum. In particular, for $0 < p < 1$, there is a non p-convex space which is a twisted sum of two p-convex spaces.

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