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On the representation of measurable functions by multiple trigonometric series

by

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Abstract. It is proved that for every almost everywhere finite measurable function f defined on a k-torus T_k can be represented by a k-fold trigonometric series that is convergent to f almost everywhere by square summation. For $k = 2$, we show that there exists a function F, continuous on T_2 and such that the result of term-by-term mixed differentiation of its double Fourier series is a double trigonometric series convergent to f almost everywhere by square summation. For higher dimensions we have a similar result.

1. Introduction. Answering a question posed by Lusin, Men'shov [2], [4] proved that, for any measurable almost everywhere finite 2π-periodic function, there is a trigonometric series convergent to the given function almost everywhere. Bary [2] strengthened Men'shov's result, by showing that for any $f(x)$, measurable and finite almost everywhere on $T_1 = [0, 2\pi]$, there exists a function $F(x)$ continuous on T_1 and such that $F'(x) = f(x)$ almost everywhere and the result of term-by-term differentiation of the Fourier series of $F(x)$ is a trigonometric series convergent to $f(x)$ almost everywhere. This result is very deep, since even for integrable functions the Fourier series cannot in general be taken as the apparatus for representing summable functions [3].

In connection with the above question for functions of several variables, it is natural to ask whether every measurable almost everywhere finite function f defined on the k-torus T_k can be represented by a k-fold trigonometric series convergent to f almost everywhere and summed either by squares or by rectangles [5].

In the present paper, we give a confirmation of Men'shov's theorem on the existence of such a representation by a multiple trigonometric series summed by squares. For notational simplicity, we treat the question explicitly for the case of dimension 2.

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2. Preliminaries and notation. Consider the 2-torus $T_2 = [0, 2\pi] \times \times [0, 2\pi]$ of points (x, y). For $F \in L^1(T_2)$, let $S_{nm}F(x, y)$ be the rectangular partial sum of the double Fourier series $SF(x, y)$ of F at (x, y), where $n, m \geqslant 0$. We shall make use of the standard equality

$$S_{nm}F(x, y) = \frac{1}{\pi^2} \int_{T_2} F(u, v) D_n(u-x) D_m(v-y)\, du dv,$$

where

$$D_n(\cdot) = \frac{\sin(n+\frac{1}{2})(\cdot)}{2\sin\frac{\cdot}{2}}.$$

It is easy to see the mixed differentiation

$$\frac{\partial^2 S_{nm}F(x, y)}{\partial x \partial y} = \frac{1}{\pi^2} \int_{T_2} F(u, v) \frac{\partial D_n(u-x)}{\partial u} \frac{\partial D_m(v-y)}{\partial v}\, du dv. \tag{2.1}$$

As is known (see [2], pp. 406–410), there exists continuous monotonic function $g(x)$, $g(0) = 0$, $g(2\pi) = 1$, constant in all the intervals contiguous to some perfect set of measure zero in $[0, 2\pi]$ for which

$$\lim_{n\to\infty} \int_0^{2\pi} e^{-int} dg(t) = 0;$$

hence, as shown in the proof of a theorem in [2], pp. 366–367, the Fourier–Stieltjes series of g converges to zero almost everywhere. We shall fix such a function in this paper.

To any continuous function h on T_2 we can assign a function F_h as follows:

$$\begin{aligned} F_h(u, v) = {} & \int_0^v \int_0^u h(t, s)\, dt ds - g(v) \int_0^{2\pi} \int_0^u h(t, s)\, dt ds - \\ & - g(u) \int_0^v \int_0^{2\pi} h(t, s)\, dt ds + g(u) g(v) \int_0^{2\pi} \int_0^{2\pi} h(t, s)\, dt ds, \end{aligned} \tag{2.2}$$

where $(u, v) \in T_2$ and g is the function stated above.

Also, for convenience, in the following sections we use the same letter C for an absolute constant, which may be different from case to case.

3. Some essential lemmas. Some basic tools are necessary for our main result. First we rewrite a modification lemma due to Men'shov:

LEMMA 1. *Suppose $[c, d] \subset [0, 2\pi]$, $\gamma \in \boldsymbol{R}$, $\varepsilon > 0$ and let $\nu > 8$ be an integer. Then there exists a function $\psi(x)$ and a closed set D such that*

(3.1) *$\psi(x)$ is a continuous piecewise linear function on $[0, 2\pi]$ and $\psi(x) = 0$ outside $[c, d]$;*

$$|\psi(x)| \leqslant 2\nu|\gamma| \quad in \quad 0 \leqslant x \leqslant 2\pi; \tag{3.2}$$

$$\left| \int_0^\xi \psi(x)\, dx \right| < \varepsilon, \quad 0 \leqslant \xi \leqslant 2\pi; \tag{3.3}$$

$$\psi(x) = \gamma \quad in \quad D, \tag{3.4}$$

where $D \subset [c, d]$ and $\mu_1(D) > (d-c)(1-5/\nu)$ (μ_1 the Lebesgue measure in $\boldsymbol{R}$);

$$\begin{aligned} \left| \int_0^{2\pi} \psi(u) D_n(u-x)\, du \right| &\leqslant C\nu|\gamma| \quad (n \geqslant 0), \\ \left| \int_0^{2\pi} \psi(u) D_n(u-x)\, du \right| &\leqslant Cn^2\varepsilon \quad (n \geqslant 1) \end{aligned} \tag{3.5}$$

for each $x \in [0, 2\pi]$, where C is an absolute constant.

Proof. The proof of this lemma is essentially the same as in [1], pp. 488–504.

The following lemma is an generalization of the above on T_2.

LEMMA 2. *Suppose $[c, d] \times [a, b] \subset T_2$, $\gamma \in T$, $\varepsilon > 0$ and let $\nu > 8$ be an integer. Then there exists a continuous function $\varphi(x, y)$ and a closed set Λ such that*

$$\varphi(x, y) = \psi(x) \lambda(y), \tag{3.1'}$$

where $\psi(x)$ is a function constructed as in Lemma 1 and $\lambda(y)$ is a continuous piecewise linear function in $[0, 2\pi]$ vanishing outside $[a, b]$;

$$|\varphi(x, y)| \leqslant 2\nu|\gamma| \quad in \quad 0 \leqslant x, y \leqslant 2\pi; \tag{3.2'}$$

$$\left| \int_0^\xi \int_0^\eta \varphi(x, y)\, dx dy \right| < 2\pi\varepsilon, \quad 0 \leqslant \xi, \eta \leqslant 2\pi; \tag{3.3'}$$

$$\varphi(x, y) = \gamma \quad in \quad \Lambda, \tag{3.4'}$$

where $\Lambda \subset [c, d] \times [a, b]$ and $\mu_2(\Lambda) > (d-c)(b-a)(1-5/\nu)$ (μ_2 the Lebesgue measure in $\boldsymbol{R}^2$);

(3.5′) *the rectangular partial sums $\{S_{nm}\varphi(x, y)\}$ converge uniformly, and for each $(x, y) \in T_2$,*

$$\begin{aligned} |S_{nm}\varphi(x, y)| &\leqslant C\nu|\gamma| \quad (n \geqslant 0,\ m \geqslant 0), \\ |S_{nm}\varphi(x, y)| &\leqslant Cn^2\varepsilon \quad (n \geqslant 1,\ m \geqslant 0); \end{aligned}$$

where C is an absolute constant.

Proof. Let $\psi(x)$ and D be, respectively, the function and the closed set constructed by Lemma 1.

Choose δ such that

$$0 < \delta < \frac{\mu_1(D) - (d-c)(1-5/\nu)}{2\mu_1(D)}(b-a).$$

Define $\lambda\colon [0, 2\pi] \to \boldsymbol{R}$ as follows:

$$\lambda(y) = \begin{cases} (y-a)/\delta, & a \leqslant y \leqslant a+\delta, \\ 1, & a+\delta \leqslant y \leqslant b-\delta, \\ (b-y)/\delta, & b-\delta \leqslant y \leqslant b, \\ 0, & y \notin [a, b]. \end{cases} \tag{3.6}$$

Set $\varphi(x, y) = \psi(x)\lambda(y)$, $(x, y) \in T_2$ and $\Delta = D \times [a+\delta, b-\delta]$. Note that, as in [2], pp. 488–489, for each m and y we have

$$\Big|\int_a^b \lambda(v) D_m(v-y)\,dv\Big| \leqslant C.$$

So it is easy to see that $\varphi(x, y)$ and Δ have the desired properties (3.1′)–(3.5′).

The next lemma is a fundamental tool for our main result, and it seems interesting in itself.

LEMMA 3. *Let h be a continuous function on T_2, and let F_h be defined as in* (2.2). *Then we have*

(3.6′) *$F_h(x, y) = 0$ on the boundary of T_2 and F_h is continuous on T_2, and* $\|F\|_\infty \leqslant 4\sup\{|\int_0^\xi\int_0^\eta h(t, s)\,dtds|\colon 0 \leqslant \xi, \eta \leqslant 2\pi\}$, *and also*

(3.7′) *if the rectangular partial sums $\{S_{nm}h(x, y)\}$ (resp. the square partial sums $\{S_{mm}h(x, y)\}$) converge uniformly, then $\left\{\frac{\partial^2 S_{nm}F_h(x, y)}{\partial x\partial y}\right\}$ (resp. $\left\{\frac{\partial^2 S_{mm}F_h(x, y)}{\partial x\partial y}\right\}$) converges to $h(x, y)$ almost everywhere.*

Proof. The conclusion (3.6′) follows immediately from the definition of F_h. It remains to prove (3.7′).

From (2.1), integrating by parts and using the fact that $F_h(x, y) = 0$ on the boundary of T_2, we obtain

$$\frac{\partial^2 S_{nm}F_h(x, y)}{\partial x\partial y} = \frac{1}{\pi^2}\int_{T_2} F_h(u, v)\frac{\partial D_n(u-x)}{\partial u}\frac{\partial D_m(v-y)}{\partial v}\,dudv$$

$$= \frac{1}{\pi^2}\Big[\int_0^{2\pi}\int_0^{2\pi} h(u, v) D_n(u-x) D_m(v-y)\,dudv -$$

$$- \int_0^{2\pi} D_n(u-x)\,dg(u)\int_0^{2\pi}\int_0^{2\pi} h(u, v) D_m(v-y)\,dudv -$$

$$- \int_0^{2\pi} D_m(v-y)\,dg(v)\int_0^{2\pi}\int_0^{2\pi} h(u, v) D_n(u-x)\,dudv +$$

$$+ \int_0^{2\pi} D_n(u-x)\,dg(u)\int_0^{2\pi} D_m(v-y)\,dg(v)\int_0^{2\pi}\int_0^{2\pi} h(u, v)\,dudv\Big].$$

Note that

$$\int_0^{2\pi}\int_0^{2\pi} h(u, v) D_m(v-y)\,dudv \quad\text{and}\quad \int_0^{2\pi}\int_0^{2\pi} h(u, v) D_n(u-x)\,dudv$$

converge uniformly.

So the conclusions follow from the fact that the Fourier–Stieltjes series of g converges to 0 almost everywhere.

4. Main theorem. In this section we shall use the tools of Section 3 to prove our main result.

THEOREM 1. *For any function f, measurable and finite almost everywhere on T_2, there exists an F continuous on T_2 and such that the result of term-by-term mixed differentiation of the double Fourier series $SF(x, y)$ of F is a double trigonometric series convergent to f almost everywhere by square summation, that is, for a.e. $(x, y) \in T_2$,*

$$\lim_{m\to\infty}\frac{\partial^2 S_{mm}F(x, y)}{\partial x\partial y} = f(x, y).$$

Proof. Suppose $\sigma_n = 1/[5\pi^2 2^{n+2}]$, $\nu_n = [5\pi^2 2^{n+2}]+1$.

We divide the proof into four steps:

Step I. There exists a function α_1 continuous on T_2 and such that

$$\alpha_1(x, y) = f(x, y) \quad\text{in}\quad Y_1, \tag{4.a}$$

where $Y_1 \subset T_2$ is closed and $\mu_2(Y_1) > 4\pi^2 - \frac{1}{2}$.

Since α_1 is continuous, there exists a step function

$$\beta_1 = \sum_{s=1}^{e_1}\gamma_s\chi_{\Delta_s}, \qquad \|\beta_1 - \alpha_1\|_\infty \leqslant \sigma_2^2 \quad\text{and}\quad \|\beta_1\|_\infty \leqslant \|\alpha_1\|_\infty, \tag{4.b}$$

where $\{\Delta_s\}_{s=1}^{e_1}$ is a sequence of non-overlapping rectangles and $\bigcup_{s=1}^{e_1}\Delta_s = T_2$.

Set $\Delta_s = [c_s, d_s] \times [a_s, b_s]$ $(s = 1, 2, \ldots, e_1)$.

Let $1 = n_1 < n_2 < \ldots < n_s < \ldots < n_{e_1} < n_{e_1+1}$ be a sequence of natural numbers which we will define inductively, see (4.8′) and (4.h). Assume that

$$\varepsilon^s = \frac{1}{2^s n_s^2} \qquad (s = 1, 2, \ldots, e_1).$$

On the basis of Lemma 2, in which we suppose that

$$[c, d]\times[a, b] = [c_s, d_s]\times[a_s, b_s], \qquad \varepsilon = \varepsilon_s, \qquad \nu = \nu_1, \qquad \gamma = \gamma_s,$$

for each $s = 1, 2, \ldots, e_1$, we can find a continuous function $\varphi_s(x, y)$ and a closed set Δ_s such that

(4.1′) $\varphi_s(x, y) = \psi_s(x)\lambda_s(y)$, where $\psi_s(x)$ is a function as in Lemma 1 corresponding to $[c, d] = [c_s, d_s]$, $\varepsilon = \varepsilon_s$, $\gamma = \gamma_s$, $\nu = \nu_1$, and $\lambda_s(y)$ is a function like $\lambda(y)$ in the proof of Lemma 2 with $[a, b] = [a_s, b_s]$;

(4.2′) $$|\varphi_s(x, y)| \leqslant 2\nu_1|\gamma_s| \quad \text{in} \quad 0 \leqslant x, y \leqslant 2\pi;$$

(4.3′) $$\left|\int_0^\xi\int_0^\eta \varphi_s(x, y)\,dx\,dy\right| < 2\pi\varepsilon_s, \qquad 0 \leqslant \xi, \eta \leqslant 2\pi;$$

(4.4′) $$\varphi_s(x, y) = \gamma_s \text{ in } \Delta_s \text{ where}$$

$$\Delta_s \subset [c_s, d_s]\times[a_s, b_s] \quad \text{and} \quad \mu_2(\Delta) > (d_s - c_s)(b_s - a_s)\left(1 - \frac{5}{\nu_1}\right);$$

(4.5′) the square partial sums $\{S_{mm}\varphi_s(x, y)\}$ converge uniformly, and for each $(x, y) \in T_2$

$$|S_{mm}\varphi_s(x, y)| \leqslant C\nu_1|\gamma_s| \qquad (m \geqslant 0),$$
$$|S_{mm}\varphi_s(x, y)| \leqslant Cm^2|\varepsilon_s| \qquad (m \geqslant 1).$$

Let $F_s = F_{\varphi_s}$ as in (2.2). Then

(4.6′) $F_s(x, y) = 0$ on the boundary of T_2 and $F_s(x, y)$ is continuous on T_2, and $\|F_s\|_\infty \leqslant 8\pi\varepsilon_s$;

(4.7′) $$\lim_{m\to\infty} \frac{\partial^2 S_{mm}F_s(x, y)}{\partial x\,\partial y} = \varphi_s(x, y) \qquad \text{for a.e. } (x, y) \in T_2.$$

By (4.7′) and Egoroff's theorem, for $s = 1, 2, \ldots, e_1 - 1$, there exists a closed set $E_s \subset T_2$, $\mu_2(E_s) > 4\pi^2 - 1/s^2$ and $n_{s+1} > n_s$ such that

(4.8′) $$\left|\sum_{j=1}^{s}\left[\frac{\partial^2 S_{mm}F_j(x, y)}{\partial x\,\partial y} - \varphi_j(x, y)\right]\right| < \frac{1}{s} \quad \text{in } E_s \text{ for each } m \geqslant n_{s+1}.$$

Set

(4.c) $$P_1 = Y_1 \cap \Big(\bigcup_{s=1}^{e_1}\Delta_s\Big).$$

Then P_1 is closed and $\mu_2(P_1) > 4\pi^2 - 1$, since $\mu_2(Y_1) > 4\pi^2 - \frac{1}{2}$ and $\sum_{s=1}^{e_1}\mu_2(\Delta_s) > 4\pi^2\left(1 - \frac{5}{\nu_1}\right) > 4\pi^2 - \frac{1}{2}$.

Note that

(4.d) $$\left|f(x, y) - \sum_{s=1}^{e_1}\varphi_s(x, y)\right| \leqslant \sigma_2^2 \quad \text{in} \quad P_1.$$

Since P_1 is closed, there exists a $\delta > 0$ such that the set

$$G = \{(x, y) \in T_2 \colon d((x, y), P_1) > \delta\}$$

is open and

$$\mu_2(G) > \left(1 - \frac{1}{4\pi^2 2^3}\right)\mu_2(T_2\sim P_1).$$

We can choose finite non-overlapping closed rectangles $\Delta'_{e_1+1}, \ldots, \Delta'_l$ in G such that

(4.e) $$\sum_{s=e_1+1}^{l'}\mu_2(\Delta'_s) > \left(1 - \frac{1}{4\pi^2 2^3}\right)\mu_2(T_2\sim P_1).$$

Therefore $T_2 \sim \bigcup_{s=e_1+1}^{l'}\Delta'_s$ can be covered by a finite number of non-overlapping rectangles $\Delta'_{l'+1}, \ldots, \Delta'_l$.

Let $f_1 = f - \sum_{s=1}^{e_1}\varphi_s$. There exists a $q > 1$ such that

(4.f) $$\mu_2(f_1^{-1}[-q, q] \cap \Delta'_s) > \mu_2(\Delta'_s)(1 - \sigma_2), \qquad e_1 < s \leqslant l'.$$

Now select an integer $n_{e_1+1} > n_{e_1}$ and closed sets X_1, E_{e_1} such that

(4.g) $$\left|\int_0^{2\pi} D_m(u - x)\,dg(u)\right|, \left|\int_0^{2\pi} D_m(v - y)\,dg(v)\right| < \frac{\sigma_2^2}{q} \quad \text{in} \quad X_1$$

and

(4.h) $$\left|\sum_{s=1}^{e_1}\left[\frac{\partial^2 S_{mm}F_s(x, y)}{\partial x\,\partial y} - \varphi_s(x, y)\right]\right| < \frac{1}{e_1} \quad \text{in} \quad E_{e_1},$$

where $X_1 \subset P_1$, $m \geqslant n_{e_1+1}$ and $\mu_2(X_1) > 4\pi^2 - 1$, $\mu_2(E_{e_1}) > 4\pi^2 - 1/e_1^2$.

From (4.e), (4.f), we have

$$\mu_2\Big(\bigcup_{s=e_1+1}^{l'} f^{-1}[-q, q] \cap \Delta'_s\Big) > \mu_2(T_2\sim P_1) - \frac{1}{2^2}.$$

Also, note that

$$\mu_2\Big(f^{-1}[-\sigma_2^2,\sigma_2^2]\cap\big(T_2\sim\bigcup_{s=e_1+1}^{l'}\Delta_s'\big)\Big)\geqslant\mu_2(P_1).$$

Hence, for each $s=e_1+1,\ldots,l$, there is a continuous function α_s' on T_2 and a closed set $Y_s'\subset\Delta_s'$ such that $\alpha_s'(x,y)=0$ for each $(x,y)\notin\Delta_s'$,

(4.i) $\quad\|\alpha_s'\|_\infty\leqslant q,\ e_1<s\leqslant l';\quad \|\alpha_s'\|_\infty\leqslant\sigma_2^2,\ l_1'<s\leqslant l,$

$$\alpha_s'(x,y)=f_1(x,y)\ \text{in}\ Y_s',$$

where

$$\mu_2\Big(\bigcup_{s=e_1+1}^{l}Y_s'\Big)>4\pi^2-\frac{1}{2^2}.$$

Let $Y_2=\bigcup_{s=e_1+1}^{l}Y_s'$ and $\alpha_2=\sum_{s=e_1+1}^{l}\alpha_s'$. Then α_2 is continuous on T_2 and

(4.a′) $$\alpha_2=f_1=f-\sum_{s=1}^{e_1}\varphi_s\quad\text{in}\quad Y_2.$$

For $e_1<s\leqslant l_1$, there exists a step function

$$\beta_s'=\sum_{i=1}^{j_s}\gamma_{si}\chi_{\Delta_{si}}$$

such that $\|\beta_s'-\alpha_s'\|_\infty\leqslant\sigma_3^2$ and $\|\beta_s'\|_\infty\leqslant\|\alpha_s'\|_\infty$, where $\{\Delta_{si}\}_{i=1}^{j_s}$ is a sequence of non-overlapping rectangles $\bigcup_{i=1}^{j_s}\Delta_{si}=\Delta_s'$ and if $\bar\Delta_{si}=[c_{si},d_{si}]\times[a_{si},b_{si}]$ then

(4.j)
$$\max\{d_{si}-c_{si},b_{si}-a_{si}\}<\frac{\sigma_2^2\sin(\delta/2\sqrt{2})}{q}\quad\text{for each } e_1<s\leqslant l',\ 1\leqslant i\leqslant j_s.$$

We enumerate the rectangles Δ_{si}, $1\leqslant i\leqslant j_s$ $(e_1<s\leqslant l)$ in the following way:

For $e_1<s\leqslant l'$ we arrange them in the order $\Delta_{e_1'+1},\ldots,\Delta_{e_1'}$; and for $l'<s\leqslant l$ in the order $\Delta_{e_1'+1},\ldots,\Delta_{e_2}$.

Now we may write the sum of the step functions as β_2,

(4.b′) $$\beta_2=\sum_{s=e_1+1}^{l_1}\beta_s'=\sum_{s=e_1+1}^{e_2}\gamma_s\chi_{\Delta_s}$$

and

$$\|\beta_2-\alpha_2\|_\infty\leqslant\sigma_3^2,\qquad\|\beta_2\|_\infty\leqslant\|\alpha_2\|_\infty.$$

Step II. By the same procedure as in Step I, for the step function β_2 we can define: a sequence of integers $n_{e_1+1}<n_{e_1+2}<\ldots<n_{e_2}$, a sequence of positive numbers $\{\varepsilon_s=1/2^sn_s^2\}_{s=e_1+1}^{e_2}$, two sequences of continuous functions $\{\varphi_s\}_{s=e_1+1}^{e_2}$, $\{F_s\}_{s=e_1+1}^{e_2}$, and two sequences of closed sets $\{E_s\}_{s=e_1+1}^{e_2-1}$, $\{\Delta_s\}_{s=e_1+1}^{e_2}$ such that for each $s=e_1+1,\ldots,e_2$ properties (4.1′)–(4.8′) hold only if ν_1 is replaced by ν_2.

Now set

(4.c′) $$P_2=Y_2\cap\Big(\bigcup_{s=e_1+1}^{e_2}\Delta_s\Big).$$

Then P_2 is closed and $\mu_2(P_2)>4\pi^2-\frac{1}{2}$, since

$$\mu_2(Y_2)>4\pi^2-\frac{1}{2^2}\quad\text{and}\quad\sum_{s=e_1+1}^{e_2}\mu_2(\Delta_s)>\sum_{s=e_1+1}^{e_2}\mu_2(\Delta_s)\Big(1-\frac{5}{\nu_2}\Big)>4\pi^2-\frac{1}{2^2}.$$

(4.d′) $$\Big|f(x,y)-\sum_{s=1}^{e_2}\varphi_s(x,y)\Big|=\Big|f_1(x,y)-\sum_{s=e_1+1}^{e_2}\varphi_s(x,y)\Big|\leqslant\sigma_3^2\quad\text{in}\quad P_2.$$

We now show that for each $(x,y)\in X_1$

(4.k) $$\left|\frac{\partial^2S_{mm}F_s(x,y)}{\partial x\partial y}\right|\leqslant\frac{C}{2^2}\quad\text{whenever } e_1<s\leqslant e_2 \text{ and } m\geqslant n_{e_1+1}.$$

Write

(4.l) $$\frac{\partial^2S_{mm}F_s(x,y)}{\partial x\partial y}=\frac{1}{\pi^2}\Big[\sum_{i=1}^{4}W_s^i(x,y)\Big]$$

where

$$W_s^1(x,y)=\int_0^{2\pi}\int_0^{2\pi}\varphi_s(u,v)D_m(u-x)D_m(v-y)\,du\,dv,$$

$$W_s^2(x,y)=-\int_0^{2\pi}D_m(u-x)\,dg(u)\int_0^{2\pi}\int_0^{2\pi}\varphi_s(u,v)D_m(v-y)\,du\,dv,$$

$$W_s^3(x,y)=-\int_0^{2\pi}D_m(v-y)\,dg(v)\int_0^{2\pi}\int_0^{2\pi}\varphi_s(u,v)D_m(u-x)\,du\,dv,$$

and

$$W_s^4(x,y)=\int_0^{2\pi}D_m(u-x)\,dg(u)\int_0^{2\pi}D_m(v-y)\,dg(v)\int_0^{2\pi}\int_0^{2\pi}\varphi_s(u,v)\,du\,dv.$$

Since $d(X_1,\Delta_s)>\delta$ as $s=e_1+1,\ldots,e_1'$, if $\bar\Delta_s=[c_s,d_s]\times[a_s,b_s]$, we have the following two cases:

Case 1. $x \notin [c_s - \delta/\sqrt{2},\, d_s + \delta/\sqrt{2}]$. Then

$$|W_s^1(x, y)| \leqslant C \Big| \int_{c_s}^{d_s} \psi_s(u) D_m(u-x)\, du \Big|$$

$$\leqslant C\nu_2|\gamma_s| \frac{1}{\sin(\delta/2\sqrt{2})}(d_s - c_s) \qquad [\text{see } (3.2)]$$

$$\leqslant C\nu_2|\gamma_s| \frac{1}{\sin(\delta/2\sqrt{2})} \frac{\sigma_2^2 \sin(\delta/2\sqrt{2})}{q} \qquad [\text{see } (4.\text{j})]$$

$$\leqslant \frac{C}{2^4}, \qquad \text{since } |\gamma_s| \leqslant q \qquad [\text{see } (4.\text{i})],$$

$$|W_s^i(x, y)| \leqslant C \frac{\sigma_2^2}{q} \nu_2 |\gamma_s| \leqslant \frac{C}{2^4} \qquad (i = 2, 3) \qquad [\text{see } (4.\text{g})]$$

and

$$|W_s^4(x, y)| \leqslant \left(\frac{\sigma_2^2}{q}\right)^2 2\pi\varepsilon_s \leqslant \frac{C}{2^4}.$$

Hence in this case we have

$$\left| \frac{\partial^2 S_{mm} F_s(x, y)}{\partial x \partial y} \right| \leqslant \frac{C}{2^2}.$$

Case 2. $y \notin [a_s - \delta/\sqrt{2},\, b_s + \delta/\sqrt{2}]$. Here we can treat $|W_s^i(x, y)|$ $(i = 2, 3, 4)$ in the same way as above, and for $W_s^1(x, y)$ we have

$$|W_s^1(x, y)| = \Big| \int_0^{2\pi} \psi_s(u) D_m(u-x)\, du \Big| \Big| \int_{a_s}^{b_s} \lambda_s(v) D_m(v-y)\, dv \Big|$$

$$\leqslant C\nu_2|\gamma_s|(b_s - a_s) \frac{1}{\sin(\delta/2\sqrt{2})} \leqslant \frac{C}{2^4} \qquad [\text{see } (3.5) \text{ and } (3.\text{i})].$$

For $e_1' < s \leqslant e_2$ we have

$$|W_s^i(x, y)| \leqslant C\nu_2|\gamma_s| \leqslant C\nu_2\sigma_2^2 \leqslant C/2^4 \qquad (i = 1, 2, 3) \qquad [\text{see } (4.\text{i})]$$

and

$$|W_s^4(x, y)| \leqslant \sigma_2^4 2\pi\varepsilon_s \leqslant C/2^4.$$

Hence (4.k) is proved.

Step III. Inductively, we can obtain:

(i) a sequence of positive numbers $\{\varepsilon_s\}$,

(ii) strictly increasing sequences of natural numbers $\{n_s\}$, $\{e_k\}$,

(iii) a sequence of rectangles $\{\Delta_s\}$,

(iv) three sequences of continuous functions $\{\varphi_s\}$, $\{F_s\}$, $\{\alpha_k\}$, and a sequence of step functions $\{\beta_k\}$,

(v) a sequence of real numbers $\{\gamma_s\}$,

(vi) five sequences of closed sets $\{Y_k\}$, $\{\Lambda_s\}$, $\{P_k\}$, $\{X_k\}$ and $\{E_s\}$, such that for $e_{k-1} < s \leqslant e_k$ we have the following:

(4.1″) $\varphi_s(x, y) = \psi_s(x)\lambda_s(y)$ where $\psi_s(x)$ is a function as in Lemma 1 corresponding to $[c, d] = [c_s, d_s]$, $\varepsilon = \varepsilon_s = 1/2^s n_s^2$, $\gamma = \gamma_s$, $\nu = \nu_k$; and $\lambda_s(y)$ is a function like $\lambda(y)$ in the proof of Lemma 2 with $[a, b] = [a_s, b_s]$, $\Delta_s = [c_s, d_s] \times [a_s, b_s]$, so that $\bigcup_{s=e_{k-1}+1}^{e_k} \Delta_s = T_2$;

(4.2″) $$|\varphi_s(x, y)| \leqslant 2\nu_k|\gamma_s| \quad \text{in} \quad 0 \leqslant x, y \leqslant 2\pi;$$

(4.3″) $$\Big| \int_0^\xi \int_0^\eta \varphi_s(x, y)\, dx dy \Big| < 2\pi\varepsilon_s, \quad 0 \leqslant \xi, \eta \leqslant 2\pi;$$

(4.4″) $$\varphi_s(x, y) = \gamma_s \text{ in } \Lambda_s, \text{ where}$$

$$\Lambda_s \subset \Delta_s \quad \text{and} \quad \mu_2(\Lambda_s) > \mu_2(\Delta_s)\Big(1 - \frac{5}{\nu_k}\Big);$$

(4.5″) the square partial sums $\{S_{mm}\varphi_s(x, y)\}$ converge uniformly, and for each $(x, y) \in T_2$

$$|S_{mm}\varphi_s(x, y)| \leqslant C\nu_k|\gamma_s| \qquad (m \geqslant 0),$$

$$|S_{mm}\varphi_s(x, y)| \leqslant Cm^2|\varepsilon_s| \qquad (m \geqslant 1);$$

(4.6″) $F_s(x, y) = F_{\varphi_s}(x, y) = 0$ on the boundary of T_2 and is continuous on T_2 and $\|F_s\|_\infty \leqslant 8\pi\varepsilon_s$;

(4.7″) $$\lim_{m\to\infty} \frac{\partial^2 S_{mm} F_s(x, y)}{\partial x \partial y} = \varphi_s(x, y) \quad \text{for a.e. } (x, y) \in T_2;$$

(4.8″) $$\Big| \sum_{j=1}^{s} \Big[\frac{\partial^2 S_{mm} F_j(x, y)}{\partial x \partial y} - \varphi_j(x, y) \Big] \Big| < \frac{1}{s} \quad \text{as } m \geqslant n_{s+1} \text{ and } (x, y) \in E_s,$$

where $E_s \subset T_2$ and $\mu_2(E_s) > 4\pi^2 - 1/s^2$;

(4.a″)

$$\alpha_k = f - \sum_{s=1}^{e_{k-1}} \varphi_s \quad \text{in} \quad Y_k, \quad \text{where} \quad \mu_2(Y_k) > 4\pi^2 - 1/2^k, \text{ and } e_0 = 1;$$

(4.b″) $$\beta_k = \sum_{s=e_{k-1}+1}^{e_k} \gamma_s \chi_{\Delta_s}, \quad \|\beta_k - \alpha_k\|_\infty \leqslant \sigma_{k+1}^2, \quad \|\beta_k\|_\infty \leqslant \|\alpha_k\|_\infty;$$

(4.c″) $$P_k = Y_k \cap \Big(\bigcup_{s=e_{k-1}+1}^{e_k} \Lambda_s \Big), \quad \mu_2(P_k) > 4\pi^2 - 1/2^{k-1};$$

(4.d'') $$\left|f(x,y)-\sum_{s=1}^{e_k}\varphi_s(x,y)\right|\leqslant\sigma_{k+1}^2 \quad \text{in} \quad P_k;$$

(4.k') $X_k\subset P_k$, $\mu_2(X_k)>4\pi^2-1/2^{k-1}$ and for each $(x,y)\in X_k$
$$\left|\frac{\partial^2 S_{mm}F_{s'}(x,y)}{\partial x\partial y}\right|\leqslant\frac{C}{2^{k+1}} \quad \text{whenever } e_k<s'\leqslant e_{k+1}, \text{ and } m\geqslant n_{e_k+1}.$$

Step IV. Let $F=\sum_{s=1}^{\infty}F_s$. Then, by (4.6'') $\sum_{s=1}^{\infty}F_s$ converges uniformly on T_2, and so F is continuous on T_2. Therefore, for each m, we have
$$\frac{\partial^2 S_{mm}F(x,y)}{\partial x\partial y}=\sum_{s=1}^{\infty}\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}.$$

Suppose $\Omega=\varliminf_{n\to\infty}E_n\cap\varliminf_{n\to\infty}X_n$. Then
$$\mu_2(\Omega)=4\pi^2,$$
since $\mu_2(E_n)>4\pi^2-1/n^2$ and $\mu_2(X_n)>4\pi^2-1/2^{n-1}$.

We show that for each $(x,y)\in\Omega$,

(4.9) $$\lim_{m\to\infty}\frac{\partial^2 S_{mm}F(x,y)}{\partial x\partial y}=f(x,y).$$

First, $(x,y)\in\Omega$ implies $(x,y)\in X_n$ for n large enough, and so $(x,y)\in P_n$. Observe that (4, d'') holds and that at most one term of the sum $\sum_{s=e_{k-1}+1}^{e_k}\varphi_s(x,y)$ is non-zero. We conclude that
$$f(x,y)=\sum_{s=1}^{\infty}\varphi_s(x,y).$$

Next, for $(x,y)\in\Omega$ there exists a j_0 such that $(x,y)\in E_n\cap X_n$ whenever $n\geqslant j_0$.

Given $\varepsilon>0$, there exists a k_0 such that (i) $1/(k_0-1)<\varepsilon$, (ii) $k_0-1\geqslant j_0$, (iii) if $j>e_{k_0}$, then $\left|\sum_{s=j}^{\infty}\varphi_s(x,y)\right|<\varepsilon$.

Now
$$\left|\frac{\partial^2 S_{mm}F(x,y)}{\partial x\partial y}-f(x,y)\right|=\left|\sum_{s=1}^{\infty}\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}-\sum_{s=1}^{\infty}\varphi_s(x,y)\right|$$
$$\leqslant\left|\sum_{s=1}^{j-1}\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}-\sum_{s=1}^{j-1}\varphi_s(x,y)\right|+\left|\frac{\partial^2 S_{mm}F_j(x,y)}{\partial x\partial y}\right|+$$
$$+\left|\sum_{s=j+1}^{\infty}\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}\right|+\left|\sum_{s=j}^{\infty}\varphi_s(x,y)\right|.$$

If $m>n_{e_{k_0}}$, then there exists a j, $e_k<j\leqslant e_{k+1}$ $(k\geqslant k_0)$ such that $n_j<m\leqslant n_{j+1}$
$$\left|\frac{\partial^2 S_{mm}F(x,y)}{\partial x\partial y}-f(x,y)\right|\leqslant\frac{1}{j-1}+\frac{C}{2^{k+1}}+\varepsilon+\left|\sum_{s=j+1}^{\infty}\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}\right|,$$
(see (4.8'') and (4.k')).

Write
$$\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}=\frac{1}{\pi^2}\Big[\sum_{i=1}^{4}W_s^i(x,y)\Big] \quad \text{as in (4.1).}$$
$$|W_s^i(x,y)|\leqslant Cm^2\varepsilon_s\leqslant Cm^2\frac{1}{2^s n_s^2}\leqslant\frac{C}{2^s} \quad (i=1,3),$$
$$|W_s^2(x,y)|\leqslant C\left|\int_0^{2\pi}\varphi_s(u)\,du\right|\leqslant C\varepsilon_s\leqslant C/2^s,$$
$$|W_s^4(x,y)|\leqslant\left|\int_0^{2\pi}\int_0^{2\pi}\varphi_s(u,v)\,du\,dv\right|\leqslant C/2^s.$$
So
$$\sum_{s=j+1}^{\infty}\left|\frac{\partial^2 S_{mm}F_s(x,y)}{\partial x\partial y}\right|\leqslant\sum_{s=j+1}^{\infty}\frac{C}{2^s}\leqslant C\varepsilon.$$

Therefore we obtain the proof of (4.9), and the proof of our theorem is complete.

5. Some remarks. The basic ideas for proving the theorem in the last section are the lemmas stated in Section 3. So if we generalize these lemmas to higher dimensions, the only difficulty being that the notation becomes slightly complicated, we have the following theorem.

THEOREM 2. *For any function f defined on the k-torus $T_k=[0,2\pi]\times\times[0,2\pi]\times\ldots\times[0,2\pi]$, measurable and finite a.e., there exists a continuous function F on T_k such that*
$$\lim_{m\to\infty}\frac{\partial^k S_{mm\ldots m}F(x_1,x_2,\ldots,x_k)}{\partial x_1\partial x_2\ldots\partial x_k}=f(x_1,x_2,\ldots,x_k)$$
for a.e. $(x_1,x_2,\ldots,x_k)\in T_k$, where $S_{mm\ldots m}F(x_1,x_2,\ldots,x_k)$ is the square partial sum for the multiple Fourier series of F at $(x_1,x_2,\ldots,x_k)$.

We still do not know whether every almost everywhere finite measurable f defined on T_k can be represented by a k-fold $(k\geqslant 2)$ trigonometric series convergent to f a.e., summed by rectangles or spheres. But as a parallel to the proof of our theorems, we can prove the following theorems:

THEOREM 3. *For any function f defined on the k-torus T_k, measurable and finite a.e., there exists a continuous function F on T_k such that*

$$\lim_{\substack{m_j \geqslant m_1 \to \infty \\ (j=2,3,\ldots,k)}} \frac{\partial^k S_{m_1 m_2 \ldots m_k} F(x_1, x_2, \ldots, x_k)}{\partial x_1 \partial x_2 \ldots \partial x_k} = f(x_1, x_2, \ldots, x_k)$$

for a.e. $(x_1, x_2, \ldots, x_k) \in T_k$, *where* $S_{m_1 m_2 \ldots m_k} F(x_1, x_2, \ldots, x_k)$ *is the rectangular partial sum of the multiple Fourier series of* F *at* $(x_1, x_2, \ldots, x_k)$.

THEOREM 4. *Let f be a function as in Theorem 3. Then there exists a continuous function F on T_k such that*

$$\lim_{\substack{m_i/m_j \leqslant \lambda,\, m_i \to \infty \\ (i,j=1,2,\ldots,k)}} \frac{\partial^k S_{m_1 m_2 \ldots m_k} F(x_1, x_2, \ldots, x_k)}{\partial x_1 \partial x_2 \ldots \partial x_k} = f(x_1, x_2, \ldots, x_k)$$

for a.e. $(x_1, x_2, \ldots, x_k) \in T_k$, *where λ is a fixed positive constant not less than* 1.

Therefore, any finite a.e. measurable function defined on T_k can be represented by a multiple trigonometric series a.e. in the sense of restricted summability ([6], p. 68; [7], p. 308).

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