Proof. Let P be a $\mathscr{C}$-hereditary function module property, $J\colon C(K, X)\to C(L, X)$ an isometric isomorphism. By 2.9 and 2.4, $(K\times\hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K)$ and $(L\times\hat{K}, (X_{(q,k)})_{(q,k)\in L\times\hat{K}}, \hat{X}_L)$ are equivalent so that, by definition, $K\times P(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X}) = P(K\times\hat{K}, (X_{(p,k)})_{(p,k)\in K\times\hat{K}}, \hat{X}_K)\cong P(L\times\hat{K}, (X_{(q,k)})_{(q,k)\in L\times\hat{K}}, \hat{X}_L) = L\times P(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$.

4.2. Examples. (a) $\hat{K}$ is always contained in $\mathscr{P}_{\mathscr{C}}(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ so that we always may conclude that $\hat{K}\times K\cong\hat{K}\times L$ (this has first been noted, for M-finite Banach spaces, in [2], th. 4.1).

(b) For an M-finite Banach space $X = M_1^{n_1}\oplus_\infty\cdots\oplus_\infty M_r^{n_r}$, $C(K, X)\cong C(L, X)$ implies that $\{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}\times K\cong\{(\varrho, 1), \ldots, (\varrho, n_\varrho)\}\times L$ ($\varrho = 1, \ldots, r$) (cf. 3.5). This is just the assertion of Theorem 4.4 in [2] for the compact case.

(c) X_s has the Banach–Stone property for the class of all nonvoid compact Hausdorff spaces for every $|s|\in\,]0, 1[$. Note that $P^a(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ consists of a single point for every function module representation $(\hat{K}, (X_k)_{k\in\hat{K}}, \hat{X})$ of X_s and $a = s$.

Analogously one can treat the case of arbitrary G-spaces (cf. 2.5).

References

[1] E. M. Alfsen, E. G. Effros, *Structure in real Banach spaces II*, Ann. of Math. 96 (1972), pp. 129–173.

[2] E. Behrends, *An application of M-structure to theorems of the Banach–Stone type*, in: North Holland Comp., Notas de mathematica (1977).

[3] — *On the Banach–Stone theorem*, Math. Ann. 233 (1978), pp. 261–272.

[4] — *The centralizer of the tensor product of Banach spaces*, Pacific J. Math. 81 (1979), pp. 291–301.

[5] F. Cunningham, *M-structure in Banach spaces*, Proc. Cambridge Philos. Soc. 63 (1967), pp. 613–629.

[6] F. Cunningham, N. Roy, *Extreme functionals on an upper semicontinuous function space*, Proc. Amer. Math. Soc. 42 (1974), pp. 461–465.

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The localization principle for double Fourier series

by

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Abstract. Definitive results are obtained for localization by square and rectangular sums for the Fourier series of functions of 2 variables. For this purpose functions of Λ bounded variation, ΛBV, and in particular, harmonic bounded variation, HBV, are defined for functions of 2 variables. It is shown that if $f\in$ HBV, then localization holds for rectangular sums. However, if $\Lambda\mathrm{BV}\not\subset\mathrm{HBV}$, there is an $f\in\Lambda$BV for which localization fails even for square sums.

This constrasts of course with the 1 variable case, where localization holds for all summable functions. It differs as well from the case $n\geqslant 3$ where previously obtained definitive results are in a Sobolev space framework.

The Riemann localization principle for periodic functions of one variable asserts that if an integrable function vanishes identically on an open interval, then the partial sums of its Fourier series converge uniformly to zero on any compact subset of that interval. For functions of several variables, strong additional assumptions are required in order that the principle of localization may hold. Indeed, if we consider convergence of the rectangular partial sums, $S_n = S_{n_1,n_2,\ldots,n_m}$, of the Fourier series of an integrable function defined on $[-\pi, \pi]^m$, $m > 1$, localization may fail even if the function is continuous. Here, by convergence of $\{S_n\}$, we mean the existence of $\lim S_n$ as $\min\{n_i\}\to\infty$.

There are various alternatives one may pursue to obtain localization theorems. Among these are:

(1) to require that $f = 0$ on a larger set,

(2) to make additional global requirements on f,

(3) to replace convergence by other limiting procedures,

(4) to replace rectangular partial sums by other sums of terms of the Fourier series,

and various combinations of these [14], Chap. 17.

For example, if we require that $f = 0$ not only in the given interval, but on every line in the direction of a coordinate axis and intersecting

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the interval, a set which is called a *cross-neighborhood*, then localization holds for the rectangular partial sums in the original interval. On the other hand, if we replace *convergence* of the rectangular partial sums by *(C, 1)-summability*, then localization holds for bounded functions.

More recently, Igari [6] has shown that, for *square partial sums* (S_n with $n_1 = n_2 = \ldots = n_m$), the localization principle fails in the class of continuous functions, while for *square* $(C, 1)$ *means*, the localization principle holds in L^p if $p \geq m-1$, but fails if $p < m-1$.

Tonelli [10] introduced a notion of bounded variation which yielded a pointwise convergence theorem for functions of two variables. This theorem implies a *pointwise localization principle*: If a function of this class vanishes on an open set, then the rectangular partial sums of its Fourier series converge to zero at each point of the set. He obtains the usual (uniform) localization principle only with additional restrictive hypotheses which are unnecessary in view of the results of this paper.

Cesari [1] improved on Tonelli's results with his introduction of the notion of *generalized bounded variation* which guarantees localization and a.e. convergence and has had many other fruitful applications. This notion may be expressed as follows [5], [9] for a function f defined on an interval in $\boldsymbol{R}^m$:

f is measurable and, corresponding to each coordinate direction, there is an equivalent function which is of bounded variation on a.e. line in that direction, and whose total variation on those lines is an integrable function of the remaining $(m-1)$ variables.

This class contains the Sobolev space W_1^1, which suggested to Goffman and Liu [3] an approach to obtaining a localization principle for $m > 2$. They have shown that the localization principle for square partial sums holds in $\boldsymbol{R}^m$ for $f \in W_p^1$ if $p \geq m-1$, but fails to hold if $p < m-1$. Liu [7] extended this to rectangular sums, but in this case localization holds if $p > m-1$ and fails if $p \leq m-1$.

Another way in which one may attempt to generalize the Cesari–Tonelli result is by enlarging the class of functions. A method which suggests itself is that of replacing ordinary bounded variation (on the lines in the coordinate directions) with other notions of bounded variation.

Let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive real numbers such that $\sum 1/\lambda_n$ diverges. A function g defined on $[a, b] \subset \boldsymbol{R}^1$ is said to be of *Λ-bounded variation* (ΛBV) if $\sum_1^\infty |g(a_n) - g(b_n)|/\lambda_n$ converges for every sequence of non-overlapping intervals $[a_n, b_n] \subset [a, b]$. The supremum of such sums is the *total Λ-variation* of g on $[a, b]$. If $\lambda_n = n$, we say that f is of *harmonic bounded variation* (HBV). The convergence and summability properties of Fourier series of functions of these classes have been studied recently [11], [12], [13]. In particular, we note that the conclusion of the Dirichlet–Jordan theorem holds for functions in HBV, but not for larger ΛBV classes. The class HBV contains properly the various classes of functions of generalized bounded variation introduced by Wiener, L. C. Young, Garsia and Sawyer, and Salem, for which a generalized Dirichlet–Jordan theorem was known to hold.

In § 1 of this paper we shall show that if the Cesari variation is generalized by replacing ordinary variation by harmonic variation, then, in $\boldsymbol{R}^2$, the localization principle for rectangular partial sums holds for integrable functions of that class.

In proving this result we will make certain measurability and continuity assumptions. In § 2 we will show that these assumptions cause no loss in generality and that the measurable functions corresponding to each coordinate direction can be chosen so that the total variation on lines in that direction is minimized. We define the class $V_{\Lambda,a}^p$ to consist of those $f \in L^p$, $p \geq 1$, on an interval in $\boldsymbol{R}^m$, to which there correspond equivalent f_i, $i = 1, \ldots, m$. such that, on almost every line in the ith coordinate direction, $f_i \in \Lambda$BV and V_i, the total Λ-variation of f_i on these lines, is in L^a, $a \geq 1$, as a function of the remaining $(m-1)$-variables. For $m > 2$, we assume further that each f_i, restricted to almost any line in the ith coordinate direction, has, at each point, a value between the upper and lower limits at that point. We also give an equivalent definition of the space which avoids the introduction of the m equivalent functions.

In § 3 we show that a norm may be defined on $V_{\Lambda,a}^p$ so that it is a Banach space with the property that $f \in V_{\Lambda,a}^p$ and $f = g$ a.e. if and only if $g \in V_{\Lambda,a}^p$ and $\|f-g\| = 0$. We are then able to show that, in $\boldsymbol{R}^2$, the localization theorem of § 1 is best possible in the sense that if ΛBV is not contained in HBV, then there is a function in $V_{\Lambda,1}^1$ for which the localization principle for *square* partial sums does not hold.

§ 1. If $f(x, y)$ is a function on $I = [-\pi, \pi]^2$, let $V_x(f(x, y), [a, b])$ denote the total harmonic variation of f as a function of x (with y fixed) on the interval $[a, b]$; $v_y(\ldots)$ is similarly defined.

Our positive theorem on localization may be stated as follows:

THEOREM 1. *Let f be an integrable function on I and let there exist functions g and h, equivalent to f, such that $V_x(g(x, y), [-\pi, \pi])$ and $V_y(h(x, y), [-\pi, \pi])$ are finite a.e. and are integrable functions of y and x, respectively. Then the localization principle for rectangular partial sums holds for f.*

In the proof of this result we will make several assumptions that are essential to our argument. In § 2 it will be made clear that these assumptions result in no loss of generality.

We assume that

(i) for a.e. y, $g(x, y)$ is right continuous as a function of x in $[-\pi, \pi)$ and left continuous at $x = \pi$,

(ii) $V_x(g(x, y), [a, b])$ is a measurable function of y for any interval $[a, b] \subset [-\pi, \pi]$,

and the analogous statements for the function h.

Proof of Theorem 1. Let $D_m(s) = \sin(m+\frac{1}{2})s/\sin\frac{1}{2}s$. We must show that if $f(x, y) = 0$ for $(x, y) \in [-\delta, \delta]^2$, where $0 < \delta < \pi$, then for any $\delta' \in (0, \delta)$,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s, y+t) D_m(s) D_n(t)\, ds\, dt \to 0$$

uniformly for $(x, y) \in [-\delta', \delta']^2$. The integral in question can be written as the sum of integrals over the following domains:

$$\begin{Bmatrix} \pi \geqslant |x| \geqslant \delta \\ \pi \geqslant |y| \geqslant \delta \end{Bmatrix}, \quad \begin{Bmatrix} \pi \geqslant |x| \geqslant \delta \\ \delta \geqslant |y| \geqslant a \end{Bmatrix}, \quad \begin{Bmatrix} \delta \geqslant |x| \geqslant a \\ \pi \geqslant |y| \geqslant \delta \end{Bmatrix},$$

$$\begin{Bmatrix} \pi \geqslant |x| \geqslant \delta \\ a \geqslant |y| \geqslant 0 \end{Bmatrix}, \quad \begin{Bmatrix} a \geqslant |x| \geqslant 0 \\ \pi \geqslant |y| \geqslant \delta \end{Bmatrix}$$

where $a \in (0, \delta)$. For a fixed a, we may show that the integral over E, the union of the first three domains, tends uniformly to zero as a consequence of much the same arguments as are used to establish the localization principle for cross neighborhoods.

We have

$$\iint_E f(x+s, y+t) D_m(s) D_n(t)\, ds\, dt$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s, y+t)\gamma(s, t) \sin(m+\tfrac{1}{2})s \sin(n+\tfrac{1}{2})t\, ds\, dt$$

where $\gamma(s, t)$ is of period 2π in each variable and, in $[-\pi, \pi]^2$, equals $(\sin\frac{1}{2}s \sin\frac{1}{2}t)^{-1}$ in E and zero elsewhere. Proceeding in a manner similar to that employed in the one variable case [14], Chap. 2, Lemma 6.4, we can show that this integral tends uniformly to zero as $m, n \to \infty$.

Our proof of the theorem will be complete when we show that a, M and N can be chosen to make the integrals over the remaining portions arbitrarily small *uniformly* for $(x, y) \in [-\delta', \delta']^2$ when $m > M$ and $n > N$.

We will discuss only the integral over $[0, a] \times [\delta, \pi]$. The remainder can be divided into integrals over seven similar domains and can be estimated in the same way.

For $x < x' < x+a$, write

$$\int_{\delta}^{\pi}\int_{0}^{a} f(x+s, y+t) D_m(s) D_n(t)\, ds\, dt$$

$$= \int_{\delta}^{\pi} D_n(t)\, dt \int_0^a [f(x+s, y+t) - f(x', y+t)] D_m(s)\, ds + \int_0^a D_m(s)\, ds \times$$

$$\times \int_{\delta}^{\pi} f(x', y+t) D_n(t)\, dt = P + Q.$$

Noting that $g(x, y) = f(x, y)$ a.e., we will write

$$\psi = g(x+s, y+t) - g(x', y+t),$$

indicating the dependence of ψ on any of its variables only to the extent necessary to clarify the argument. Then, for $a < 3\pi/(m+1/2)$, we have

$$\left|\int_0^a \psi D_m(s)\, ds\right| \leqslant 3\pi^2 \sup_{0<s<a} |\psi| \leqslant 3\pi^2 V_s\big(g(x+s, y+t), [0, a]\big)$$

for a.e. t. For $a \geqslant 3\pi/(m+1/2)$, let k be the greatest integer such that $(2k+1)\pi/(m+1/2) \leqslant a$. Then

$$\int_0^a \psi D_m(s)\, ds = \int_0^{\pi/(m+1/2)} + \int_{(2k+1)\pi/(m+1/2)}^{a} + \int_{\pi/(m+1/2)}^{(2k+1)\pi/(m+1/2)} = I_1 + I_2 + I_3.$$

For a.e. t,

$$|I_1| \leqslant \pi^2 \sup_{0<s<\pi/(m+1/2)} |\psi| \leqslant \pi^2 V_s\big(g(x+s, y+t), [0, a]\big)$$

and, similarly,

$$|I_2| \leqslant 2\pi^2 V_s\big(g(x+s, y+t), [0, a]\big).$$

Now

$$|I_3| = \frac{1}{m+\frac{1}{2}} \sum_{i=1}^{2k} \int_{i\pi}^{(i+1)\pi} \psi\left(\frac{s}{m+\frac{1}{2}}\right) \frac{\sin s}{\sin s/(2m+1)}\, ds$$

$$= \frac{1}{m+\frac{1}{2}} \sum_{i=1}^{2k} \int_0^{\pi} \psi\left(\frac{s+i\pi}{m+\frac{1}{2}}\right) \frac{\sin(s+i\pi)}{\sin\left(\dfrac{s+i\pi}{2m+1}\right)}\, ds$$

$$= \frac{1}{m+\frac{1}{2}} \sum_{i=1}^{2k} (-1)^i \int_0^{\pi} \psi\left(\frac{s+i\pi}{m+\frac{1}{2}}\right) \frac{\sin s}{\sin\left(\dfrac{s+i\pi}{2m+1}\right)}\, ds$$

$$= \frac{1}{m+\frac{1}{2}}\sum_{i=1}^{k}\int_{0}^{\pi}\left[\frac{\psi\left(\frac{s+2i\pi}{m+\frac{1}{2}}\right)}{\sin\left(\frac{s+2i\pi}{2m+1}\right)}-\frac{\psi\left(\frac{s+(2i-1)\pi}{m+\frac{1}{2}}\right)}{\sin\left(\frac{s+(2i-1)\pi}{m+\frac{1}{2}}\right)}\right]\sin s\,ds$$

$$= \frac{1}{m+\frac{1}{2}}\sum_{i=1}^{k}\int_{0}^{\pi}\frac{\psi\left(\frac{s+2i\pi}{m+\frac{1}{2}}\right)-\psi\left(\frac{s+(2i-1)\pi}{m+\frac{1}{2}}\right)}{\sin\left(\frac{s+(2i-1)\pi}{2m+1}\right)}\sin s\,ds+$$

$$+\frac{1}{m+\frac{1}{2}}\sum_{i=1}^{k}\int_{0}^{\pi}\psi\left(\frac{s+2i\pi}{m+\frac{1}{2}}\right)\left[\frac{\sin\left(\frac{s+(2i-1)\pi}{2m+1}\right)-\sin\left(\frac{s+2i\pi}{2m+1}\right)}{\sin\left(\frac{s+(2i-1)\pi}{2m+1}\right)\sin\left(\frac{s+2i\pi}{2m+1}\right)}\right]\sin s\,ds$$

$$= I_3'+I_3''.$$

The absolute value of the term in brackets in I_3'' is

$$\leqslant \frac{\pi/(2m+1)}{\left(\frac{2}{\pi}\right)^2\left(\frac{(2i-1)\pi}{2m+1}\right)^2} = \frac{\pi}{4}\cdot\frac{2m+1}{(2i-1)^2}.$$

Hence, for a.e. t,

$$|I_3''| \leqslant \frac{1}{m+1}\sum_{i=1}^{k}\sup_{0<s<a}|\psi|\cdot\frac{\pi}{4}\cdot\frac{2m+1}{(2i-1)^2}\leqslant \pi V_s\big(g(x+t,\,y+s),\,[0,\,a]\big).$$

If $\operatorname{osc}_s(\gamma,[a,b])$ denotes the oscillation of γ, as a function of s, over the interval $[a,b]$, we have

$$\left|\int_{0}^{\pi}\left[\psi\left(\frac{s+2i\pi}{m+\frac{1}{2}}\right)-\psi\left(\frac{s+(2i-1)\pi}{m+\frac{1}{2}}\right)\right]\frac{\sin s}{\sin\left(\frac{s+(2i-1)\pi}{2m+1}\right)}\,ds\right|$$

$$\leqslant \operatorname{osc}_s\left(g(x+s,\,y+t),\left[\frac{(2i-1)\pi}{m+\frac{1}{2}},\,\frac{(2i+1)\pi}{m+\frac{1}{2}}\right]\right)\int_{0}^{\pi}\frac{\sin s}{\frac{2}{\pi}\,\frac{(2i-1)\pi}{2m+1}}\,ds$$

$$= \frac{2m+1}{2i-1}\operatorname{osc}_s(\ldots).$$

Thus

$$|I_3'| \leqslant 2\sum_{i=1}^{k}\operatorname{osc}_s\left(g(x+s,\,y+t),\left[\frac{(2i-1)\pi}{m+\frac{1}{2}},\,\frac{(2i+1)\pi}{m+\frac{1}{2}}\right]\right)\Big/(2i-1)$$
$$\leqslant 2V_s\big(g(x+s,\,y+t),\,[0,\,a]\big)$$

for a.e. t. We have now shown that there is a constant C such that for a.e. t, for every $x'\in(x,\,x+a)$, and for every m,

$$\left|\int_{0}^{a}\psi D_m(s)\,ds\right| < CV_s\big(g(x+s,\,y+t),\,[0,\,a]\big).$$

Then

$$|P|\leqslant C\int_{\delta}^{\pi}V_s\big(g(x+s,\,y+t),\,[0,\,a]\big)\,|D_n(t)|\,dt$$

$$\leqslant \frac{\pi}{\delta}C\int_{-\pi}^{\pi}V_s\big(g(x+s,\,y+t),\,[0,\,a]\big)\,dt$$

$$= \frac{\pi}{\delta}C\int_{-\pi}^{\pi}V_s\big(g(x+s,\,t),\,[0,\,a]\big)\,dt,$$

since g is periodic. Now the integrand in this last expression is a non-negative measurable function of t bounded above by the integrable function $V_s\big(g(s,t),\,[-\pi,\,\pi]\big)$. For a.e. t, $V_s\big(g(x+s,\,t),\,[0,\,a]\big)\searrow 0$ as $a\searrow 0$ [13], Theorem 3. Hence, given $\varepsilon>0$, there is an $a(\varepsilon,x)>0$ such that

$$|P|<\varepsilon$$

for every y, m, and n, if $0<a\leqslant a(\varepsilon,x)$ and $x<x'<x+a$.

If $x\leqslant\bar{x}<\bar{x}'<x+a/2$ and a is such that $\bar{x}'-\bar{x}<a\leqslant a/2$, then

$$\left|\int_{\delta}^{\pi}D_n(t)\,dt\int_{0}^{a}[g(\bar{x}+s,\,y+t)-g(\bar{x}',\,y+t)]D_m(s)\,ds\right|$$

$$\leqslant\frac{\pi}{\delta}C\int_{-\pi}^{\pi}V_s\big(g(\bar{x}+s,\,t),\,[0,\,a]\big)\,dt\leqslant\frac{\pi}{\delta}\int_{-\pi}^{\pi}V_s\big(g(x+s,\,t),\,[0,\,a]\big)\,dt<\varepsilon$$

if $0<a\leqslant a(\varepsilon,x)$. For every $x\in[-\delta',\,\delta']$ choose a positive $a(x)\leqslant\frac{1}{2}\min\{a(\varepsilon,x),\,\delta-\delta'\}$. Then the collection of intervals, $\{(x,\,x+a(x))\mid x\in(-\delta',\,\delta')\}$ has a finite subset which, with $[-\delta',\,-\delta'+a(-\delta')]$ and $[\delta',\,\delta'+a(\delta')]$ is a covering of $[-\delta',\,\delta']$. Choose $\Delta>0$ to be less than one-half the minimum distance between *distinct* endpoints of intervals in this covering and so that $2\delta'/\Delta$ is an integer. Consider the intervals

$[-\delta'+i\Delta, -\delta'+(i+1)\Delta]$, $i = 0, 1, \ldots, 2\delta'/\Delta$. In each interval choose $x_i > -\delta'+i\Delta$ such that $f(x_i, y) \in L^1(y)$. If $x \in [-\delta', \delta']$, then for some i, $0 < x_i - x < 2\Delta$. There is then an interval of the covering, whose end-points we will denote by ξ and $\xi + a(\xi)$, such that $\xi \leqslant x < x_i < \xi + a(\xi)$. Then

$$\left|\int_\delta^\pi D_n(t)\int_0^{2\Delta}[g(x+s, y+t)-g(x_i, y+t)]D_m(s)\,ds\right| \leqslant \varepsilon$$

since $\xi \leqslant x < x_i < \xi + a(\xi) \leqslant \xi + \dfrac{a(\varepsilon, \xi)}{2}$ and $x_i - x < 2\Delta \leqslant a(\varepsilon, \xi)/2$.

Choosing $a = 2\Delta$, we see that, for any $(x, y) \in [-\delta', \delta']^2$,

$$|P| < \varepsilon$$

if x' is chosen to be the appropriate x_i.

Choosing $x' = x_i$, we have

$$|Q| = \left|\int_0^a D_m(s)\,ds\int_\delta^\pi f(x_i, y+t)D_n(t)\,dt\right| \leqslant \pi^2\left|\int_\delta^\pi f(x_i, y+t)D_n(t)\,dt\right| < \varepsilon$$

if $n > N_i(\varepsilon)$ for $-\delta' \leqslant y \leqslant \delta'$ by the Riemann localization principle for functions of one variable. Setting

$$N(\varepsilon) = \max\{N_i(\varepsilon) |\ i = 0, \ldots, 2\delta'/\Delta\},$$

we have

$$|Q| < \varepsilon$$

for $(x, y) \in [-\delta', \delta']^2$ if $n > N(\varepsilon)$. ∎

§ 2. Our goals in this section are to show that:

(i) the assumptions (i) and (ii) of § 1 cause no loss in generality,

(ii) we may dispense with the functions f_i in the definition of $V^p_{\Lambda,a}$.

If we were to assume f to be continuous, the delicate considerations of this section would become unnecessary. However, the functions we consider may be highly discontinuous. Indeed, there are functions of generalized bounded variation in the sense of Cesari all of whose equivalent functions are nowhere continuous.

The following two lemmas concerning functions in ΛBV have recently been established [8]. Note that functions in ΛBV have only simple discontinuities. A function f, with a simple discontinuity at x, is said to have an *internal saltus* at x if $\liminf_{t\to x} f(t) \leqslant f(x) \leqslant \limsup_{t\to x} f(t)$.

LEMMA 1. *If a: $[a, b] \to \mathbf{R}^1$ is in ΛBV and has an internal saltus at each point of discontinuity, then the total Λ-variation of a is independent of the values of a at the points of discontinuity.*

LEMMA 2. *If a: $[a, b] \to \mathbf{R}^1$ is in ΛBV and $\bar{a}(x) = a(x)$ at each point of continuity of a and has an internal saltus at each point of discontinuity, then $\bar{a}$ is in ΛBV and its total Λ-variation is less than that of a.*

Let I be an interval in $\mathbf{R}^m$ and let $l_i = l_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ be the (non-empty) intersection of I and the line through $(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_m)$ in the direction of the ith coordinate axis. If f: $I \to \mathbf{R}^1$, $V_i(f)$ will denote the total Λ-variation of f on an l_i, so that $V_i(f)$ is a function of the $(m-1)$ variables x_j, $j \neq i$. We may set $V_i(f) = \infty$ for those l_i on which f is not in ΛBV. When f restricted to l_i is continuous at x, we will say that f is *linearly continuous* at x on l_i or has x as a point of *linear continuity*. If f is right or left continuous at x, or discontinuous at x, analogous terminology is employed.

A function f is in *class* V_Λ if it is measurable and there exist *corresponding functions* f_i, $i = 1, \ldots, m$, equivalent to f, such that $V_i(f_i) < \infty$ a.e. for every i.

A function f is in *class* $V^p_{\Lambda,a}$, $p \geqslant 1$, $a \geqslant 1$, if $f \in V_\Lambda$, $f \in L^p$, $V_i(f_i) \in L^a$ for every i, and if, for $m > 2$, each f_i has an internal saltus at its points of linear discontinuity on each l_i for which $V_i(f_i) < \infty$.

Suppose now that f is a measurable function on I and $V_i(f) < \infty$ a.e. Let L be the union of the l_i on which $V_i(f) = \infty$. Restricting f to an l_i in L^C, f has only simple discontinuities, each of which is an approximate discontinuity of f as a function of x_i. It is known that the set of points in I at which a measurable function is not approximately continuous in a particular variable is a set of measure zero [2], 3. 13 (4). Letting A be the set of points at which f is not approximately continuous as a function of x_i, we see that the points at which f is discontinuous as a function of x_i are contained in $A \cup L$. We have, as a consequence, the following result.

LEMMA 3. *If f is measurable and $V_i(f) < \infty$ a.e., then f is continuous a.e. as a function of x_i.*

The next result is essential to what follows, but its proof is quite technical and so we defer it to the end of this section. It must be emphasized that this result is for $m = 2$. At the end of its proof we will make some remarks on the case $m > 2$.

Let f_i, $i = 1, 2$, be the corresponding functions of $f \in V_\Lambda$. We will say that a real number a has *property Δ_i* if one of the following is satisfied:

(i) on almost every l_i, f_i is linearly right continuous at $x_i = a$,

(ii) on almost every l_i, f_i is linearly left continuous at $x_i = a$.

Let $V_i(f, [a, b])$ denote the total Λ-variation of a function f on a segment $a \leqslant x_i \leqslant b$ of an l_i.

THEOREM 2.1. *Let $f \in V_\Lambda$ and f_i, $i = 1, 2$, be the corresponding functions. If an f_i has an internal saltus at each of its points of linear disconti-*

nuity on the l_i for which $V_i(f_i) < \infty$, then the function $V_i(f_i, [a, b])$ is measurable for every $[a, b]$ such that both a and b have property Δ_i.

If we make assumption (i) of § 1, then this theorem shows that assumption (ii) is satisfied.

The continuity condition in this theorem may not seem natural, but the following simple example shows that it is.

Let E be a non-measurable subset of $[0, 1]$. Define f on $[0, 1]\times[0, 1]$ by

$$f(x, y) = \begin{cases} 1 & \text{if } \quad x > \tfrac{1}{2} \text{ or, if } x = \tfrac{1}{2} \text{ and } y = E, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is measurable, satisfies the saltus condition, and its total Λ-variation is $1/\lambda_1$ on each horizontal line and 0 on all but one vertical line, but $V_1(f, [0, 1/2])$ is not measurable.

We are now able to prove a theorem which justifies our assumption (i) of § 1.

THEOREM 2.2. *Let $f \in V_\Lambda$ and let f_i, $i = 1, \ldots, m$, be the corresponding functions. If $\bar{f}_i(x) = f_i(x)$ at each point of linear continuity on the l_i for which $V_i(f_i) < \infty$ and has an internal saltus at the points of linear discontinuity on those l_i, then $\bar{f}_i$, $i = 1, \ldots, m$, are also corresponding functions for f.*

If $f \in V^p_{\Lambda,\alpha}$, then $\|V_i(\bar{f}_i)\|_\alpha$, $\alpha = 1, \ldots, m$, are minimal values for these norms over the class of corresponding functions.

Proof of Theorem 2.2. From Lemma 3 we see that $\{x|\ \bar{f}_i(x) = f_i(x)\}$ has measure zero, so that $\bar{f}_i$ is equivalent to f. If l_i is such that $V_i(f_i) < \infty$, then by Lemma 2, $V_i(\bar{f}_i) \leqslant V_i(f_i)$. Hence the $\bar{f}_i$, $i = 1, \ldots, m$, are corresponding functions.

Let $f \in V^p_{\Lambda,\alpha}$ and g_i, $i = 1, \ldots, m$ be another set of corresponding functions. For any i, on almost every l_i, $V_i(f_i) < \infty$, $V_i(g_i) < \infty$, and $f_i(x) = f(x) = g_i(x)$ a.e. (x_i). It is elementary then that, for those l_i, f_i and g_i have the same set of non-removable linear discontinuities and, therefore, that $\bar{f}_i(x) = g_i(x)$ at each point of linear continuity of g_i. Then, by Lemma 2, $V_i(\bar{f}_i) \leqslant V_i(g_i)$ for those l_i. If $m = 2$, by Theorem 2.1, $V_i(\bar{f}_i)$ is measurable. For $m > 2$, $V_i(\bar{f}_i) = V_i(f_i) = V_i(g_i)$ a.e. Hence $\|V_i(\bar{f}_i)\|_\alpha \leqslant \|V_i(g_i)\|_\alpha$. ∎

The following lemma provides the basis for our simplification of the definitions of V_Λ and $V^p_{\Lambda,\alpha}$.

LEMMA 4. *If f is measurable and, for almost every l_i, there is a function in ΛBV on l_i, with total Λ-variation V, which is equivalent to f on l_i, then there exists an $\bar{f} = f$ a.e. such that $V_i(\bar{f}) < V < \infty$ a.e. and, on the l_i for which $V_i(\bar{f}) < \infty$, $\bar{f}$ has an internal saltus at each of its points of linear discontinuity.*

Proof. Without loss of generality, we may set $i = 1$ and write $x = x_1$, $y = (x_2, \ldots, x_m)$.

There is a set Z of $(m-1)$-dimensional measure zero such that, for $y \notin Z$, there is a function $g_y(x) \in \Lambda\mathrm{BV}$ and $f(x, y) = g_y(x)$ a.e. (x). We may assume that $V_1(f) = \infty$ if $y \in Z$.

Let $[a, b]$ be the projection of I on the X-axis. For $y \in Z$, let $\bar{f}(x, y) = f(x, y)$. For $y \notin Z$, let

$$\bar{f}(x, y) = \begin{cases} g_y(a+), & x = a, \\ g_y(b-), & x = b, \\ \tfrac{1}{2}[g(x-, y)+g(x+, y)], & x \in (a, b). \end{cases}$$

If $I_h(x) = [x-h, x+h]\cap[a, b]$, $h > 0$, we see that, for $y \notin Z$ and $I_h = I_h(x)$,

$$\bar{f}(x, y) = \lim_{h\to 0}\frac{1}{|I_h|}\int_{I_h} g_y(t)\,dt = \lim_{h\to 0}\frac{1}{|I_h|}\int_{I_h} f(t, y)\,dt.$$

Suppose $y \notin Z$ and f is approximately continuous in x at (x_0, y). Then there is a set $E \subset \boldsymbol{R}^1$, having density one at x_0, and such that

$$g_y(t) = f(x_0, y)+o(1)$$

as $t \to x_0$ within E. Since $g_y(t)$ is bounded, as $h \to 0$ we have, for $I_h = I_h(x_0)$,

$$\frac{1}{|I_h|}\int_{I_h} g_y(t)\,dt = \frac{1}{|I_h|}\int_{I_h\cap E}\ldots+\frac{1}{|I_h|}\int_{I_h\cap E^c}\ldots$$

$$= \big(f(x_0, y)+o(1)\big)\frac{|I_h\cap E|}{|I_h|}+O(1)\frac{|I_h\cap E^c|}{|I_h|} = f(x_0, y_0)+o(1).$$

Thus $\bar{f}(x_0, y) = f(x_0, y)$. Since f is a.e. approximately continuous in x and $[a, b]\times Z$ has measure zero, we have $\bar{f} = f$ a.e. For $y \notin Z$, we note that $\bar{f}(x, y) = g_y(x)$ at each point of continuity of $g_y(x)$ and is either continuous or has an internal saltus elsewhere. By Lemma 2, $V_1(\bar{f}) \leqslant V(g_y)$, the total Λ-variation of g_y, for $y \notin Z$. ∎

The next theorem shows that we can dispense with the notion of "corresponding functions" in the definition of V_Λ and $V^p_{\Lambda,\alpha}$.

THEOREM 2.3. *$f \in V_\Lambda$ if and only if f is measurable and, for each i, $i = 1, \ldots, m$, on almost every l_i, f is equivalent to a function in ΛBV.*

$f \in V^p_{\Lambda,\alpha}$ if and only if $f \in L^p$ and, for each i, $i = 1, \ldots, m$, on almost every l_i, f is equivalent to a function in ΛBV (having, if $m > 2$, an internal saltus at points of discontinuity) whose total Λ-variation, as a function of the remaining $(m-1)$-variables, is in L^α.

Proof of Theorem 2.3. It is clear that functions in V_Λ or $V^p_{\Lambda,a}$ have the indicated properties. We will show the converse.

If, on almost every l_i, f is equivalent to a function in ΛBV, then Lemma 4 asserts the existence of $\bar{f}_i = f$ a.e. in I and such that $V_i(\bar{f}_i) < \infty$ a.e. Thus $\bar{f}_i$, $i = 1, \ldots, m$, are corresponding functions and $f \in V_\Lambda$.

If we assume also that the total Λ-variation V_i of the function in ΛBV, which is equivalent to f on l_i, is in L^a, then by Lemma 4, if $m = 2$, $V_i(\bar{f}_i) \leqslant V_i$ a.e. and Theorem 2.1 implies that $V_i(\bar{f}_i)$ is measurable. Hence $\|V_i(\bar{f}_i)\|_a \leqslant \|V_i\|_a$. If $m > 2$, then Lemma 1 implies that $V_i(\bar{f}_i) = V_i$ and, therefore, $V_i(\bar{f}_i) \in L^a$. ∎

We turn now to the proof of Theorem 2.1.

A function f will be said to be *almost continuous* at a point if there is a set of measure zero such that the restriction of f to the complement of that set is continuous at the point. *Almost lower* (and *upper*) *semicontinuity* are similarly defined.

LEMMA 5. *If f is measurable, equivalent to g in I, and $V_i(g) < \infty$ a.e., then f is almost continuous in x_i a.e.*

Proof. By Lemma 3, g is continuous in x_i a.e., say on the set A. Let $B = \{x \mid f(x) = g(x)\}$. Let C be the union of the l_i on which $f(x) = g(x)$ a.e. If $x \in A \cap B \cap C$, a set of full measure, then f is almost continuous at x in the variable x_i.

COROLLARY. *A function in V_Λ is almost continuous in each x_i a.e.*

LEMMA 6. *Let $f \in V_\Lambda$ on $I = I_1 \times I_2$ with corresponding functions f_i, $i = 1, 2$. If an f_i has an internal saltus at each of its points of linear discontinuity on the l_i for which $V_i(f_i) < \infty$, then for $[a, b] \subset I_i$ and such that, on a.e. l_i, f_i is right continuous at a and left continuous at b, the function $V_i(f_i, [a, b])$ is almost lower semicontinuous a.e.*

Proof. Let us choose $i = 1$ and write $(x, y) = (x_1, x_2)$ and $g(x, y) = f_1(x_1, x_2)$. Let $A \subset I_2$ be the set of $y_0 \in I_2$ such that, as a function of x, $g(x, y_0) \in \Lambda$BV, is right continuous at $x = a$ and left continuous at $x = b$, and, as a function of y, $g(x, y)$ is almost continuous at y_0 for almost every x. By the corollary to Lemma 5, we see that A has full measure. Fix $y_0 \in A$. Then

$$V(y_0) = V_1\big(g(x, y_0), [a, b]\big) < \infty.$$

For any $\varepsilon > 0$, according to Lemma 1 there are non-overlapping intervals $[a_j, b_j] \subset [a, b]$, $j = 1, \ldots, n$, such that $g(x, y_0)$ is continuous in x and almost continuous in y at each (a_j, y_0), (b_j, y_0) and

$$\sum_1^n |g(a_j, y_0) - g(b_j, y_0)|/\lambda_j > V(y_0) - \varepsilon/2.$$

There is a δ and a set $Z \subset I_2$ of measure zero such that, if $B = \{y \mid V_1(g) < \infty\}$, then $y \in B - Z$ and $|y - y_0| < \delta$ imply

$$V(y) \geqslant \sum_1^n |g(a_j, y) - g(b_j, y)|/\lambda_j \geqslant V(y_0) - \varepsilon.$$

Thus V is almost lower semicontinuous at y_0.

Lemma 7. *If a function φ is defined on an interval in $\boldsymbol{R}^1$ and is almost lower semicontinuous a.e., then φ is measurable.*

Proof. Suppose φ is almost lower semicontinuous except on Z, a set of measure zero. Let $\{I_n\}$ be the intervals with rational endpoints. Suppose $k \in \boldsymbol{R}^1$ and $x_0 \notin Z$ such that $\varphi(x_0) > k$. Then for some $n = n(k, x_0)$, there is a set Z_n of measure zero such that $\varphi(x) > k$ for $x \in I_n - Z_n$. Thus, if $\mathcal{N} = \{n(k, x) \mid \varphi(x) > k\}$ and $Z' = \{x \mid \varphi(x) > k\} \cap Z$, then

$$\{x \mid \varphi(x) > k\} = Z' \cup \Big(\bigcup_{n \in \mathcal{N}} (I_n - Z_n)\Big).$$

which is measurable.

Proof of Theorem 2.1. We use the notation of the proof of Lemma 6 and consider $g = f_1$. If we suppose that for almost every y for which $V_1(g) < \infty$, $g(x, y)$, as a function of x, is right continuous at (a, y) and left continuous at (b, y), then by Lemma 6, $V_1(g, [a, b])$ is almost lower semicontinuous a.e. and is, therefore, by Lemma 7, also measurable. There are three other cases. We will consider only one; the others may be treated analogously.

Suppose that, for almost every y for which $V_1(g) < \infty$, $g(x, y)$, as a function of x, is right continuous at (a, y) and at (b, y).

By Lemma 3, the set A of x for which $g(x, y)$ is continuous in x for almost every y has full measure. Choose $b_n \in A$, $n = 1, 2, \ldots$, so that $b_n \searrow b$. For each n, $V_n = V_1(g, [a, b_n])$ is a measurable function of y. Let $Z_n \subset I_2$ be the set of measure zero consisting of those y for which $g(x, y)$ is not continuous in x at $x = b_n$. Let $Z_0 \subset I_2$ be the set of measure zero consisting of those y for which $g(x, y)$ is not right continuous at $x = b$. Then if $y \notin \bigcup_0^\infty Z_n$, we have $V_n(y) \searrow V(y)$; hence V is measurable. ∎

A result analogous to Theorem 2.1, but for $m > 2$, would require a condition implying that the function is almost continuous as a function of $(m-1)$ variables on almost all coordinate hyperplanes. Such a condition has appeared in [3] and [4] in work related to absolutely continuous functions, i.e., to functions in Sobolev spaces W^1_p. Functions in W^1_p are $(m-1)$ continuous if $p > m-1$. In our present setting it is not known what further conditions on f imply almost $(m-1)$ continuity.

§ 3. In the first section of this paper we showed that the localization principle for rectangular partial sums hold for the class $V^1_{H,1}$ in $\boldsymbol{R}^2$ $(H = \{n\})$. The following theorem asserts that this result is best possible in a certain sense.

THEOREM 3.1. *If ΛBV is not contained in* HBV, *then the localization property for square partial sums does not hold for the class $V^1_{\Lambda,1}$ in $\boldsymbol{R}^2$.*

In proving this result we use the fact that $V^1_{\Lambda,1}$, with a suitable norm, is a Banach space. This is a consequence of our final result.

For $f \in V^p_{\Lambda,\alpha}$, choose corresponding functions f_i, $i = 1, \ldots, m$, to be right continuous for $-\pi \leqslant x_i < \pi$ and left continuous at $x_i = \pi$ on almost every l_i. Let

$$\|f\|_{\Lambda,p,\alpha} = \|f\|_p + \sum_{i=1}^{m} \|V_i(f_i)\|_\alpha .$$

Note that if $f \in V^p_{\Lambda,\alpha}$ with f_i so chosen and $g = f$ a.e., then the f_i are suitable corresponding functions for g, $g \in V^p_{\Lambda,\alpha}$, and $\|f-g\|_{\Lambda,p,\alpha} = 0$. Conversely, if $g \in V^p_{\Lambda,\alpha}$ and $\|f-g\|_{\Lambda,p,\alpha} = 0$, then $f = g$ a.e. and $f \in V^p_{\Lambda,\alpha}$. We see then that we may consider the elements of $V^p_{\Lambda,\alpha}$ to be equivalence classes of a.e. equal functions.

Our final result is the following

THEOREM 3.2. *$V^p_{\Lambda,\alpha}$ is a Banach space with norm $\|\cdot\|_{\Lambda,p,\alpha}$.*

We turn now to the proof of Theorem 3.1 assuming Theorem 3.2 to be valid. The following lemma has been established recently [8]. We include its proof for the sake of completeness.

LEMMA 8. *If ΛBV is not contained in* HBV, *then* $\sum_1^n 1/k \neq O\left(\sum_1^n 1/\lambda_k\right)$.

Proof. The hypothesis implies that there is a real sequence $\{a_n\}$, $a_n \searrow 0$, such that $\sum_1^\infty a_n/\lambda_n < \infty$, but $\sum_1^\infty a_n/n = \infty$. If we suppose that

$$\sum_1^n 1/k \leqslant C \sum_1^n 1/\lambda_k$$

for all n, then

$$\sum_1^n a_n/k = \sum_1^{n-1} \Big(\sum_1^k 1/j\Big)(a_k - a_{k+1}) + \Big(\sum_1^n 1/k\Big) a_n$$
$$\leqslant C \sum_1^{n-1} \Big(\sum_1^k 1/\lambda_j\Big)(a_k - a_{k+1}) + C\Big(\sum_1^n 1/\lambda_k\Big) a_n = C \sum_1^n a_k/\lambda_k,$$

implying $\sum_1^\infty a_n/n < \infty$, contrary to our hypothesis. ■

Proof of Theorem 3.1. Let I denote the square $[-\pi, \pi]^2$, I' the square $[-\pi/2, \pi/2]^2$, and h the characteristic function of $I - I'$.

If $S_{nn}(X, f)$ is the nth square partial sum of the Fourier series of f at $X = (x, y)$, define a linear operator on $V^1_{\Lambda,1}$ by

$$T_n(g) = S_{nn}(0, h\cdot g).$$

If the localization principle for square partial sums holds in $V^1_{\Lambda,1}$, then $T_n(g) \to 0$ as $n \to \infty$ for every $g \in V^1_{\Lambda,1}$, implying that $\{T_n\}$ is a bounded sequence of operators. Let

$$f_n = \operatorname{signum} D_n(x) D_n(y).$$

Then

$$\|f_n\|_{H,1,1} = 4\pi^2 + 8\pi \sum_1^{2n} 1/k,$$

$$\|f_n\|_{\Lambda,1,1} = 4\pi^2 + 8\pi \sum_1^{2n} 1/\lambda_k .$$

Let $g_n = f_n / \|f_n\|_{\Lambda,1,1}$. There is a $C > 0$ such that

$$T_n(f) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} h(x, y)\, |D_n(x) D_n(y)|\, dx\, dy \geqslant C \log n$$

for large n. Thus

$$T_n(g_n) \geqslant \frac{C \log n}{4\pi^2 + 8\pi \sum_1^{2n} 1/\lambda_n} \neq O(1)$$

as $n \to \infty$, in view of Lemma 8. Thus $\{T_n\}$ is not a bounded sequence. ■

Proof of Theorem 3.2. We will show only that $V^p_{\Lambda,\alpha}$ is complete. It is clear that $V^p_{\Lambda,\alpha}$ is a linear space and verification of the properties of the norm is straightforward. We treat the case $m = 2$, since it is entirely typical. The facts used concerning the space ΛBV are known [11], [13].

Consider $\{f_n\}$ Cauchy convergent in $V^p_{\Lambda,\alpha}$ on $I = [-\pi, \pi]^2$. Let $g_n = (f_n)_1$, $h_n = (f_n)_2$ be the corresponding functions, chosen as above.

If f is an L^p-limit of $\{f_n\}$, let $\{F_n\}$ be a subsequence of $\{f_n\}$ converging a.e. to f and $\{G_n\}$ the corresponding subsequence of $\{g_n\}$.

For every $k \in Z^+$ there is a least n_k in Z^+ such that

$$\int V_1^\alpha(G_n - G_m)\, dy < 1/3^k \quad \text{if} \quad m, n \geqslant n_k .$$

Thus

$$\int V_1^\alpha(G_{n_{k+1}} - G_{n_k}) dy < 1/3^k \quad \text{for} \quad k = 1, 2, \ldots$$

Let

$$E_k = \{y \mid V_1(G_{n_{k+1}} - G_{n_k}) > 2^{-k/\alpha}\}.$$

Then

$$|E_k| < (2/3)^k$$

and

$$(*) \qquad \sum_1^\infty V_1(G_{n_{k+1}} - G_{n_k})$$

converges of $y \notin \bigcup_{k>N} E_k$ for some N. Since $\left|\bigcup_{k>N} E_k\right| \to 0$ as $N \to \infty$, the series $(*)$ converges for almost every y. Hence, for $k > j$ and almost every y, as $k, j \to \infty$,

$$(**) \qquad V_1(G_{n_k} - G_{n_j}) \leqslant V_1(G_{n_{j+1}} - G_{n_j}) + \ldots + V_1(G_{n_k} - G_{n_{k-1}}) \to 0.$$

Now $F_n \to f$ a.e. implies $G_n \to f$ a.e. Thus, for almost every y, $\{G_{n_k}\}$ converges for some x, which, with $(**)$, implies that $\{G_{n_k}\}$ converges in ΛBV as a function of x, for almost every y, to a ΛBV function $g(x, y)$. For such y, g is right continuous for $-\pi \leqslant x < \pi$ and left continuous at $x = \pi$, since convergence in ΛBV implies uniform convergence. Since, for almost every y,

$$\lim_{k\to\infty} V_1(g_n - G_{n_k}) = V_1(g_n - g)$$

and

$$\lim_{k\to\infty} V_1(G_{n_k}) = V_1(g),$$

we see that $V_1(g_n - g)$ and $V_1(g)$ are measurable functions. Given $\varepsilon > 0$, we have

$$\int V_1^a(g_n - g)\,dy \leqslant \liminf_{k\to\infty} \int V_1^a(g_n - G_{n_k})\,dy < \varepsilon$$

if n is sufficiently large. We must show that $V_1(g) \in L^a$, but

$$\int V_1^a(g)\,dy \leqslant \liminf_{k\to\infty} \int V^a(G_{n_k})\,dy < \infty$$

since

$$\int |V_1(G_{n_k}) - V_1(G_{n_j})|^a\,dy \leqslant \int V_1^a(G_{n_k} - G_{n_j})\,dy \to 0$$

as $k, j \to \infty$, implying that $\{V_1(G_{n_k})\}$ converges in L^a.

In the same fashion, we may show that there is an $h = f$ a.e., with the appropriate continuity properties and such that

$$\|V_2(h_n - h)\|_a \to 0$$

and $V_2(h) \in L^a$. Then $f \in V^p_{\Lambda,a}$ with corresponding functions g and h,

$$\|f\|_{\Lambda,p,a} = \|f\|_p + \|V_1(g)\|_a + \|V_2(h)\|_a$$

and

$$\|f_n - f\|_{\Lambda,p,a} \to 0,$$

as was to be shown. ■

References

[1] L. Cesàri, *Sulle funzioni di due variabli a variazone limitata secondo Tonelli e sulla convergenza delle relative serie doppie di Fourier*, Rend. Sem. Mat. R. Univ. Roma 1 (1937), pp. 277–294.

[2] H. Federer, *Geometric measure theory*, Springer, New York 1969.

[3] C. Goffman and F. C. Liu, *On the localization property of square partial sums for multiple Fourier series*, Studia Math. 44 (1972), pp. 61–69.

[4] C. Goffman and W. P. Ziemer, *Higher dimensional mappings for which the area formula holds*, Ann. of Math. 29 (1970), pp. 482–488.

[5] R. E. Hughs, *Functions of BVC type*, Proc. Amer. Math. Soc. 12 (1961), pp. 698–701.

[6] S. Igari, *On the localization property of multiple Fourier series*, J. Approximation Theory 1 (1968), pp. 182–188.

[7] F. C. Liu, *On the localization of rectangular partial sums for multiple Fourier series*, Proc. Amer. Math. Soc. 34 (1972), pp. 90–96.

[8] S. J. Perlman and D. Waterman, *Some remarks on functions of Λ-bounded variation*, to appear.

[9] J. B. Serrin, Jr., *On the differentiability of functions of several variables*, Arch. Rational Mech. Anal. 7 (1961), pp. 359–372.

[10] L. Tonelli, *Serie trigonometriche*, Zanichelli, Bologna 1928, Cap. 9.

[11] D. Waterman, *On convergence of Fourier series of functions of generalized bounded variation*, Studia Math. 44 (1972), pp. 107–117.

[12] — *On the summability of Fourier series of functions of Λ-bounded variation*, Studia Math. 55 (1976), pp. 87–95.

[13] — *On Λ-bounded variation*, ibid. 57 (1976), pp. 33–45.

[14] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, Cambridge 1959, Chap. 17.

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