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MEAN WITH RESPECT TO A MAP

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An *n-mean* on a space X is a continuous function (map) $m: X^n \to X$, where $X^n = X \times X \times \ldots \times X$ (the n-fold Cartesian product of X), satisfying the following two conditions:

1. $m(x, x, \ldots, x) = x$ for every $x \in X$;

2. $m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for each n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$, σ being an element of the symmetric group S_n of n elements, i.e., σ is a permutation of the set $\{1, 2, \ldots, n\}$.

A space with a mean (admitting a mean) is called an ***m*-space** (for m-spaces see [1]-[8]). We now introduce a new idea, i.e. a mean with respect to a map.

Definition 1. Let X and Y be topological spaces and let $f: X \to Y$ be a map. We say that $m: X^n \to Y$ is an ***n*-mean with respect to *f*** if m is continuous and satisfies the following two conditions:

1. $m(x, x, \ldots, x) = f(x)$ for every $x \in X$;

2. $m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every $(x_1, x_2, \ldots, x_n)$ in X^n and for every $\sigma \in S_n$.

If $X = Y$ and $f = \mathrm{id}_X$ (identity map on X), then Definition 1 coincides with the usual definition of a mean on X. For a constant map f, m always exists, and this is the trivial case where m is constant. If X is homeomorphic to Y and f is a homeomorphism, then a mean with respect to f induces a mean on X, namely $f^{-1}m$, so that both X and Y are m-spaces. For further clarification we give the following two examples:

Example 1. Let $X = I = [0, 1]$, $Y = S^1$ (the unit circle in the plane) and let $f: I \to S^1$ be the map $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Let $\mu: I^2 \to I$ be the arithmetic mean. If $m = f\mu$, then m is an n-mean with respect to f as it can easily be verified.

Example 2. Let X and Y be finite polyhedra, Y convex, and let $f: X \to Y$ be a piecewise linear map such that $f(X)$ is not a convex subset of Y. We define $m(x_1, x_2, \ldots, x_n)$ to be the barycenter of $f(x_1), f(x_2), \ldots, f(x_n)$. Then m is an n-mean with respect to f.

It should be noted that in each of these two examples either X or Y is an m-space. In such cases the existence of a mean with respect to a map f is guaranteed by the following

PROPOSITION 1. *Let X and Y be topological spaces and let $f: X \to Y$ be continuous. If either X or Y admits a mean, then there exists a mean with respect to f.*

Proof. Suppose that X admits a mean $\mu: X^n \to X$ and that $f: X \to Y$ is continuous. The map $m = f\mu$ is obviously a mean with respect to f.

If, on the other hand, Y admits a mean μ', then $m' = \mu' f^n$ is a mean with respect to f.

In view of Proposition 1 a question arises as to whether a mean with respect to f exists if neither X nor Y is an m-space. The affirmative answer to this question is provided by the following example:

Let $X = Y = S^1$, where $S^1 = \{z \in C: |z| = 1\}$. Let $f: S^1 \to S^1$ be defined by $f(z) = z^2$. Since S^1 is an Abelian topological group, the multiplication $m(z_1, z_2) = z_1 z_2$ is continuous and $m(z_1, z_2) = m(z_2, z_1)$. Moreover, $m(z, z) = f(z)$ so that m is a mean with respect to f.

The converse of Proposition 1 is not true as the above-given example shows, since S^1 is not an m-space [1]. However, in Proposition 2 we give necessary conditions on f in order that Y is an m-space, and in Proposition 3 we examine conditions under which a covering space X is an m-space if there is a mean with respect to the projection map $p: X \to Y$.

PROPOSITION 2. *Let μ be an n-mean with respect to a map $f: X \to Y$, where f is open and onto. If for any two n-tuples $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ in X^n such that $f(x_i) = f(x'_{\sigma(i)})$ for some permutation $\sigma \in S_n$ we have $\mu(x_1, x_2, \ldots, x_n) = \mu(x_1', x_2', \ldots, x_n')$, then Y admits an n-mean.*

Proof. Since f is onto, for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$. Let $(y_1, y_2, \ldots, y_n) \in Y^n$ and define $m: Y^n \to Y$ by

$$m(y_1, y_2, \ldots, y_n) = \mu(x_1, x_2, \ldots, x_n), \qquad \text{where } y_i = f(x_i).$$

We shall show that m is an n-mean on Y.

First we show m is well defined. Let $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ be two n-tuples in X^n such that $f(x_i) = f(x_i')$. Then $\mu(x_1, x_2, \ldots, x_n) = \mu(x_1', x_2', \ldots, x_n')$ since $f(x_i) = f(x_i')$. Here we take σ to be the identity permutation in S_n. Hence $m(y_1, y_2, \ldots, y_n) = \mu(x_1, x_2, \ldots, x_n)$ is a unique point in Y and m is well defined.

To show continuity we observe that, by the definition of m, we have $\mu = mf^n$. If U is an open subset of Y, then $\mu^{-1}(U) = (f^n)^{-1}m^{-1}(U)$ is open in X^n by the continuity of μ. Since f is open, f^n is also open, and since f is onto, $f^n(f^n)^{-1}m(U) = m^{-1}(U)$ is open. Therefore, f is continuous.

We now show that m satisfies the conditions for a mean.

1. If $y \in Y$ and $x \in X$ are such that $f(x) = y$, then

$$m(y, y, \ldots, y) = \mu(x, x, \ldots, x) = f(x) = y.$$

2. Let $(y_1, y_2, \ldots, y_n) \in Y^n$ and let $(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})$ be a permutation of $(y_1, y_2, \ldots, y_n)$. Let $(x_1, x_2, \ldots, x_n)$ and $(x'_1, x'_2, \ldots, x'_n)$ be two n-tuples in X^n such that $f(x_i) = y_i$ and $f(x'_i) = y_{\sigma(i)}$. From the latter equality we obtain $f(x'_{\tau(i)}) = y_i$, where $\tau = \sigma^{-1}$, and therefore $f(x_i) = f(x'_{\tau(i)})$. By hypothesis, $\mu(x_1, x_2, \ldots, x_n) = \mu(x'_1, x'_2, \ldots, x'_n)$. Thus

$$m(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) = \mu(x_1, x_2, \ldots, x_n) = m(y_1, y_2, \ldots, y_n).$$

Since this is true for every permutation σ, m satisfies the symmetric property of a mean, therefore m is an n-mean on Y.

Definition 2. Let X and Y be pathwise connected and locally arcwise connected spaces and let $p: X \to Y$ be continuous. The pair (X, p) is called a *covering space* of Y if

1. p is onto,

2. for each $x \in X$, there exists an open set U in X containing x such that $p^{-1}(U)$ is a disjoint union of open sets, each of which maps homeomorphically onto U by p.

PROPOSITION 3. *Let X be a covering space of Y and let $p: X \to Y$ be the projection map. If μ is a mean with respect to p and $\mu_*\pi_1(X^n) \subset p_*\pi_1(X)$, then X admits a mean (μ_* and p_* are the homomorphisms induced by μ and p, respectively).*

Proof. Let Y_0 be a point in Y and let $* \in p^{-1}(Y_0)$ be a base point of X. By the definition of a covering space, X is pathwise connected and locally arcwise connected, therefore X^n is pathwise connected and locally arcwise connected with $(*, *, \ldots, *)$ as a base point. Since μ is a mean with respect to p, we have $\mu(*, *, \ldots, *) = p(*)$, and because of the condition $\mu_*\pi_1(X^n) \subset p_*\pi_1(X)$ there is a unique lifting $m: X^n \to X$ of μ such that $m(*, *, \ldots, *) = *$ and the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ X^n \arrow[ur, "m"] \arrow[r, "\mu"'] & Y \end{tikzcd}$$

commutes. The map m is a mean on X. To show this we first prove that $m(x, x, \ldots, x) = x$ for every $x \in X$. Let $x \in X$ and let w be a path from $*$ to x. We have $w(0) = *$ and $w(1) = x$. If $i: I \to I^n$ is the imbedding $i(t) = (t, t, \ldots, t)$, we let

$$\varphi = mw^n i = m(w \times w \times \ldots \times w)i: [0, 1] \to X.$$

We have

$$\varphi(0) = m(w \times w \times \ldots \times w)i(0) = m(w(0), w(0), \ldots, w(0))$$
$$= m(*, *, \ldots, *) = *$$

and

$$p\varphi(t) = pm(w(t), w(t), \ldots, w(t)) = \mu(w(t), w(t), \ldots, w(t)) = pw(t)$$

by the commutativity of the diagram above and the fact that μ is a mean with respect to p. Since $p\varphi$ and pw agree at one point, namely 0, we have

$$\varphi = w$$

and

$$x = w(1) = \varphi(1) = m(w(1), w(1), \ldots, w(1)) = m(x, x, \ldots, x).$$

To show that

$$m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

we let $(x_1, x_2, \ldots, x_n)$, $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \in X^n$ and $\sigma \in S_n$. Let us define $m: X^n \to X$ by

$$\tilde{m}(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

We observe that

$$\tilde{m}(*, *, \ldots, *) = *$$

and

$$p\tilde{m}(x_1, x_2, \ldots, x_n) = pm(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$
$$= \mu(x_1, x_2, \ldots, x_n).$$

But this means that $\tilde{m}$ is a lifting of μ. Since

$$m(*, *, \ldots *) = \tilde{m}(*, *, \ldots, *)$$

by the uniqueness of the lifting $m = \tilde{m}$, we have

$$m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

Thus m is a mean on X.

COROLLARY 1. *Let X and Y be as in Proposition 3. Let Y be an m-space with a mean m such that $(mp^n)_* \pi_1(X^n) \subset p_* \pi_1(X)$. Then X also admits a mean.*

Proof. If we let $mp^n = \mu$, then μ is obviously continuous. Moreover,

$$\mu(x, x, \ldots, x) = m(p(x), p(x), \ldots, p(x)) = p(x)$$

and

$$\mu(x_1, x_2, \ldots, x_n) = m(p(x_1), p(x_2), \ldots, p(x_n)) = m(p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}))$$
$$= \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \quad \text{for all } \sigma \in S_n.$$

Thus μ is a mean with respect to p, and all assumptions of Proposition 3 are satisfied. Therefore, X admits a mean.

COROLLARY 2. *If Y in Corollary 1 is an m-space and X is its universal covering space, then X admits a mean.*

Proof. Since X is the universal covering of Y, $\pi_1(X)$ and $\pi_1(X^n)$ are both zero. Also $\mu = mp^n$ is easily shown to be a mean with respect to p, and $\mu_* \pi_1(X^n) \subset p_* \pi_1(X)$ since $\pi_1(X) = \pi_1(X^n) = 0$. By Proposition 3, X admits a mean.

REFERENCES

[1] G. Aumann, *Über Räume mit Mittelbildungen*, Mathematische Annalen 119 (1943), p. 210-215.

[2] P. Bacon, *Compact means in the plane*, Proceedings of the American Mathematical Society 22 (1969), p. 242-246.

[3] — *Unicoherence in means*, Colloquium Mathematicum 21 (1970), p. 211-215.

[4] B. Eckmann, *Räume mit Mittelbildungen*, Commentarii Mathematici Helvetici 28 (1954), p. 329-340.

[5] — T. Ganea and P. Hilton, *Generalized means*, Studies in Mathematical Analysis and Related Topics, Stanford University Press, 1962.

[6] G. J. Michaelides, *A note on topological m-spaces*, Colloquium Mathematicum 32 (1975), p. 193-197.

[7] — *Complements of solenoids in S^3 are m-spaces*, Fundamenta Mathematicae 97 (1977), p. 71-77.

[8] K. Sigmon, *Acyclicity of compact means*, Michigan Mathematical Journal 16 (1969), p. 111-115.

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