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A THEOREM ON EXPANSIONS INTO INVERSE SYSTEMS OF METRIC SPACES

BY

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1. Preliminaries. In this note we show that there exists a functor S from a category of completely regular spaces into a category of inverse systems of metric spaces. The functor S is injectively adjoint to the functor of the limits of inverse systems of metric spaces. The maps considered here are assumed to be uniformly continuous. We use the notion of uniformity in the covering sense. A family which satisfies all axioms of uniformity except the axiom of separation is said to be a *pseudouniformity*. If X is a completely regular space, then by $\mathscr{U}_X^*$ we denote the greatest uniformity inducing the topology of X. Expressions $P \succ Q$ and $P \overset{*}{\succ} Q$ stand for a refinement and a star-refinement, respectively. Some symbols and notation are taken from [1].

There exists a functor h (see [1]) from the category of pseudouniform spaces into the category of uniform spaces such that for each pseudouniform space $(X, \mathscr{U})$ there exists a uniform map $h: (X, \mathscr{U}) \to (hX, h\mathscr{U})$ satisfying two conditions:

(i) $h^{-1}h\mathscr{U} = \mathscr{U}$, where $h^{-1}h\mathscr{U} = \{h^{-1}Q: Q \in h\mathscr{U}\}$;

(ii) for each uniform map $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ into a uniform space there exists a uniform map $g: (hX, h\mathscr{U}) \to (Y, \mathscr{V})$ such that $f = gh$.

A *uniform feathering* of a space X in a space $Y \supset X$ is a countable family $\mathscr{P}$ of coverings of X consisting of open sets in Y such that

$$\mathscr{P}|X \subset \mathscr{U}_X^* \quad \text{and} \quad [x]_{\mathscr{P}} = \bigcap \{\mathrm{st}(x, P): P \in \mathscr{P}\} \subset X$$

for each $x \in X$.

2. Properties of a covering type. Let $\mathscr{A}$ be a countable family of relations defined on $\mathscr{U}_X^*$. A pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ is said to be an $\mathscr{A}$-*pseudouniformity* if for each $a \in \mathscr{A}$ and each $P \in \mathscr{U}$ there exists $P' \in \mathscr{U}$ such that $(P', P) \in a$ and for each $P' \in \mathscr{U}$ the relation $(P', P) \in a$ implies $P' \overset{*}{\succ} P$.

A topological property A of a space X is said to be of a *covering-countable type* if there exists a countable family $\mathscr{A}$ of relations defined on $\mathscr{U}_X^*$ such that

(a) the greatest uniformity $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity;

(b) for each $\mathscr{A}$-pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ with a countable base the space hX with topology induced by the uniformity $h\mathscr{U}$ has the property A.

Let X be a subspace of a space Y. For each pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ we denote by $\mathrm{ext}_Y\mathscr{U}$ the set of all extensions, open in Y, of open coverings belonging to $\mathscr{U}$. Let $\mathscr{B}$ be a countable family of relations defined on the family $\mathrm{ext}_Y\mathscr{U}_X^*$.

A pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ is said to be a *$\mathscr{B}$-pseudouniformity* if for each $b \in \mathscr{B}$ and $P \in \mathrm{ext}_Y\mathscr{U}$ there exists $P' \in \mathrm{ext}_Y\mathscr{U}$ such that $(P', P) \in b$ and for each $P' \in \mathrm{ext}_Y\mathscr{U}$ the relation $(P', P) \in b$ implies $P'|X \succ_* P|X$ and $\mathrm{cl}_Y P' \succ P$.

We say that a topological property B is *inherited* from a space Y onto $X \subset Y$ by small layers if there exists a countable set $\mathscr{B}$ of relations defined on $\mathrm{ext}_Y\mathscr{U}_X^*$ such that $\mathscr{U}_X^*$ is a $\mathscr{B}$-uniformity and for each $\mathscr{B}$-pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ the set

$$[x]_{\mathrm{ext}_Y\mathscr{U}} = \bigcap\{\mathrm{st}(x, P) \colon P \in \mathrm{ext}_Y\mathscr{U}\}$$

has the property B and $[x]_{\mathrm{ext}_Y\mathscr{U}} \subset X$ for each $x \in X$.

A map $f\colon X \to Y$ is said to be a *B-map* if for each $y \in Y$ the counterimage $f^{-1}y$ has the property B.

We shall give two examples of properties of a covering type. Other examples can be found in [2].

PROPOSITION 1. *If X is a completely regular space, then* $\dim X = n$ *is a property of the covering-countable type.*

Proof. Define a set $\mathscr{A} = \{a_1, a_2\}$ of relations on $\mathscr{U}_X^*$ as follows:
$(P', P) \in a_1$ iff $P' \succ_* P$ and $\mathrm{ord} P' \leqslant n+1$,

$(P', P) \in a_2$ iff $P' \succ_* P$ and there is no covering $P'' \in \mathscr{U}_X^*$ such that $\mathrm{ord} P'' < n$ and $P'' \succ P$.

Since the uniformity $\mathscr{U}_X^*$ has a base consisting of all locally finite and functionally open coverings of X, we have $\dim\mathscr{U}_X^* = n$ iff $\dim X = n$. Thus $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity. Now, we shall verify condition (b) of the definition of a property of the covering-countable type. From the construction of the functor h it follows that $\dim\mathscr{U} = \dim h\mathscr{U}$ (see property (a) of the functor h). But, if the uniformity $h\mathscr{U}$ has a countable base, then $\dim hX = \dim h\mathscr{U}$, where the topology of the space hX is induced by $h\mathscr{U}$.

A space X is *cohomologically locally connected* in a dimension not greater than n, $n \leqslant \infty$, and in a group of coefficients G (written: $X \in \mathrm{clc}_G^n$)

if for each neighbourhood U of a point x there exists a neighbourhood $V \subset U$ of x such that the homomorphism of reduced cohomology Alexander-Čech groups $H^k(U; G) \to H^k(V; G)$, induced by the embedding $V \subset U$, is trivial for each $k \leqslant n$.

PROPOSITION 2. *For each paracompact p-space X, $X \in \mathrm{clc}_G^n$ and $H^k(X; G) = A_k$ for each $k \leqslant n$ is a property of the covering-countable type.*

Proof. Let $\mathscr{P} = \{P_n: n = 1, 2, \ldots\}$ be a feathering of X in the Čech-Stone compactification βX. Define relations a_n^k on $\mathscr{U}_X^*$ by the condition: $(P', P) \in a_m^k$ iff $P' \overset{*}{>} P$, $\mathrm{cl}_{\beta X}\tilde{P}' \succ P_m \wedge P$ (where $\tilde{P}$ denotes the greatest extension of $P \in \mathscr{U}_X^*$ open in βX) and for each $u' \in P'$ there exists $u \in P$ such that $u' \subset u$ and the homomorphism $H^k(u; G) \to H^k(u'; G)$ is trivial.

Put $\mathscr{A} = \{a_m^k: k \leqslant n, m < \infty\}$. Notice that $\mathscr{U}_X^*$ is an $\mathscr{A}$-uniformity. Now, let $\mathscr{U} \subset \mathscr{U}_X^*$ be an $\mathscr{A}$-pseudouniformity with a countable base. The condition $\mathrm{cl}_{\beta X}\tilde{P}' \succ P_m \wedge \tilde{P}, P' \overset{*}{>} P$, ensures that a family $\{\mathrm{st}(x, P) : P \in \mathscr{U}\}$ is a base of neighbourhoods of the set $[x]_{\mathscr{U}} \subset X$, because

$$[x]_{\mathscr{U}} = \bigcap \{\mathrm{st}(x, P): P \in \mathscr{U}\} = \bigcap \{\mathrm{cl}_{\beta X}\mathrm{st}(x, P): P \in \mathscr{U}\}$$

and βX is a compact space. This implies that for each neighbourhood $U[x]$ of the set $[x]$ there exists a neighbourhood $V[x] \subset U[x]$ of $[x]$ such that the homomorphism $H^k(U[x]; G) \to H^k(V[x]; G)$ is trivial. Hence, for each $x \in X$ and $k \leqslant n$ we have $H^k([x]; G) = 0$ (cf. [3], Theorem 6.6.2).

Now let us consider the space hX with topology induced by the uniformity $h\mathscr{U}$. Since the family $\{\mathrm{st}(x, P): P \in \mathscr{U}\}$ is a base of neighbourhoods of the set $[x]_{\mathscr{U}}$, the map $h: X \to hX$ is perfect and $H^k(h^{-1}h(x); G) = 0$, $x \in X$, $k \leqslant n$. By the Vietoris-Begle theorem (cf. [3], Theorem 6.9.15), the map induces the isomorphism $H^k(hX; G) \to H^k(X; G)$, $k \leqslant n$. Hence we obtain immediately $hX \in \mathrm{clc}_G^n$ and $H^k(hX; G) = H^k(X) = A_k$, $k \leqslant n$.

3. MAIN LEMMA. *Let X be a completely regular space and assume that $Y_i \supset X$ $(i = 1, 2, \ldots)$ are spaces in which X has uniform featherings, A_i $(i = 1, 2, \ldots)$ are properties of X of the covering-countable type, and B_i $(i = 1, 2, \ldots)$ are properties which are inherited from Y_i onto X by small layers. Moreover, let $\mathscr{A}_i$ and $\mathscr{B}_i$ be countable families of relations on $\mathscr{U}_X^*$ corresponding to the properties A_i and B_i.*

Then for each pair of pseudouniformities $\mathscr{U}_1, \mathscr{U}_2 \subset \mathscr{U}_X^*$, having countable bases, there exists an $\mathscr{A}_i$- and $\mathscr{B}_i$-pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$ with a countable base such that $\mathscr{U}_1 \cup \mathscr{U}_2 \subset \mathscr{U}$.

Proof. Define by induction countable families $\mathscr{W}_k \subset \mathscr{U}_X^*$. Let $\mathscr{W}_0 = \mathscr{V}_1 \cup \mathscr{V}_2$, where $\mathscr{V}_1 \subset \mathscr{U}_1$ and $\mathscr{V}_2 \subset \mathscr{U}_2$ are countable bases consisting of open coverings. Fix an extension $P(i)$, open in Y_i, of a covering $P \in \mathscr{W}_0$.

Suppose that the countable families $\mathscr{W}_k \subset \mathscr{U}_X^*$ are defined for each $k \leqslant n$ and assume that extensions $P(i)$, open in Y_i, of P are given for each $P \in \mathscr{W}_k$, $k \leqslant n$.

For each pair $P_1, P_2 \in \bigcup\{\mathscr{W}_k\colon k = 0, 1, \ldots, n\}$ choose a countable family $\mathscr{W}(P_1, P_2) \subset \mathscr{U}_X^*$ such that for each relation $a \in A_i$ $(i = 1, 2, \ldots)$ there exists $P \in \mathscr{W}(P_1, P_2)$ such that $(P, P_1 \cap P_2) \in a$ and for each relation $b \in \mathscr{B}_i$ $(i = 1, 2, \ldots)$ there exists $P' \in \mathscr{W}(P_1, P_2)$ having an extension $P(i)$ open in Y_i and such that

$$\big(P'(i), P_1(i) \wedge P_2(i)\big) \in b.$$

Put

$$\mathscr{W}_{n+1} = \bigcup\big\{\mathscr{W}(P_1, P_2)\colon P_1, P_2 \in \bigcup\{\mathscr{W}_k\colon k = 0, 1, \ldots, n\}\big\}$$

and fix an extension $P(i)$, open in Y_i, of $P \in \mathscr{W}_{n+1}$ (for those P for which the extensions $P(i)$ have not been fixed yet).

Put $\mathscr{W} = \bigcup\{\mathscr{W}_n\colon n = 0, 1, \ldots\}$. The countable family $\mathscr{W}$ is a base for a pseudouniformity $\mathscr{U} \subset \mathscr{U}_X^*$, $\mathscr{U} \supset \mathscr{U}_1 \cup \mathscr{U}_2$ and, moreover, $\mathscr{U}$ is both an $\mathscr{A}_i$- and $\mathscr{B}_i$-pseudouniformity for each $i = 1, 2, \ldots$

4. A category of inverse systems. Let *Top* be a category of topological spaces and continuous maps. Denote by inv*Top* a class obtained from the category *Top* in the following way: an object X of the class inv*Top* is an inverse system $X = \{X_s, p_s^{s'}, E_X\}$ of topological spaces with the bonding maps $p_s^{s'}\colon X_{s'} \to X_s$ onto, and a morphism $f\colon X \to Y$ between two objects $X = \{X_s, p_s^{s'}, E_X\}$ and $Y = \{Y_t, p_t^{t'}, E_Y\}$ belonging to the class inv*Top* is a family of continuous maps

$$f = \{f_t^s\colon X_s \to Y_t,\, t \in E_Y\}$$

satisfying the conditions

(1) for each $t \in E_Y$ there exists $f_t^s \in f$ for some $s \in E_X$;

(2) for each $f_t^s \in f$ and for each $s' \geqslant s$ and $t' \leqslant t$, $f_{t'}^{s'} = p_{t'}^t f_t^s p_s^{s'} \in f$.

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in the class inv*Top*. Define a composition $gf\colon X \to Z$ by

$$gf = \{(gf)_u^s : u \in E_Z \text{ and some } s \in E_X\},$$

where $(gf)_u^s = g_u^t f_t^s$ for some $t \in E_Y$, $g_u^t \in g$, $f_t^s \in f$. The composition $(gf)_u^s$ does not depend on the choice of $t \in E_Y$. Indeed, consider compositions $g_u^{t'} f_{t'}^s$ and $g_u^{t''} f_{t''}^s$. According to (1) and (2), there exist $g_u^t \in g$ for some $t \geqslant t', t''$ and $f_t^{s'} \in f$ for some $s' \geqslant s$. By (2) we have

$$g_u^t f_t^{s'} = g_u^{t''} f_{t''}^s p_s^{s'} \quad \text{and} \quad g_u^t f_t^{s'} = g_u^{t'} f_{t'}^s p_s^{s'}.$$

Since $p_s^{s'}$ is onto, $g_u^{t'} f_{t'}^s = g_u^{t''} f_{t''}^s$.

LEMMA 1. *The class* inv*Top is a category.*

Proof. Put $p_X : X \to X$,

$$p_X = \{p_s^{s'}\colon s, s' \in E_X\} \quad \text{for each } X = \{X_s, p_s^{s'}, E_X\} \in \mathrm{inv}Top.$$

The morphism p_X is the identity in the class $\mathrm{inv}\,Top$, since from (2) it follows immediately that, for each two morphisms $f\colon X \to Y$ and $g\colon Z \to X$, $fp_X = f$ and $p_X g = g$.

Now, consider three morphisms, $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, belonging to the class $\mathrm{inv}\,Top$. In order to see that $h(gf) = (hg)f$ it suffices to verify that $h_v^{u'}(gf)_{u'}^s = (hg)_v^{t'} f_{t'}^s$ for $t' \in E_Y$, $u' \in E_Z$. Put $(gf)_{u'}^s = g_{u'}^{t''} f_{t''}^s$ and $(hg)_v^{t'} = h_v^{u''} g_{u''}^{t'}$ for some $t'' \in E_Y$ and $u'' \in E_Z$. According to (1) and (2), there exist $h_v^u \in h$, $g_u^t \in g$ and $f_t^{s'} \in f$ for some $u \geq u'$, u'', $t \geq t'$, t'' and $s' \geq s$. Then by (2) we have

$$h_v^u g_u^t f_t^{s'} = h_v^{u''} g_{u''}^{t'} f_{t'}^s p_s^{s'} \quad \text{and} \quad h_v^u g_u^t f_t^{s'} = h_v^{u'} g_{u'}^{t''} f_{t''}^s p_s^{s'}.$$

Since $p_s^{s'}$ is onto, $h_v^{u''} g_{u''}^{t'} f_{t'}^s = h_v^{u'} g_{u'}^{t''} f_{t''}^s$, i.e., $(hg)_v^{t'} f_{t'}^s = h_v^{u'}(gf)_{u'}^s$.

Two morphisms $f, g\colon X \to Y$ belonging to the category $\mathrm{inv}\,Top$ are *in a relation* $\sim$, $f \sim g$, if for each $t \in E_Y$ and $s' \in E_X$ there exists $s \in E_X$, $s \geq s'$, such that $f_t^s = g_t^s$.

LEMMA 2. *The relation* $\sim$ *is an equivalence relation.*

Proof. It is obvious that $f \sim f$ and that $f \sim g$ implies $g \sim f$. Suppose that $f_t^{s'} = g_t^{s'}$ and $g_t^{s''} = h_t^{s''}$. Choose $s \geq s'$, s''. Then

$$f_t^s = f_t^{s'} p_{s'}^s = g_t^{s'} p_{s'}^s = g_t^s = g_t^{s''} p_{s''}^s = h_t^{s''} p_{s''}^s = h_t^s.$$

Thus we have proved that $f \sim g$ and $g \sim h$ imply $f \sim h$.

LEMMA 3. *If* $f \sim \bar{f}\colon X \to Y$ *and* $g \sim \bar{g}\colon Y \to Z$, *then* $gf \sim \bar{g}\bar{f}$.

Proof. For each $u \in E_Z$ there exists $t \in E_Y$ such that $g_u^t = \bar{g}_u^t$ and there exists $s \in E_X$ such that $f_t^s = \bar{f}_t^s$. Hence $(gf)_u^s = (\bar{g}\bar{f})_u^s$. Thus $gf \sim \bar{g}\bar{f}$.

For each $f\colon X \to Y$ in $\mathrm{inv}\,Top$ put $[f] = \{\bar{f}\colon \bar{f} \sim f\}$ and define a composition by $[g][f] = [gf]$. From Lemma 3 it follows that the composition is well defined and, moreover,

$$[f][p_X] = [p_Y][f] = [f] \quad \text{and} \quad ([f][g])[h] = [f]([g][h]).$$

Hence we obtain

LEMMA 4. *The class* $\overline{\mathrm{inv}\,Top}$ *with the objects from the category* $\mathrm{inv}\,Top$ *and the morphisms* $[f]$, *where* f *is a morphism from the category* $\mathrm{inv}\,Top$, *is a category.*

One can also prove

LEMMA 5. *Let* $X = \{X_s, p_s^{s'}, E_X\} \in \mathrm{inv}\,Top$ *and let* $E' \subset E_X$ *be a cofinal subset of* E_X. *Then the object* $X' = \{X_s, p_s^{s'}, s, s' \in E'\}$ *is isomorphic to the object* X *in the category* $\overline{\mathrm{inv}\,Top}$.

A limit of an inverse system $\{X_s, p_s^{s'}, E_X\} \in \mathrm{inv}\,Top$ is the topological space

$$X_* = \varprojlim X = \{x \in \mathsf{P}\{X_s\colon s \in E_X\} : p_s^{s'} x(s') = x(s) \ \text{for} \ s' \geq s,\ s, s' \in E_X\}$$

with topology induced from the product $\mathsf{P}\{X_s\colon s \in E_X\}$.

For each morphism $f \in \operatorname{inv} Top$, $f\colon X \to Y$, put $f_*\colon X_* \to Y_*$ to be a map between topological spaces defined by $(f_* x)(t) = f_t^s x(s)$. The map f_* is well defined because it does not depend on the choice of $s \in E_X$. Indeed, for $s \geqslant s', s''$ we have

$$f_t^{s'} x(s') = f_t^{s'} p_{s'}^s x(s) = f_t^s x(s)$$

and, similarly,

$$f_t^{s''} x(s'') = f_t^s x(s).$$

Hence $f_t^{s'} x(s') = f_t^{s''} x(s'')$. Moreover, $f_* x \in Y$, since

$$p_{t'}^t (f_* x)(t) = p_{t'}^t f_t^s x(s) = f_{t'}^s x(s) = f_* x(t).$$

The continuity of $f_t^s \in f$ implies the continuity of the map f_*.

Notice that if $f \sim g$, then $f_* = g_*$, because for each $t \in E_Y$ there exists $s \in E_X$ such that $f_t^s = g_t^s$, and hence

$$(f_* x)(t) = f_t^s x(s) = g_t^s x(s) = (g_* x)(t).$$

The converse implication does not, in general, hold. However, we have

LEMMA 6. *If X is an inverse system $\{X_s, p_s^{s'}, E_X\}$ of compact spaces X_s with the bonding maps $p_s^{s'}\colon X_{s'} \to X_s$ onto, then for each two morphisms $f, g\colon X \to Y$ in the category* $\operatorname{inv} Top$ *the relation $f_* = g_*$ implies $f \sim g$.*

It can be verified that $(fg)_* = f_* g_*$ and $(p_X)_* = 1_{X_*}$. Hence we obtain

LEMMA 7. *The operation $*\colon \operatorname{inv} Top \to Top$ ($\overline{\operatorname{inv}} Top \to Top$), which assigns to each object $X \in \operatorname{inv} Top$ its limit $X_* \in Top$ and to each morphism $f \in \operatorname{inv} Top$ ($[f] \in \overline{\operatorname{inv}} Top$) a map $f_* \in Top$, is a functor from the category* $\operatorname{inv} Top$ ($\overline{\operatorname{inv}} Top$) *into the category Top.*

In literature one can find the following definition of a mapping between inverse systems:

A mapping $(\varphi, f)\colon X \to Y$ between inverse systems of topological spaces $X = \{X_s, p_s^{s'}, E_X\}$ and $Y = \{Y_t, p_t^{t'}, E_Y\}$ is a set of maps $(\varphi, f) = \{\varphi, f_t\colon t \in E_Y\}$, where the map $\varphi\colon E_Y \to E_X$ is weakly monotonic and the map $f_t\colon X_{\varphi(t)} \to Y_t$ is such that $p_{t'}^t f_t = f_{t'} p_{\varphi(t')}^{\varphi(t)}$ for each pair $t, t' \in E_Y$, $t \geqslant t'$.

Each mapping $(\varphi, f)\colon X \to Y$ induces a map $(\varphi, f)_*\colon X_* \to Y_*$ between the limits.

The following lemmas can be easily proved:

LEMMA 8. *For each mapping $(\varphi, f)\colon X \to Y$ there exists a morphism $f_\varphi\colon X \to Y$ in the sense of the category* $\operatorname{inv} Top$ *such that $(\varphi, f)_* = f_*\colon X_* \to Y_*$ and $f_t = (f_\varphi)_{\varphi(t)}^{\varphi(t)}$.*

LEMMA 9. *Let $f\colon X \to Y$ be a morphism in the category* $\operatorname{inv} Top$. *If each element of the set E_Y has finitely many predecessors, then there exists a mapping $(\varphi, f)\colon X \to Y$ such that $f_\varphi = f$.*

5. An inverse expansion theorem. In order to formulate a theorem on expansions of spaces and maps, consider a new category $Top(*, *)$. An *object* of the category is a triple (X, a, b), where X is a completely regular space, a is a countable family of properties of X of the covering-countable type, and b is a countable family of properties of type B which are inherited by small layers. A morphism $f\colon (X, a, b) \to (Y, c, d)$ is a continuous map between X and Y.

Denote by $\mathrm{inv}\,Top(m)$ a category of inverse systems of metric spaces and uniformly continuous maps with morphisms defined as in the category $\mathrm{inv}\,Top$.

THEOREM. *There exists a functor S from the category $Top(*, *)$ into the category $\mathrm{inv}\,Top(m)$ and there exists a natural map $i: I \to S_*$ of the identity functor defined on $Top(*, *)$ into the functor $S_* = \varprojlim S$ such that the following conditions are satisfied:*

(1) *Each metric space $X_\alpha \in S(X, a, b)$ has property A for each $A \in a$.*

(2) *For each $(X, a, b) \in Top(*, *)$, $i_X\colon (X, a, b) \to S_*(X, a, b)$ is a dense embedding which is onto whenever the uniformity $\mathscr{U}^*_X$ is complete.*

(3) *The composition $p_\alpha i_X$ is a B-map for each $B \in b$ (where $p_\alpha\colon S_*(X, a, b) \to X_\alpha$, $X_\alpha \in S(X, a, b)$, is the projection) and $p_\alpha i_X$ is a perfect map whenever there exists $B \in b$ such that B is the compactness.*

(4) *For each map $f\colon (X, a, b) \to Y_*$, where $Y \in \mathrm{inv}\,Top(m)$, there exists a unique (in the sense of $\overline{\mathrm{inv}}\,Top$) map $f'\colon S(X, a, b) \to Y$ such that $f = f'_* i_X$.*

A *natural map* $i\colon \mathscr{F} \to \mathscr{G}$ between functors defined on a category $\mathscr{C}$ is a class of morphisms $\{i_X\colon \mathscr{F}(X) \to \mathscr{G}(X)\colon X \in \mathscr{C}\}$ such that, for each morphism $f\colon X \to Y$, $\mathscr{G}(f) i_X = i_Y \mathscr{F}(f)$.

Notice that condition (4) of the Theorem means that S is injectively adjoint to $\varprojlim$. The functor S is unique in the category $\overline{\mathrm{inv}}\,Top(*, *)$, since each two adjoint functors are unique up to isomorphism.

Proof of the Theorem. Let $(X, a, b) \in Top(*, *)$ and let $\mathscr{A}_i$ and $\mathscr{B}_i$ be sets of relations which are determined by $A_i \in a$ and $B_i \in b$, respectively. For each $A_i \in a$ and $B_i \in b$ the uniformity $\mathscr{U}^*_X$ is an $\mathscr{A}_i$- and $\mathscr{B}_i$-uniformity. The Main Lemma implies that a set E_X of all $\mathscr{A}_i$- and $\mathscr{B}_i$-pseudouniformities $\alpha \subset \mathscr{U}^*_X$, $A_i \in a$, $B_i \in b$, having countable bases, is directed with respect to the inclusion $\supset$ and, moreover, $\mathscr{U}^*_X = \bigcup E_X$. According to property (ii) of the functor h, the diagram

$$\begin{tikzcd}
 & (X, \alpha) \arrow[r, "h_\alpha"] \arrow[d] & (h_\alpha X, h\alpha) \arrow[d, "p^\alpha_\beta"] \\
(X, \mathscr{U}^*_X) \arrow[ur] \arrow[dr] & & \\
 & (X, \beta) \arrow[r, "h_\beta"] & (h_\beta X, h\beta)
\end{tikzcd} \qquad \alpha \supset \beta,\ \alpha, \beta \in E_X,$$

of pseudouniform spaces and uniform maps is commutative.

Put $X_a = h_a X$ with metric induced by the uniformity ha. We have obtained an inverse system $S(X, a, b) = \{X_a, p^a_\beta, a, \beta \in E_X\}$ of metric spaces X_a, $a \in E_X$, and uniformly continuous maps p^a_β: $X_a \to X_\beta$ such that for each $A \in a$ the space X_a has the property A.

Maps of the form $(X, \mathscr{U}^*_X) \to (X, a) \to (h_a X, ha)$ are B-maps for each $B \in b$ and induce a dense embedding i_X: $(X, a, b) \subset S_*(X, a, b)$ which is a homeomorphism in the case where the uniformity $\mathscr{U}^*_X$ is complete.

Now, we prove that S is a functor. Let f: $(X, a, b) \to (X', a', b')$ be a map. For each pair of pseudouniformities $a \in E_X$ and $a' \in E_{X'}$ such that $a \supset f^{-1} a'$ there exists, according to property (ii) of the functor h, a unique map $f^a_{a'}$: $X_a \to X_{a'}$ such that the diagram

$$\begin{array}{ccc} (X, \mathscr{U}^*_X) & \xrightarrow{f} & (X', \mathscr{U}^*_{X'}) \\ \downarrow{\scriptstyle h_a} & & \downarrow{\scriptstyle h_a} \\ (h_a X, ha) & \xrightarrow{f^a_{a'}} & (h_{a'} X, ha') \end{array}$$

commutes.

From the Main Lemma and from the uniqueness of the maps $f^a_{a'}$, $a \in E_X$, $a' \in E_{X'}$, $a \supset f^{-1} a'$, it follows that the family

$$S(f) = \{f^a_{a'}\colon X_a \to X_{a'} : a \in E_X,\ a' \in E_{X'},\ a \supset f^{-1} a',\ X_a \in S(X, a, b),\\ X_{a'} \in S(X', a', b')\}$$

is a morphism in the sense of the category invTop. It is easy to see that $S(fg) = S(f)S(g)$ and $S(1_X) = 1_{S(X)}$. Thus conditions (1)-(3) of the Theorem are proved.

Now, we prove that the functor S is injectively adjoint to the functor $\varprojlim$. Let f: $(X, a, b) \to Y_*$ be a map, where $Y = \{Y_a, p^a_\beta, E_Y\}$, p^a_β: $Y_a \to Y_\beta$ is a uniformly continuous map between the metric spaces Y_a and Y_β. For each $a \in E_Y$ let $\bar{a}$ be the uniformity induced by the metric of the space Y. Put

$$f' = \{f^a_\beta\colon X_a \to Y_\beta,\ \beta \in E_Y,\ a \in E_X,\ a \supset f^{-1} p_\beta^{-1} \bar{\beta}\},$$

where p_β: $Y_* \to Y_\beta$ is the projection. The family f' of maps is a morphism in the sense of the category inv$Top(m)$. From the definition of f' it follows that $f'_* i_X = f$. The morphism $[f']$ is unique in the sense of the category $\overline{\text{inv}Top(m)}$, since each map $f^a_\beta \in f'$ is uniquely determined by property (ii) of the functor h.

REFERENCES

[1] W. Kulpa, *Factorization and inverse expansion theorems for uniformities*, Colloquium Mathematicum 21 (1970), p. 217-227.
[2] — *Properties of the covering type and a factorization theorem*, Fundamenta Mathematicae 107 (1980), p. 161-166.
[3] E. H. Spanier, *Algebraic topology*, New York 1966.

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