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NORMAL P-SPACES AND THE G_δ-TOPOLOGY

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1. Introduction. The present work deals with properties of P-spaces and the Baire topology (also called the G_δ-topology) of a Tihonov space. Specifically, we investigate the concept of normality in P-spaces and the preservation of paracompactness and Lindelöf degree by the Baire topology of scattered spaces.

The following summarizes the organization of the paper. Section 2 presents the basic definitions and notation used in this paper. Section 3 presents several preliminary results; in particular, we state results due to A. W. Hager, and to W. Rudin, A. Pełczyński, and Z. Semadeni, which characterize among the P-spaces and the compact spaces, respectively, the spaces satisfying the following property: each continuous real-valued function defined on the space has countable image. We also establish that each completely additive disjoint Baire family in a pseudocompact space is countable. In Section 4, we discuss the concept of normality in P-spaces. It is shown that, under a certain set-theoretic restriction, normal P-spaces with density no larger than c are collectionwise normal. An example (in ZFC) communicated to the authors by W. Fleissner shows that normal P-spaces with density larger than c need not be collectionwise normal. It is also shown that normal pseudo-$\aleph_1$-compact P-spaces are collectionwise normal. Section 5 considers the Baire topology of a Tihonov space and includes the following results: the Baire topology of a paracompact (respectively, Lindelöf) scattered space is paracompact (respectively, Lindelöf). The Lindelöf portion of this result generalizes a similar result for compact spaces due to P. R. Meyer. We also add some new equivalences to the many already known which characterize when the Baire topology of a compact space is Lindelöf. Finally, we establish the equivalence of certain properties of the Baire topology of a pseudocompact space which are related to the commuting of the Hewitt realcompactification operator ν and the formation of the Baire topology. Section 6 presents several examples which answer natural questions arising from the discussion in Sections 3-5, and Section 7 gives a list of unsettled questions.

We extend our thanks to Professors Linn Sennott and William Fleissner for several discussions on the contents of this paper.

2. Definitions and notation. Unless otherwise specified, we will employ the definitions and notation used in [5]. In particular, we consider only Tihonov (that is, completely regular and Hausdorff) spaces. The members of the σ-field generated by the zero sets of a Tihonov space X are called *Baire sets* and a function $f\colon X \to R$ is *Baire measurable* if $f^{-1}(A)$ is a Baire set for each open set A of R. Baire(X) denotes the family of real-valued Baire measurable functions. The *Baire topology* (or G_δ-topology; called the ι-topology in [10]) of a Tihonov space X is the weak topology generated by Baire(X); equivalently, the Baire topology is the topology having for a basis the family of Baire sets (or zero sets or G_δ-sets) of X. The resulting topological space is denoted by bX. We note that since the Baire topology of X is generated by the G_δ-sets of X, if A is a subset of X, the Baire topology on A is the restriction to A of the Baire topology on X. A *P-space* is a space in which each G_δ-set (or zero set) is open. It is easily established that the Baire topology is the smallest P-space topology which contains the topology of the given space. The density character of a space X is denoted by $d(X)$. If m is a cardinal number, X is *m-Lindelöf* if each open cover of X has a subcover of cardinal at most m. A space X is *almost-compact* (respectively, *almost-Lindelöf*; see [3]) if at least one of any pair of disjoint zero sets is compact (respectively, Lindelöf). X is *pseudo-$\aleph_1$-compact* (see [7]) if each locally finite family of open sets or cozero sets is countable (equivalently, if each discrete family of open sets or cozero sets is countable). X is *functionally countable* if each member of $C(X)$ has countable image. A space is *scattered* (or *dispersed*) if every non-empty subspace contains an isolated point.

3. Preliminary results. The results stated in this section will be used in the sequel. The proof of 3.1 is outlined so that it may be contrasted with the proof of 5.7. The proofs of 3.2-3.4 are included because these results have apparently not appeared in the literature.

3.1. PROPOSITION (Rudin [14], Pełczyński and Semadeni [13]). *Assume X is compact. Then the following are equivalent:*

(i) *X is functionally countable.*

(ii) $f \in C(X) \to |f(X)| < c$.

(iii) *X is scattered.*

Outline of the proof. To establish (ii) $\Rightarrow$ (iii), assume that X is not scattered. Then there exists a non-empty compact subset K which has no isolated points. For each sequence $t = (t_1, t_2, \ldots)$ of zeros and ones define a non-empty closed subset A_t of K in the following manner:

Let $B(0)$ and $B(1)$ be disjoint non-empty closed subsets of K. Let $B(i0)$ and $B(i1)$ be disjoint non-empty closed subsets of $B(i)$ for $i = 0, 1$. Continue this process inductively and put

$$A_t = \bigcap_{n=1}^{\infty} B(t_1, t_2, \ldots, t_n).$$

Let $A = \bigcup \{A_t : t \in \{0, 1\}^{\aleph_0}\}$. The mapping $f: A \to \{0, 1\}^{\aleph_0}$ defined by $f|A_t = t$ is a continuous function from the compact set A onto the Cantor set; hence f may be extended to $\hat{f} \in C(X)$ with $|\hat{f}(X)| = c$.

To prove (iii) $\Rightarrow$ (i), let f be an element of $C(X)$. If $|f(X)| > \aleph_0$, then $f(X)$ is an uncountable compact subset of R, so $f(X)$ contains a copy C of the Cantor set. Let A be a subset of $f^{-1}(C)$ such that $f|A : A \to C$ is irreducible. Then A has no isolated points.

Note. The implication (i) $\Rightarrow$ (iii) of 3.1 is found in [13], and (iii) $\Rightarrow$ (i) is found in [14].

The next proposition characterizes the functionally countable spaces among P-spaces.

3.2. Proposition (A. W. Hager — oral communication). *A P-space is functionally countable if and only if it is pseudo-$\aleph_1$-compact.*

Proof. Assume that $\mathscr{U}$ is an uncountable discrete family of non-empty open subsets of X and let $\{U_s\}_{s\in A}$ be a subfamily of $\mathscr{U}$, where $A \subseteq R$ and $|A| = \aleph_1$. For each $s \in A$, choose $x_s \in U_s$ and $f_s \in C(X)$ such that $f_s(x_s) = s$ and $f_s|(X - U_s) = 0$. Then $f = \sum f_s \in C(X)$ (since $\mathscr{U}$ is discrete) and $|f(X)| = \aleph_1$.

To prove the converse, if X is not functionally countable, choose $f \in C(X)$ such that $|f(X)| > \aleph_0$. Since X is a P-space, $\{f^{-1}(r) : r \in R\}$ is an uncountable discrete cozero family.

Remarks. (i) Examples of pseudo-$\aleph_1$-compact spaces are pseu do compact spaces, Lindelöf spaces, and spaces which satisfy the countable chain condition.

(ii) The proof of 3.2 shows that every functionally countable space is pseudo-$\aleph_1$-compact; hence paracompact functionally countable spaces are Lindelöf (see [7]).

(iii) A metric space is functionally countable if and only if it is countable. To see this, suppose (M, d) is a functionally countable metric space. M must be separable since a non-separable metric space has an uncountable closed (and hence C-embedded) discrete subset which admits a continuous real-valued function with uncountable image. If M is zero-dimensional, M can be viewed as a subspace of R (see [6]); hence the injection $M \to R$ has countable image, so M is countable. Therefore, we are done if we can show that M is zero-dimensional. But if M is not zero-dimensional, there is a point $p \in M$ such that the function $f: M \to R$ defined by $f(x) = d(p, x)$ has uncountable image — in fact, $|f(M)| = c$ — contradicting the functional countability of M. Thus M is countable. It follows that a space X is functionally countable if and only if every continuous metric image of X is countable.

3.3. LEMMA. *Let X be a set and let $\mathscr{B}$ be a collection of subsets of X with the following properties:*

(i) $\{\emptyset, X\} \subset \mathscr{B}$,

(ii) *$\mathscr{B}$ is closed under finite unions and countable intersections,*

(iii) *each subfamily of $\mathscr{B}$ which has the finite intersection property also has the countable intersection property.*

Let $\hat{\mathscr{B}}$ be the closure of $\mathscr{B}$ under countable intersections and countable unions. Assume $f: X \to R$ is a mapping such that $f^{-1}(H) \in \hat{\mathscr{B}}$ for each open subset H of R.

Then $f(X)$ is a Souslin set in R (that is, a member of the family of sets generated by the Souslin operation from the closed sets of R).

Note that (i) $\Rightarrow$ (iii) says that $\mathscr{B}$ is a semi-compact paving in the sense of [1].

Proof of the lemma. Define $h: X \times R \to R \times R$ by $h(x, z) = (f(x), z)$ and let π_X: $X \times R \to X$ denote the projection mapping. The assumption on the mapping f guarantees that, for each open set G in $R \times R$, we have $h^{-1}(G) \in \mathscr{S}$, $\mathscr{S}$ being the set of $S \subseteq X \times R$ which may be Souslin derived from sets of the form $B \times H$, where $B \in \mathscr{B}$ and H is open in R. Since the diagonal Δ in $R \times R$ is a G_δ-set, we have

$$h^{-1}(\Delta) = \{(x, f(x)) : x \in X\} \in \mathscr{S}.$$

Hence, by 1.3 in [1], the assumption on $\mathscr{B}$ guarantees that $\pi_X(h^{-1}(\Delta)) = f(X)$ is a Souslin set in R.

3.4. PROPOSITION. *Assume that X is pseudocompact. Then each completely additive disjoint Baire family is countable.*

Proof. Assume that $\{B_s\}_{s \in S}$ is an uncountable completely additive disjoint Baire family, where $S \subseteq R$ is not a Souslin set and $|S| = \aleph_1$. (Such a set S may always be chosen: if $c = \aleph_1$, then simply choose any $S \subseteq R$ which is not a Souslin set; if $\aleph_1 < c$, choose a set S with $|S| = \aleph_1$; then S is not a Souslin set since each uncountable Souslin set in R contains a copy of the Cantor set.) Let $\mathscr{B}$ be the family of zero sets of X. Then the $\hat{\mathscr{B}}$ defined in 3.3 is the family of Baire sets of X and $\mathscr{B}$ satisfies the conditions in 3.3 since X is pseudocompact. Define $f: X \to R$ by

$$f(x) = \begin{cases} s & \text{if } x \in B_s, \\ 0 & \text{if } x \notin \bigcup_{s \in S} B_s. \end{cases}$$

Since $\{B_s\}$ is a completely additive Baire family, $f \in \mathrm{Baire}(X)$; hence, by 3.3, $f(X) = S \cup \{0\}$ is a Souslin set, which contradicts our choice of S.

4. Normality. We first remark that a P-space need not be normal — an example is given in Section 6, but many others are known. In fact, there are compact spaces such as $[0, 1]^{2^c}$ whose Baire topology is not normal. On the other hand, assuming the continuum hypothesis (CH) we infer that each P-space of cardinal not greater than c is paracompact. (This is easily proved by an induction argument — see [7].) The results of this section fall between these two extremes and are responses to the following question: When are normal P-spaces collectionwise normal?

4.1. PROPOSITION. *Assume that X is a normal P-space such that each closed discrete set in X has cardinal at most c. Then X is collectionwise normal.*

Proof. Let $\{A_s\}_{s\in S}$ be a discrete family of closed sets in X. Since closed discrete subsets of X have at most c elements, $|S| \leqslant c$, we may assume $S \subseteq R$. Let

$$A = \bigcup_{s\in S} A_s$$

and define $f: A \to R$ by $f|A_s = s$. Then f is continuous, so by the normality of X there exists $\hat{f} \in C(X)$ such that $\hat{f}|A = f$. Since X is a P-space, each $\hat{f}^{-1}(s)$ is an open set and $A_s \subseteq \hat{f}^{-1}(s)$, so the result follows.

4.2. COROLLARY. *Assume $2^{c^+} > 2^c$. If X is a normal P-space with $d(X) \leqslant c$, then X is collectionwise normal.*

Proof. Since X is normal, $d(X) \leqslant c$, and $2^{c^+} > 2^c$, each closed discrete set in X has cardinal at most c, so 4.1 may be applied.

Remarks. (i) Professor William Fleissner has pointed out that a modification of Bing's example G yields an example (in ZFC) of a normal P-space (with density greater than c) which is not collectionwise normal.

(ii) Assume $V = L$. Then any normal space (whether a P-space or not) in which each closed set has character at most c is collectionwise normal. To see this, let $\{F_s\}$ be a discrete family of closed sets in X and let Y be the quotient space obtained by identifying each F_s with a point y_s. One may verify that Y is normal. Y has character at most c, so, by [2], Y is collectionwise Hausdorff. Therefore, the closed discrete set $\{y_s\}$ can be separated by open sets in Y, so the family $\{F_s\}$ can be separated by open sets in X. Hence X is collectionwise normal.

4.3. PROPOSITION. *The following statements are equivalent for a P-space X:*

(i) *X is normal and pseudo-$\aleph_1$-compact.*

(ii) *X is normal and each closed discrete subset of X is countable.*

(iii) *X is normal and each uncountable subset of X has a limit point.*

(iv) *X is collectionwise normal and pseudo-$\aleph_1$-compact.*

Proof. The implications (iv) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iii) are immediate. To establish (ii) $\Rightarrow$ (iv), first note that (ii) and 4.1 imply that X is collectionwise normal. Since (ii) guarantees that each discrete cozero family is countable, X is also pseudo-$\aleph_1$-compact, so (iv) is established. To complete the proof we need only to establish (i) $\Rightarrow$ (ii). Assume that there exists a closed discrete subset F of X of cardinal $\aleph_1$ and let $f : F \to R$ be a one-to-one mapping. Since X is normal, f extends to $\hat{f} \in C(X)$, and since X is pseudo-$\aleph_1$-compact, 3.2 implies that the image of f is countable, which is a contradiction.

5. The Baire topology. Before proceeding to the main results of this section, we recall the idea of dispersal order. Given a Tihonov space X, define

$$D_1(X) = \{x \in X\colon x \text{ is a limit point of } X\}.$$

Assume that $D_\beta(X)$ has been defined for $\beta < \alpha$. If α is a limit ordinal define

$$D_\alpha(X) = \bigcap_{\beta<\alpha} D_\beta(X).$$

If α is not a limit ordinal, define

$$D_\alpha(X) = D_1\big(D_{\alpha-1}(X)\big)$$

(the set of all limit points of $D_{\alpha-1}(X)$). One may easily verify that X is scattered if and only if there exists an ordinal α such that $D_\alpha(X) = \emptyset$. If X is scattered, define $\alpha_X = \inf\{\alpha : D_\alpha(X) = \emptyset\}$ to be the dispersal order of X.

5.1. THEOREM. *Assume X is a paracompact scattered space. Then bX is a paracompact space.*

Proof. Assume that the conclusion is false and put

$$\tau = \inf\{\alpha_X\colon X \text{ is paracompact, scattered, and } bX \text{ is not paracompact}\}.$$

Choose a paracompact scattered space X such that $\tau = \alpha_X$ and bX is not paracompact. We will first show that τ is not a limit ordinal. Assume the contrary; then

$$D_\alpha = D_\alpha(X) \neq \emptyset \text{ for each } \alpha < \tau \quad \text{and} \quad \bigcap_{\alpha<\tau} D_\alpha = \emptyset.$$

Choose a σ-discrete cozero refinement

$$\mathscr{U} = \bigcup_{i=1}^{\infty} \mathscr{U}_i$$

of the open cover $\{X - D_\alpha\colon \alpha < \tau\}$ and choose $U \in \mathscr{U}_i$. Since $U \subset X - D_\alpha$ for some $\alpha < \tau$, we have $D_\alpha(U) \subseteq U \cap D_\alpha = \emptyset$; hence $\alpha_U \leqslant \alpha < \tau$, U paracompact (since U is an F_σ-set), and U scattered imply that bU is paracompact. Let $S_i = \bigcup\{U \in \mathscr{U}_i\}$, $i = 1, 2, \ldots$ Each bS_i is the discrete union of clopen paracompact spaces, so each bS_i is a clopen paracompact subspace of bX. It follows that bX is paracompact, which is a contradiction. Thus τ is a non-limit ordinal, so we may write $\tau = \gamma+1$. Since $D_1(D_\gamma) = D_\tau = \emptyset$, $D = D_\gamma$ is a closed discrete subset of X. Let $\mathscr{U}$ be a cover of X by zero sets (of X). For each $x \in D$, choose a cozero set C_x of X containing x such that the family $\{C_x\}$ is discrete in X. Then $F = X - \bigcup C_x$ is a zero set with $D_\gamma(F) \subseteq F \cap D = \emptyset$. Hence $a_F \leqslant \gamma$ implies that bF is a clopen paracompact subspace of bX. For each $x \in D$, choose $Z_x \in \mathscr{U}$ that contains x. Since $C_x - Z_x$ is a cozero set in X and $a_{C_x - Z_x} \leqslant \gamma$ (in view of $D_\gamma(C_x - Z_x) \cap \cap D = \emptyset$), each $b(C_x - Z_x)$ is a clopen paracompact subspace of bX. Thus $\mathscr{U}$ restricted to $S = F \cup \bigcup \{(C_x - Z_x) : x \in D\}$ has a locally finite open refinement $\mathscr{U}'$ in bS, so $\mathscr{U}$ has the locally finite open refinement $\mathscr{U}' \cup \{C_x \cap Z_x : x \in D\}$. We have therefore shown that bX is paracompact, which contradicts the choice of τ and establishes the theorem.

5.2. THEOREM. *Suppose X is scattered. Then X is m-Lindelöf if and only if bX is m-Lindelöf. In particular, if X is Lindelöf and scattered, then bX is Lindelöf.*

Proof. Since bX has a stronger topology than X, we infer that if bX is m-Lindelöf, then X is m-Lindelöf. For the converse, assume that the conclusion is false and put

$$\tau = \inf\{a_X\colon X \text{ is } m\text{-Lindelöf, scattered, and } bX \text{ is not } m\text{-Lindelöf}\}.$$

Choose an m-Lindelöf scattered space X such that $\tau = a_X$ and bX is not m-Lindelöf. We claim that τ is not a limit ordinal. If τ is a limit ordinal, then

$$\bigcap_{a<\tau} D_a = \emptyset \quad \text{with } D_a = D_a(X) \neq \emptyset \text{ for each } a < \tau.$$

Since X is m-Lindelöf, $\{D_a\colon a < \tau\}$ does not have the m-intersection property, so τ may be written as the supremum of at most m ordinal predecessors: $\tau = \sup a_i$, $i \in I$, $|I| \leqslant m$. Choose a cozero cover $\{U_s : s \in S\}$ which refines $\{X - D_{a_i}\}$ with $|S| \leqslant m$. Now $D_{a_i}(U_s) \subseteq D_{a_i} \cap U_s = \emptyset$ if $U \subseteq X - D_{a_i}$, so $a_{U_s} \leqslant a_i < \tau$ and U_s m-Lindelöf imply that bU_s is m-Lindelöf. Then $X = \bigcup U_s$ and $|S| \leqslant m$ imply that bX is m-Lindelöf, which is a contradiction. Hence we may write $\tau = \gamma+1$. Since D_γ is closed and discrete, we may write $D_\gamma = \{x_s\colon s \in S\}$ with $|S| \leqslant m$. Let $\mathscr{U}$ be a cover of X by zero sets of X. For each x_s, choose $Z_s \in U$ which contains x_s and a cozero set C_s such that $x_s \in C_s$ and $x_t \notin C_s$ for $s \neq t$. As in the proof of 5.1, $F = X - \bigcup C_s$ is a closed set, and the definition of τ guarantees that bF is m-Lindelöf. Similarly, each $C_s - Z_s$ is a cozero set, so the definition of τ guarantees that $b(C_s - Z_s)$ is m-Lindelöf. Hence $\mathscr{U}$ restricted to $F \cup \{C_s - Z_s\}$ has a subcover $\mathscr{U}'$ of cardinal at most m, so $\mathscr{U}$ has the subcover $\mathscr{U}' \cup \{Z_s\}$ of cardinal at most m. Therefore, bX is m-Lindelöf, which contradicts our choice of τ and establishes the theorem.

5.3. COROLLARY. *If X is an m-Lindelöf scattered space and each point is a G_δ-set, then $|X| \leqslant m$.*

Proof. The assumptions assure that bX is m-Lindelöf and discrete.

Remarks. (i) If a_X is a countable ordinal, one may establish a stronger version of Theorem 5.1, namely: *if X is scattered, countably paracompact, and collectionwise normal, then bX is paracompact.*

(ii) The conclusion of 5.2 for compact scattered spaces has been previously established by P. R. Meyer. It should be noted that the proof technique in [12] (based on transfinite induction with respect to dispersal order) strongly uses the observation that the dispersal order of a compact scattered space always is a non-limit ordinal, the fact which is not valid for Lindelöf spaces (there are countable subsets of the real line which have dispersal order ω_0).

Dr. R. Telgársky has informed us that, using the proof technique of [16], the proof of 5.1 can be shortened. He also points out that Theorem 5.2 was announced by Gewand [4] and that it follows from results of [17] and [18] dealing with topological games.

(iii) Since a topology stronger than a realcompact topology is realcompact, if X is (hereditarily) realcompact, then bX is also (hereditarily) realcompact.

(iv) Professor Kenneth Kunen has noted in a letter that under the continuum hypothesis, if X is compact, then bX is ω_1-Lindelöf if and only if bX is paracompact. Williams and Fleischman [19] have shown that if X is a product of finitely many compact linearly ordered spaces, then bX is $2^{\aleph_0}$-Lindelöf.

(v) The statement of Theorem 5.2 can be strengthened as follows: *Suppose a is an infinite cardinal and let $b_a X$ be the topology having for a base the intersections of at most a open sets of X. If X is m-Lindelöf and scattered and $a \leqslant m$, then $b_a X$ is m-Lindelöf.*

The proof is a modification of the proof of 5.2. A corollary is the following improvement on 5.3:

Suppose X is an m-Lindelöf scattered space with pseudocharacter a (that is, every point is an intersection of a open sets). Then $|X| \leqslant m + a$.

5.4. PROPOSITION. *A space X is almost-Lindelöf if and only if X is Lindelöf and $|\nu X - X| \leqslant 1$.*

Proof. Assume that X is almost-Lindelöf. We will first show that $|\nu X - X| \leqslant 1$. If $p, q \in \nu X - X$, $p \neq q$, let Z_p and Z_q be disjoint zero sets of X such that $p \in Z_p$ and $q \in Z_q$. Then $Z_p \cap X$ and $Z_q \cap X$ are disjoint zero sets of X neither of which is realcompact, contradicting the fact that X is almost-Lindelöf. To show that νX is Lindelöf, write $\nu X = X \cup \{p\}$. If $\mathscr{U}$ is a cozero cover of νX, choose $C \in \mathscr{U}$ such that C contains p. Since X is almost-Lindelöf and p is not an element of the zero set $X-C$ of X, $X-C$ is Lindelöf. Hence $\mathscr{U}$ restricted to $X-C$ has a countable subcover $\mathscr{U}'$, so $\mathscr{U}$ has the countable subcover $\mathscr{U}'\cup\{C\}$.

To prove the converse, assume νX is Lindelöf and $|\nu X-X|\leqslant 1$. If Z_1 and Z_2 are disjoint zero sets of X, then at most one of the sets $\mathrm{Cl}(Z_1)$ and $\mathrm{Cl}(Z_2)$ intersects $\nu X-X$. Hence at least one of the sets Z_1 and Z_2 is closed in νX and is therefore Lindelöf.

5.5. PROPOSITION. (i) *If bX is almost-Lindelöf, then* $C(bX)=\mathrm{Baire}(X)$.
(ii) *If* $C(bX)=\mathrm{Baire}(X)$, *then* $\nu bX=b\nu X$.

Proof. (i) Suppose bX is almost-Lindelöf and Z is a zero set of bX. Since bX is a P-space, Z and $X-Z$ are disjoint zero sets of bX. Therefore, either Z or $X-Z$ is Lindelöf. Since every open set of bX is a union of zero sets of X, either Z or $X-Z$ is a countable union of zero sets of X. Hence Z is a Baire set of X.

(ii) Since X is a G_δ-dense subspace of νX, bX is a G_δ-dense subspace of $b\nu X$, and $b\nu X$ is realcompact by Remark (iii) in 5.3. Hence, to prove $\nu bX=b\nu X$, it suffices to show that bX is C^*-embedded in $b\nu X$. Let Z_1 and Z_2 be disjoint zero sets of bX. Then, since $\mathrm{Baire}(X)=C(bX)$, Z_1 and Z_2 are zero sets of Baire functions on X, so Z_1 and Z_2 are Baire sets of X. Each Baire set of X is the restriction of a Baire set of νX. Therefore, there are disjoint Baire sets $\hat{Z}_1$ and $\hat{Z}_2$ of νX such that $\hat{Z}_i\cap X=Z_i$ for $i=1,2$. Since $\hat{Z}_1$ and $\hat{Z}_2$ are zero sets of $b\nu X$, the Urysohn extension theorem implies that bX is C^*-embedded in $b\nu X$.

5.6. PROPOSITION. *Suppose X is pseudocompact. If* $C(bX)=\mathrm{Baire}(X)$, *then bX is pseudo-$\aleph_1$-compact.*

Proof. Let $\{A_s\colon s\in S\}$ be a discrete family of cozero sets in bX. Let $A_s=\mathrm{coz}(f_s)$, where $f_s\in C(bX)$. If $S'\subseteq S$, then

$$\bigcup\{A_s : s\in S'\}=\mathrm{coz}(f_{S'}),\qquad \text{where } f_{S'}=\sum\{f_s : s\in S'\}\in C(bX),$$

so, by assumption, $\bigcup\{A_s\colon s\in S'\}$ is a Baire set in X. Hence $\{A_s\colon s\in S\}$ is a completely additive Baire family, so, by 3.4, S is countable.

The following result summarizes a number of equivalent conditions on the Baire topology of a compact space.

5.7. THEOREM. *The following statements are equivalent for a compact space X:*

(i) *X is scattered.*
(ii) *bX is Lindelöf.*
(iii) $C(bX)=\mathrm{Baire}(X)$.
(iv) *X is functionally countable.*
(v) *bX is almost-Lindelöf.*
(vi) *bX is pseudo-$\aleph_1$-compact.*

(vii) *Each closed discrete set in bX is countable.*

(viii) *Each open cover of bX has a subcover of cardinal less than c.*

Proof. (iv) $\Rightarrow$ (i) by 3.1 and (vi) $\Rightarrow$ (iv) by 3.2. Also, (v) $\Rightarrow$ (iii) by 5.5, (iii) $\Rightarrow$ (vi) by 5.6, and (i) $\Rightarrow$ (ii) by 5.2, while (ii) $\Rightarrow$ (v), (vii) $\Rightarrow$ (vi), and (ii) $\Rightarrow$ (vii) are immediate, so we have established the equivalence of (i)-(vii). Clearly, (ii) $\Rightarrow$ (viii), so it remains to show (viii) $\Rightarrow$ (i). Assume $f \in C(bX)$; then $\{f^{-1}(p) : p \in R\}$ is a disjoint open cover of bX, so, by (viii), $|f(X)| < c$. Hence, by 3.1, (i) is satisfied.

Remarks. The comments on the proof of the implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (viii) are found after Proposition 3.1. The implication (i) $\Rightarrow$ (iii) is proved in [11], 2.8, and the equivalence (i) $\Leftrightarrow$ (ii) is found in [12]. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are found in [8], Theorem 6. Finally, each of the references [11], [12], [8], [13], [15] contains additional statements equivalent to those found in Theorem 5.7.

5.8. COROLLARY. *βX is scattered if and only if X is pseudocompact and* $C(b\nu X) = \mathrm{Baire}(\nu X)$.

Proof. Assume that βX is scattered. Then X is pseudocompact (if not, by [5], 1.21, βX contains a copy of βN which is not scattered). In addition, $C(b\nu X) = \mathrm{Baire}(\nu X)$ follows from Theorem 5.7. The converse is immediate from Theorem 5.7, since $\nu X = \beta X$.

We note that even for countably compact scattered spaces, the condition $C(b\nu X) = \mathrm{Baire}(\nu X)$ in 5.8 is not equivalent to the condition $C(bX) = \mathrm{Baire}(X)$ (see Example 3 of Section 6); however, one may easily establish that the following two conditions are equivalent:

(i) $C(bX) = \mathrm{Baire}(X)$,

(ii) $C(b\nu X) = \mathrm{Baire}(\nu X)$ and $\nu bX = b\nu X$.

A number of additional conditions equivalent to the ones in 5.8 are also in [8], Theorem 9.

5.9. COROLLARY. *The following statements are equivalent:*

(i) $C(bX) = \mathrm{Baire}(X)$ *and X is pseudocompact.*

(ii) *βX is scattered and $\nu bX = b\nu X$.*

(iii) *bX is pseudo-$\aleph_1$-compact, X is pseudocompact, and $\nu bX = b\nu X$.*

Proof. (i) $\Rightarrow$ (ii) follows from 5.5 (ii) and 5.6. To establish (iii) $\Rightarrow$ (ii), observe that bX pseudo-$\aleph_1$-compact implies that $\nu bX = b\nu X = b\beta X$ is pseudo-$\aleph_1$-compact, so X is scattered by 5.7. Finally, to establish (ii) $\Rightarrow$ (i), we first note that X is pseudocompact by 5.8. Choose $f \in C(bX)$. Then f extends to a member $\hat{f} \in C(\nu bX) = C(b\nu X)$, so $\hat{f} \in \mathrm{Baire}(\nu X)$ by 5.8; hence $\hat{f}|X = f \in \mathrm{Baire}(X)$, which completes the proof.

5.10. COROLLARY. *The following statements are equivalent:*

(i) $C(bX) = \mathrm{Baire}(X)$ *and X is almost-compact.*

(ii) *bX is almost-Lindelöf and X is pseudocompact.*

Proof. To establish (ii) $\Rightarrow$ (i) it suffices (by 5.5 (i)) to show that X is almost-compact. Now, by 5.4 and (ii), $vbX = bvX = b\beta X$ contains at most one point not belonging to X; hence $|\beta X - X| \leqslant 1$, so X is almost-compact.

To establish (i) $\Rightarrow$ (ii), we first note that, by 5.9, βX is scattered and $vbX = bvX$, so $b\beta X = vbX$ is Lindelöf by 5.7. Also, $|\beta X - X| \leqslant 1$, so $|vbX - X| \leqslant 1$; hence, by 5.4, bX is almost-Lindelöf.

6. Examples. In this section we give several examples related to the results found in Sections 3-5.

Example 1. There exists an almost-compact, functionally countable, scattered space X such that bX is not pseudo-$\aleph_1$-compact and such that the Baire functions on X do not have countable image (contrast with 3.1, 3.2 and 5.7-5.9).

Let $N' = N \cup \{\infty\}$ and $D' = D \cup \{\infty'\}$ be the one-point compactifications of the discrete spaces of cardinal $\aleph_0$ and $\aleph_1$, respectively, and let $X = N' \times D' - (\infty, \infty')$. It is easily seen that X is almost compact and scattered, and X is functionally countable since $\beta X = N' \times D'$ is scattered (see 3.1). To see that bX is not pseudo-$\aleph_1$-compact (and hence not functionally countable by 3.2), observe that the uncountable closed discrete set $\{\infty\} \times D$ is open in bX.

Example 2. Assuming Lusin's Hypothesis ($2^{\aleph_0} = 2^{\aleph_1}$), there exists a perfectly normal pseudo-$\aleph_1$-compact scattered space X which contains an uncountable closed discrete subspace (contrast with 4.3).

Assuming Lusin's Hypothesis, there exists a discrete C^*-embedded subset D of $\beta N - N$ with $|D| = \aleph_1$. Now define $X = N \cup D$ with the subspace topology from βN. The space X is separable, and hence pseudo-$\aleph_1$-compact. D is an uncountable closed discrete subset of X. It is proved in [9] that X is perfectly normal.

Example 3. Let $X = [0, \omega_1)$ with the usual order topology. Then X is a countably compact scattered space, βX is scattered, and bX is discrete (since X is first-countable), so bX is not pseudo-$\aleph_1$-compact (contrast with 5.6 and 5.9).

Example 4. Let $X = [0, \omega_2)$ with the order topology. Then X is countably compact and bX is hereditarily collectionwise normal, almost-Lindelöf (hence $C(bX) = \mathrm{Baire}(X)$), and pseudo-$\aleph_1$-compact, but bX is not Lindelöf (cf. 5.7 (ii) and (iii); see also Remark (ii) in 3.2).

To prove the claims about X, first note that bX is homeomorphic to the space

$$X_0 = \{a < \omega_2 : a \text{ is not the limit of an increasing sequence of ordinals}\}$$

and X_0 is a subspace of an ordered space; hence bX is hereditarily collectionwise normal. We next show that each $f \in C(bX)$ is constant on a tail of $[0, \omega_2)$. First, observe that each uncountable subset of bX has a limit point, so bX is pseudo-$\aleph_1$-compact, and thus, by 3.2, functionally countable. Hence there exists $r \in R$ such that $S_1 = f^{-1}(r)$ is cofinal in $[0, \omega_2)$. If no tail is contained in S_1, then for some n the set

$$S_2 = f^{-1}[(-\infty, r-1/n) \cup (r+1/n, +\infty)]$$

is also cofinal in $[0, \omega_2)$. Choose an alternating increasing ω_1-sequence from the sets S_1 and S_2 and let a be a limit point of the sequence. Then $f(a) = r < r-1/n$ or $f(a) = r > r+1/n$ for some n, which is a contradiction. Hence each member of $C(bX)$ is constant on some tail of $[0, \omega_2)$. Let Z_1 and Z_2 be disjoint zero sets in bX and choose $h \in C(bX)$ such that $h|Z_1 = 0$ and $h|Z_2 = 1$. Since h is constant on some tail, either Z_1 or Z_2 is bounded, say $Z_1 \subseteq [0, a]$ for some $a < \omega_2$. But $b[0, a]$ is Lindelöf (by 5.7 since $[0, a]$ is compact and scattered), so Z_1, being closed in $b[0, a]$, is also Lindelöf. Hence bX is almost-Lindelöf. bX is not Lindelöf because X, which has a weaker topology, is not Lindelöf.

Example 5. Let X be the topological sum of two copies of the ordinal space $[0, \omega_2)$. Then X is countably compact, bX is hereditarily collectionwise normal, pseudo-$\aleph_1$-compact, and $C(bX) = \text{Baire}(X)$ (see Example 4), but bX is not almost-Lindelöf (the two copies of $b[0, \omega_2)$ are disjoint zero sets, neither of which is Lindelöf). This should be compared to 5.7 (v).

Example 6. There is a pseudo-$\aleph_1$-compact locally Lindelöf P-space which is not Lindelöf — in fact, not normal — and which contains an uncountable closed discrete subset (cf. 5.7 (ii) and (iv) and 4.3 (ii)).

Let D be a discrete space of cardinal $\aleph_1$ and let $\mathscr{E}$ be a collection of uncountable subsets of D maximal with respect to the property that any two elements have countable intersection. Then $|\mathscr{E}| > \aleph_1$. For each $E \in \mathscr{E}$ let p_E be a point not in D and let $X = D \cup \{p_E : E \in \mathscr{E}\}$. Topologize X by making points of D isolated and letting neighborhoods of p_E be $\{p_E\} \cup (E - C)$, where C is a countable subset of D. (This is a modification of Isbell's example Ψ; see ([5], 5I)). It is easy to show that X is a P-space and that the sets $\{p_E\} \cup E$ are Lindelöf. To prove that X is pseudo-$\aleph_1$-compact, suppose $f \in C(X)$ and $f(X)$ is uncountable. Then $\{f^{-1}(r) : r \in f(X)\}$ is a partition of X into uncountably many non-empty open sets. Since D is dense in X, for each $r \in f(X)$ we may choose $d_r \in D \cap f^{-1}(r)$. By the maximality of $\mathscr{E}$, there exists $E \in \mathscr{E}$ such that E contains uncountably many d_r's. But $f|E \cup \{p_E\}$ is a continuous function with uncountable image defined on a Lindelöf P-space, contradicting 3.2. Hence X is functionally countable (and therefore pseudo-$\aleph_1$-compact by 3.2). Since $\{p_E : E \in \mathscr{E}\}$ is an uncountable closed discrete subset of X, the non-normality of X follows from 4.3.

7. Some unresolved questions. In this section we list several questions which are suggested by the above-given material.

QUESTION 1. If X is normal and pseudo-$\aleph_1$-compact, does each subset of X which has cardinal c have a limit point? Is there any cardinal a such that if X is pseudo-$\aleph_1$-compact and normal, then each subset of X which has cardinal a has a limit point? (**P 1205**)

QUESTION 2. Is every Lindelöf functionally countable space a countable union of scattered (or closed scattered) subspaces? What if the space X is a P-space? If bX is Lindelöf, is X the countable union of (closed) scattered subspaces? (**P 1206**)

QUESTION 3. If X is Lindelöf and functionally countable, is bX pseudo-$\aleph_1$-compact? Does every Baire function on X have countable image? (**P 1207**)

QUESTION 4. What condition on a compact space X guarantees that bX is normal? (**P 1208**)

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