STUDIA MATHEMATICA, T. LXIX. (1980)

M-structure and the Banach–Stone theorem

by

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Abstract. We prove that, under certain conditions, the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ implies that $\Delta \times K$ is homeomorphic to $\Delta \times L$ for Δ in a family of completely regular spaces (K and L compact Hausdorff spaces, X a real Banach space).

1. Introduction. Let X be a real Banach space and $\mathscr{C}$ a class of nonvoid compact Hausdorff spaces. We say that X has the *Banach–Stone-property for* $\mathscr{C}$ if, for K and L in $\mathscr{C}$, the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ (the spaces of continuous X-valued functions on K resp. L) implies that K and L are homeomorphic. [3] contains a complete description of those Banach spaces X which have the Banach–Stone-property for all nonvoid compact Hausdorff spaces, provided that the centralizer of X is finite dimensional (the *M-finite Banach spaces*).

In the following we consider Banach spaces for which the norm topology and the strong operator topology are identical on the centralizer (the M-finite Banach spaces are obviously contained in this class). For these spaces X it is possible to calculate the most important M-structure properties of $C(K, X)$ (centralizer, function module representation) provided the corresponding properties of X are known (we note that these results are a special case of theorems concerning the M-structure of tensor products in [4]). Using this we get the following result: For every class of nonvoid compact Hausdorff spaces $\mathscr{C}$ and every Banach space X as described above there is a family of completely regular spaces such that the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ implies that $\Delta \times K$ is homeomorphic to $\Delta \times L$ for Δ in this family (K and L in $\mathscr{C}$). It turns out that this theorem is a strong generalization of parts of the results of the first-named author in [3].

2. Function modules. We will use the terminology of [3]. In particular, $X \cong Y$ for Banach spaces X and Y (resp. $K \cong L$ for topological spaces K and L) means that X and Y are isometrically isomorphic (resp. K and L are homeomorphic).

Function modules have been introduced by Cunningham ([5], [6]). Note, however, that we use a slightly different definition.

2.1. Definition. ([1], [5]) Let X be a real Banach space. A linear operator $T\colon X\to X$ is called *M-bounded* if there is a $\lambda>0$ such that the following condition is satisfied:

(C_λ) Tx is contained in every ball which contains $\pm\lambda x$ (all $x\in X$)

$Z(X)$, the *centralizer of X* means the collection of all M-bounded operators on X. It is known that there exists an isometric algebra isomorphism between $Z(X)$ and CK_X for a suitable compact Hausdorff space K_X.

2.2. Definition. I. A *function module* is a triple $\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big)$, where

(a) $\hat{K}$ is a nonvoid compact Hausdorff space;

(b) $(X_k)_{k\in\hat{K}}$ is a family of Banach spaces, indexed by the points of $\hat{K}$;

(c) $\hat{X}$ is a closed subspace of $\prod\limits_{k\in\hat{K}}^{\infty} X_k$ (the direct product of the X_k, provided with the supremum norm);

(d) $\hat{X}$ and the $(X_k)_{k\in\hat{K}}$ have the following properties:

(i) $k\mapsto\|x(k)\|$ is upper semicontinuous for every $x\in\hat{X}$;

(ii) $hx\in\hat{X}$ for $h\in C\hat{K}$, $x\in\hat{X}$ (pointwise multiplication);

(iii) $X_k=\{x(k)\mid x\in\hat{X}\}$ for $k\in\hat{K}$;

(iv) $\hat{K}=\{k\mid X_k\neq\{0\}\}^-$;

(v) $T\in Z(\hat{X})$ iff there is a function $h\in C\hat{K}$ such that $Tx=hx$ for every $x\in\hat{X}$.

II. Two function modules $\big(\hat{K}_1, (X^1_k)_{k\in\hat{K}_1}, \hat{X}_1\big)$, $\big(\hat{K}_2, (X^2_k)_{k\in\hat{K}_2}, \hat{X}_2\big)$ are called *equivalent* if there are

(i) an isometric isomorphism $J\colon\hat{X}_1\to\hat{X}_2$,

(ii) a homeomorphism $t\colon\hat{K}_1\to\hat{K}_2$,

(iii) isometric isomorphisms $u_k\colon X^1_k\to X^2_{t(k)}$ such that $u_k\big(x(k)\big)=(Jx)\big(t(k)\big)$ for every $x\in\hat{X}_1$, $k\in\hat{K}_1$.

III. A function module $\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big)$ is called a *function module representation* of a real Banach space X if $X\cong\hat{X}$.

2.3. Theorem.

(a) *Two function modules* $\big(\hat{K}_1, (X^1_k)_{k\in\hat{K}_1}, \hat{X}_1\big)$ *and* $\big(\hat{K}_2, (X^2_k)_{k\in\hat{K}_2}, \hat{X}_2\big)$ *are equivalent iff* $\hat{X}_1\cong\hat{X}_2$.

Consequently, every two function module representations of a real Banach space X are equivalent.

(b) *Every real Banach space X has a function module representation.*

Proof. (a) Let $\big(\hat{K}_1, (X^1_k)_{k\in\hat{K}_1}, \hat{X}_1\big), \big(\hat{K}_2, (X^2_k)_{k\in\hat{K}_2}, \hat{X}_2\big)$ be function modules, $J\colon\hat{X}_1\to\hat{X}_2$ an isometric isomorphism. $T\mapsto JTJ^{-1}$ is an isometric algebra isomorphism from $Z(\hat{X}_1)$ onto $Z(\hat{X}_2)$ which induces an isometric algebra isomorphism j from $C\hat{K}_1$ onto $C\hat{K}_2$ (2.2.I. (d) implies that $C\hat{K}\cong Z(\hat{X})$). Note that $J(hx)=j(h)Jx$ for $h\in C\hat{K}_1$, $x\in\hat{X}_1$. By the classical Banach–Stone theorem there exists a homeomorphism $t\colon\hat{K}_1\to\hat{K}_2$ such that $j(h)\big(t(k_1)\big)=h(k_1)$ for every $h\in C\hat{K}_1$, $k_1\in\hat{K}_1$. Let $k_0\in\hat{K}_1$ be fixed. We define $u_{k_0}\colon X^1_{k_0}\to X^2_{t(k_0)}$ by

$$u_{k_0}\big(x(k_0)\big):=(Jx)\big(t(k_0)\big)\qquad(\text{all } x\in\hat{X}_1).$$

u_{k_0} is well-defined, for let $x\in\hat{X}_1$ be an element such that $x(k_0)=0$ (we will show that $Jx\big(t(k_0)\big)=0$ in this case).

For $\varepsilon>0$, there is a neighbourhood U of k_0 such that $\|x(k)\|<\varepsilon$ for $k\in U$. Choose a function $h\in C\hat{K}_1$ such that $h(k_0)=1=\|h\|$, $h(k)=0$ for $k\notin U$. It follows that $\big\|Jx\big(t(k_0)\big)\big\|=\big\|j(h)Jx\big(t(k_0)\big)\big\|\leqslant\|j(h)Jx\|=\|J(hx)\|\leqslant\|hx\|\leqslant\varepsilon$. Similarly it can be shown that $\big\|u_{k_0}\big(x(k_0)\big)\big\|=\|x(k_0)\|$. u_{k_0} is defined on all of $X^1_{k_0}$ by 2.2.I.(d)(iii) so that $(u_k)_{k\in\hat{K}_1}$ is a family of isometric isomorphisms (the surjectivity of the u_k follows immediately from the surjectivity of J and 2.2.I.(d)(iii)). By definition, J, t, $(u_k)_{k\in\hat{K}_1}$ define an equivalence. The reverse implication is valid by definition.

(b) The following construction is due to Cunningham. Let K_X be a compact Hausdorff space with $Z(X)\cong CK_X$ and $J_X\colon Z(X)\to CK_X$ an isometric algebra isomorphism. For $x\in X$ we define $|x|\colon K_X\to\boldsymbol{R}$ (the norm resolution of x) by

$$|x|(k):=\inf\{\|Tx\|\mid T\in Z(X),\ J_X(T)(k)=1,\ J_X(T)\geqslant 0\}.$$

It is easy to see that $x\mapsto|x|(k)$ is a seminorm on X for every $k\in K_X$. Let X_k be the associated Banach space, i.e. the completion of $X/\{x\mid |x|(k)=0\}$, the quotient provided with the norm $\|[x]\|:=|x|(k)$ (all $k\in K_X$). Define $\omega\colon X\to\prod\limits_{k\in K_X}^{\infty} X_k$ by

$$\omega(x)(k):=\text{the equivalence class of } x \text{ in } X_k.$$

Then $\big(K_X, (X_k)_{k\in K_X}, \omega(X)\big)$ is a function module representation of X. $X\cong\omega(X)$ and 2.2.I.(a), (b), (c) are easily verified. (d)(i) is an immediate consequence of the definitions, (iii) is proved in [5] p. 621, (iv) follows from the surjectivity of J_X. Note that $T\mapsto\omega T\omega^{-1}$ is an isometric isomorphism between $Z(X)$ and $Z\big(\omega(X)\big)$ and that $\omega(Tx)=J_X(T)\omega(x)$ for all $T\in Z(X)$ $(x\in X)$ by which we get (d)(ii) and (v).

2.4. Corollary. *Let X and Y be Banach spaces with function module representations* $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ *resp.* $(\hat{L}, (Y_l)_{l\in\hat{L}}, \hat{Y})$. *If* $X \cong Y$, *then* $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ *and* $(\hat{L}, (Y_l)_{l\in\hat{L}}, \hat{Y})$ *are equivalent.*

2.5. Examples. (a) Let L be a locally compact Hausdorff space.

$$X := C_0 L = \{f \mid f\colon L \to \boldsymbol{R}, f \text{ continuous}, f \text{ vanishes at infinity}\}.$$

In this case the centralizer of X consists exactly of the mappings $M_h\colon f \mapsto hf$ with $h\colon L \to \boldsymbol{R}$ bounded and continuous. Then $(\beta L, (X_k)_{k\in\beta L}, X)$ is a function module representation of X, where $X_k = \boldsymbol{R}$ resp. $X_k = \{0\}$ according to whether $k \in L$ or $k \in \beta L \setminus L$.

(b) Let X be M-finite with canonical M-decomposition $X = M_1^{n_1} \oplus_\infty \cdots \oplus_\infty M_r^{n_r}$. The operators in $Z(X)$ are just the operators of the form

$$(x_1^1, \ldots, x_1^{n_1}, x_2^1, \ldots, x_2^{n_2} \ldots, x_r^1, \ldots, x_r^{n_r}) \mapsto (a_1^1 x_1^1, \ldots, a_1^{n_1} x_1^{n_1}, \ldots, a_r^1 x_r^1, \ldots, a_r^{n_r} x_r^{n_r})$$

with $a_\varrho^i \in \boldsymbol{R}$ for $\varrho = 1, \ldots, r$, $i = 1, \ldots, n_\varrho$. With $\hat{K} := \bigcup_{\varrho=1}^{r} \{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}$ and $X_{(\varrho,i)} := M_\varrho$, $(\hat{K}, (X_k)_{k\in\hat{K}}, X)$ is a function module representation of X.

(c) Let s be a fixed real number, $0 < |s| \leqslant 1$, $X_s := \{f \mid f \in C[0,1], f(0) = sf(1)\}$. In this case $Z(X_s)$ consists exactly of the operators of the form $f \mapsto hf$ with $h \in C[0,1], h(0) = h(1)$. Therefore $Z(X_s) \cong \{h \mid h \in C[0,1],\ h(0) = h(1)\} \cong C\{e^{2\pi i t} \mid t \in [0,1]\}$. Let $\hat{K} = \{e^{2\pi i t} \mid t \in [0,1]\}$, $X_{e^{2\pi i t}} = \boldsymbol{R}$ (all $t \in [0,1]$) and $\hat{X} = \{e^{2\pi i t} \mapsto f(t) \mid f \in X_s, t \in\,]0,1]\}$. Then $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ is a function module representation of X.

Similarly one can treat the case of more general G-spaces which are defined by a finite number of relations:

$$X = \{f \mid f \in CK,\ f(x_i) = s_i f(y_i),\ i = 1, \ldots, n\}$$

(if some s_i are equal to zero one has to consider Stone–Čech compactifications of suitable quotients of K).

2.6. Definition. (cf. [4]) Let X be a real Banach space. A *centralizer norming system* (cns) is a finite family $x_1, \ldots, x_n$ in X such that there exists a number $r > 0$ for which $\max\{\|Tx_1\|, \ldots, \|Tx_n\|\} \geqslant r\|T\|$ (all $T \in Z(X)$). Obviously, X has a cns iff the norm topology and the strong operator topology coincide on $Z(X)$. It is easy to see that, for function modules $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$, $\hat{X}$ has a cns iff there are $x_1, \ldots, x_n \in \hat{X}$ such that $\inf_{k\in K} \max_{i=1,\ldots,n} \|x_i(k)\| > 0$.

2.7. Examples. (a) $C_0 L$ has a cns iff L is compact.

(b) Every M-finite Banach space (in particular every reflexive space [ex. 2 in [3], Section 1)] has a cns.

(c) X_s has a cns for $0 < |s| \leqslant 1$.

2.8. Theorem. *Let* $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ *be a function module,* K *a nonvoid compact Hausdorff space. We define*

$$X_{(p,k)} := X_k \quad \textit{for} \quad (p, k) \in K \times \hat{K},$$

$$\hat{X}_K := \operatorname{lin}(\{h \otimes x \mid h \in CK, x \in \hat{X}\})^- \quad ((h \otimes x)(p, k) := h(p)x(k)).$$

If $\hat{X}$ *has a cns, then*

(a) $(K \times \hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K)$ *is a function module,*

(b) $\hat{X}_K \cong C(K, \hat{X})$.

Proof. (a) We have to verify the properties listed in 2.2.I.

(a), (b), (c), (d)(iii),(iv) are trivially satisfied.

(d) (i), (ii) are easy consequences of the well-known identity

$$C(K \times \hat{K}) = (\operatorname{lin}\{h_1 \otimes h_2 \mid h_1 \in CK,\ h_2 \in C\hat{K}\})^- \cong CK \hat{\otimes}_\varepsilon C\hat{K}.$$

It remains to prove that (d)(v) is true. Clearly, for $h \in C(K \times \hat{K})$, $z \mapsto hz$ is in the centralizer of $\hat{X}_K$ (this mapping is well-defined by (d)(ii), and it satisfies the defining property for M-bounded operators with $\lambda = \|h\|$). Conversely, let $T\colon \hat{X}_K \to \hat{X}_K$ be M-bounded. For $p \in K$, define $T_p\colon \hat{X} \to \hat{X}$ by $(T_p x)(k) := (T(\mathbf{1} \otimes x))(p, k)$ (note that $T_p x \in \hat{X}$ since $\hat{X}$ is complete and $z(p, \cdot) \in \hat{X}$ for $z \in \operatorname{lin}\{h \otimes x \mid h \in CK, x \in \hat{X}\}$). It is easy to see that every T_p is M-bounded (in fact, T_p satisfies 2.1 for the same values as T does). Consequently there exists, for $p \in K$, a function $h_p \in C\hat{K}$ such that $T_p x = h_p x$ for every $x \in \hat{X}$. Define $h_T\colon K \times \hat{K} \to \boldsymbol{R}$ by $h_T(p, k) := h_p(k)$ We will show that h_T is continuous and that $Tz = h_T z$ for $z \in \hat{X}_K$. Let $z \in \hat{X}_K$, $(p_0, k_0) \in K \times \hat{K}$, $z(p_0, k_0) = 0$. This implies that $(Tz)(p_0, k_0) = 0$ (this follows from the semicontinuity of $(p, k) \mapsto \|z(p, k)\|$ and the fact that T commutes with the operators $z \mapsto hz$: choose, for $\varepsilon > 0$, a neighbourhood U of (p_0, k_0) such that $\|z(p, k)\| \leqslant \varepsilon$ for $(p, k) \in U$ and a function $h \in C(K \times \hat{K})$ such that $h(p_0, k_0) = 1 = \|h\|$, $h(p, k) = 0$ for $(p, k) \notin U$. Then we have

$$\|(Tz)(p_0, k_0)\| = \|(hTz)(p_0, k_0)\| \leqslant \|hTz\| = \|Thz\| \leqslant \|T\|\,\|hz\| \leqslant \|T\|\varepsilon).$$

Thus, for $h \in CK$, $x \in \hat{X}$,

$$T(h \otimes x)(p_0, k_0) = h(p_0) h_{p_0}(k_0) x(k_0)$$

for every $(p_0, k_0) \in K \times \hat{K}$ (since $[h \otimes x - h(p_0)(1 \times x)](p_0, k_0) = 0$). But this means $T(h \otimes x) = h_T(h \otimes x)$ for every $h \in CK$, $x \in \hat{X}$, so that the linearity and continuity of T imply that $Tz = h_T z$ for every $z \in \hat{X}_K$ (note that h is bounded by $\|T\|$). For the proof of the continuity of h_T we first note that it is sufficient to show that $p \mapsto T_p$ is a continuous mapping from K to $Z(\hat{X})$ (since $Z(\hat{X}) \cong C\hat{K}$, $C(K, C\hat{K}) \cong C(K \times \hat{K})$). Let $x_1, \ldots, x_n \in \hat{X}$, $r > 0$ as in 2.1. For every $z \in \hat{X}_K$, $p \mapsto z(p, \cdot)$ is continuous from K to $\hat{X}$ (this follows at once from the definition of $\hat{X}_K$). In particular, for $p_0 \in K$, $\varepsilon > 0$, there is a neighbourhood U of p_0 such that

$$\|T(1 \otimes x_i)(p_0, \cdot) - T(1 \otimes x_i)(p, \cdot)\| \leqslant \varepsilon \quad \text{for} \quad i = 1, \ldots, n, \ p \in U.$$

This yields $\|T_p - T_{p_0}\| \leqslant \frac{1}{r}\varepsilon$ for every $p \in U$ so that $p \mapsto T_p$ is continuous at p_0.

(b) This is an immediate consequence of $C(K, \hat{X}) \cong CK \otimes_\varepsilon \hat{X}$ (explicitly: $C(K, \hat{X}) = (\mathrm{lin}\{h \otimes x \mid h \in CK, x \in \hat{X}\})^-$, whereby in this case $(h \otimes x)(p) := h(p)x$).

2.9. Corollary. *Let X be a real Banach space having a cns. If $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ is a function module representation of X, then $(K \times \hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K)$ is a function module representation of $C(K, X)$.*

3. Function module properties.

3.1. Definition. A *function module property* P is a rule which defines for every function module $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ a subset $P(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ of $\hat{K}$ such that the following property holds:

If $(\hat{K}_1, (X_k^1)_{k\in\hat{K}_1}, \hat{X}_1)$ and $(\hat{K}_2, (X_k^2)_{k\in\hat{K}_2}, \hat{X}_2)$ are equivalent function modules, the equivalence being defined by J, t, $(u_k)_{k\in\hat{K}_1}$ (cf. 2.2), then

$$P(\hat{K}_2, (X_k^2)_{k\in\hat{K}_2}, \hat{X}_2) = t[P(\hat{K}_1, (X_k^1)_{k\in\hat{K}_1}, \hat{X}_1)].$$

For a function module $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ let $\mathscr{P}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ be the collection of all sets $P(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ for arbitrary function module properties P.

3.2. Note. It is easy to see that $\mathscr{P}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ is always a complete Boolean algebra of subsets of $\hat{K}$.

3.3. Examples. (a) For function modules $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ we define

$$P_{X_0}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) := \{k \mid k \in \hat{K}, \ X_k \cong X_0\} \quad (X_0 \text{ a fixed Banach space}),$$

$$P_{\mathrm{continuous}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) := \{k \mid k \in \hat{K}, l \mapsto \|x(l)\| \text{ is continuous at } k \text{ for every } x \in \hat{X}\},$$

$$P^a(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) := \{k \mid k \in \hat{K}, \ a \varlimsup_{\substack{l\to k\\ l\neq k}} \|x(l)\| = \varliminf_{\substack{l\to k\\ l\neq k}} \|x(l)\| \text{ for every } x \in \hat{X}\} \quad (a \in [0, 1] \text{ a fixed number}),$$

$$P_{\mathrm{countable}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) := \{k \mid k \in \hat{K}, \text{ the neighbourhoodfilter of } k \text{ has a countable basis}\}.$$

In all these cases we have defined function module properties. This follows immediately from 2.2.II.

(b) (cf. Ex. 2.5.(b)) Let $n_1, \ldots, n_r$ be natural numbers, $M_1, \ldots, M_r$ real nonzero Banach spaces, $\hat{K} := \bigcup_{\varrho=1}^{r} \{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}$, $X_{(\varrho,i)} := M_\varrho$ ($\varrho = 1, \ldots, r$; $i = 1, \ldots, n_\varrho$), $\hat{X} := \prod_{(\varrho,i)\in\hat{K}}^{\infty} X_{(\varrho,i)}$. In this case $\mathscr{P}(\hat{K}, (X_{(\varrho,i)})_{(\varrho,i)\in\hat{K}}, \hat{X})$ is the Boolean algebra generated by the sets $\{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}$, $\varrho = 1, \ldots, r$.

3.4. Definition. Let $\mathscr{C}$ be a class of nonvoid compact Hausdorff spaces. A function module property P is called *$\mathscr{C}$-hereditary* if

$$P(K \times \hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K) = K \times P(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$$

for every $K \in \mathscr{C}$ and every function module $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ for which $\hat{X}$ has a cns (cf. 2.9).

$$\mathscr{P}_{\mathscr{C}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) := \{P(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) \mid P \text{ is a } \mathscr{C}\text{-hereditary function module property}\}$$

is a Boolean sub-algebra of $\mathscr{P}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$.

3.5. Examples. P_{X_0}, $P_{\mathrm{continuous}}$, P^a are $\mathscr{C}$-hereditary with $\mathscr{C}$ = "the class of all nonvoid compact Hausdorff spaces".

$P_{\mathrm{countable}}$ is $\mathscr{C}$-hereditary with $\mathscr{C}$ = "the class of all nonvoid first countable compact Hausdorff spaces".

This follows at once from the definition of $(K \times \hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K)$.

Note that $\mathscr{P}_{\mathscr{C}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) = \mathscr{P}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ in the case of Example 3.3(b) ($\mathscr{C}$ an arbitrary class of nonvoid compact Hausdorff spaces).

4. A theorem of the Banach–Stone type.

4.1. Theorem. *Let X be a real Banach space having a cns, $\mathscr{C}$ a class of nonvoid compact Hausdorff spaces, $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ a function module representation of X. Then, for K, L in $\mathscr{C}$, the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ implies that $\Delta \times K \cong \Delta \times L$ for every $\Delta \in \mathscr{P}_{\mathscr{C}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$.*

In particular, X has the Banach–Stone property for $\mathscr{C}$ if $\mathscr{P}_{\mathscr{C}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ contains an element consisting of a single point.

Proof. Let P be a $\mathscr{C}$-hereditary function module property, $J\colon C(K, X) \to C(L, X)$ an isometric isomorphism. By 2.9 and 2.4, $\big(K\times\hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K\big)$ and $\big(L\times\hat{K}, (X_{(q,k)})_{(q,k)\in L\times\hat{K}}, \hat{X}_L\big)$ are equivalent so that, by definition, $K\times P\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big) = P\big(K\times\hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K\big) \cong P\big(L\times\hat{K}, (X_{(q,k)})_{(q,k)\in L\times\hat{K}}, \hat{X}_L\big) = L\times P\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big)$.

4.2. Examples. (a) $\hat{K}$ is always contained in $\mathscr{P}_{\mathscr{C}}\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big)$ so that we always may conclude that $\hat{K}\times K \cong \hat{K}\times L$ (this has first been noted, for M-finite Banach spaces, in [2], th. 4.1).

(b) For an M-finite Banach space $X = M_1^{n_1}\oplus_\infty \ldots \oplus_\infty M_r^{n_r}$, $C(K, X) \cong C(L, X)$ implies that $\{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}\times K \cong \{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}\times L$ $(\varrho = 1, \ldots, r)$ (cf. 3.5). This is just the assertion of Theorem 4.4 in [2] for the compact case.

(c) X_s has the Banach–Stone property for the class of all nonvoid compact Hausdorff spaces for every $|s|\in\,]0, 1[$. Note that $P^a\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big)$ consists of a single point for every function module representation $\big(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}\big)$ of X_s and $a = s$.

Analogously one can treat the case of arbitrary G-spaces (cf. 2.5).

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Received March 16, 1978 (1409)

STUDIA MATHEMATICA, T. LXIX. (1980)

The localization principle for double Fourier series

by

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Abstract. Definitive results are obtained for localization by square and rectangular sums for the Fourier series of functions of 2 variables. For this purpose functions of Λ bounded variation, ΛBV, and in particular, harmonic bounded variation, HBV, are defined for functions of 2 variables. It is shown that if $f\in$ HBV, then localization holds for rectangular sums. However, if $\Lambda\mathrm{BV} \not\subset \mathrm{HBV}$, there is an $f\in\Lambda$BV for which localization fails even for square sums.

This constrasts of course with the 1 variable case, where localization holds for all summable functions. It differs as well from the case $n\geqslant 3$ where previously obtained definitive results are in a Sobolev space framework.

The Riemann localization principle for periodic functions of one variable asserts that if an integrable function vanishes identically on an open interval, then the partial sums of its Fourier series converge uniformly to zero on any compact subset of that interval. For functions of several variables, strong additional assumptions are required in order that the principle of localization may hold. Indeed, if we consider convergence of the rectangular partial sums, $S_n = S_{n_1,n_2,\ldots,n_m}$, of the Fourier series of an integrable function defined on $[-\pi, \pi]^m$, $m > 1$, localization may fail even if the function is continuous. Here, by convergence of $\{S_n\}$, we mean the existence of $\lim S_n$ as $\min\{n_i\}\to\infty$.

There are various alternatives one may pursue to obtain localization theorems. Among these are:

(1) to require that $f = 0$ on a larger set,

(2) to make additional global requirements on f,

(3) to replace convergence by other limiting procedures,

(4) to replace rectangular partial sums by other sums of terms of the Fourier series,

and various combinations of these [14], Chap. 17.

For example, if we require that $f = 0$ not only in the given interval, but on every line in the direction of a coordinate axis and intersecting

* Supported in part by NSF grant MCS76–21660.
** Supported in part by NSF grant MCS77–00840.