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Geometrical interpretation of the sinh-Gordon equation

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Abstract. In a three-dimensional pseudo-Riemannian manifold of constant curvature consider a spacelike (resp. timelike) surface of constant negative (resp. positive) Gaussian curvature. Then the asymptotic curves are everywhere real and distinct, and the function 2ψ (= the angle between the asymptotic directions) satisfies, relative to the Tchebycheff coordinates, a sine-Gordon (resp. sinh-Gordon) equation. An example of such a manifold is $SL(2; R)$ with the biinvariant metric.

1. Introduction. It is well known that the sine-Gordon equation (SGE)

$$u_{xx}-u_{tt} = \sin u \tag{1}$$

has a geometrical interpretation in terms of the surfaces of constant negative curvature in the three-dimensional euclidean space. We will show in this paper that by studying surfaces of constant Gaussian curvature in a three-dimensional pseudo-Riemannian manifold of constant curvature one is led to geometrical interpretations of (1) and of the sinh-Gordon equation (SHGE)

$$u_{xx}-u_{tt} = \sinh u. \tag{2}$$

2. Pseudo-Riemannian geometry. In this section we will give a review of local pseudo-Riemannian geometry, using moving frames. Let M be a smooth manifold of dimension m, with the local coordinates x^α. (In this section all small Greek indices run from 1 to m.) A pseudo-Riemannian metric in M is given by the non-degenerate quadratic differential form

$$ds^2 = \sum_{\alpha,\beta} G_{\alpha\beta}(x^1, \ldots, x^m)\,dx^\alpha\,dx^\beta, \qquad G_{\alpha\beta} = G_{\beta\alpha}. \tag{3}$$

The metric is called *Riemannian* if the form is positive definite and Lorentzian if it is of signature $+ \ldots + -$.

Let $x \in M$ and let T_x, T_x^* be respectively the tangent and cotangent spaces

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of M at x. A frame at x is an ordered set of linearly independent vectors $e_\alpha \in T_x$. The essence of the method of moving frames is to free the frames from local coordinates, a freedom which gives handsome returns. To e_α is associated a dual coframe $\omega^\beta \in T_x^*$. When they are defined over a neighbourhood, ω^β can be identified with a linear differential form. Relative to ω^β we can write

$$ds^2 = \sum g_{\alpha\beta}\, \omega^\alpha \omega^\beta, \quad g_{\alpha\beta} = g_{\beta\alpha}. \tag{3a}$$

The Levi-Civita connection is given by

$$De_\alpha = \sum \omega_\alpha^\beta e_\beta, \tag{4}$$

where the connection forms ω_α^β are determined, uniquely, by the conditions

$$d\omega^\alpha = \sum \omega^\beta \wedge \omega_\beta^\alpha, \tag{5}$$

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = dg_{\alpha\beta}. \tag{6}$$

Geometrically the first condition (5) means the "absence of torsion". In the second condition (6) the $\omega_{\alpha\beta}$ are defined by

$$\omega_{\alpha\beta} = \sum g_{\beta\gamma}\, \omega_\alpha^\gamma \tag{7}$$

and the condition means the preservation of the scalar product of vectors under parallelism. We use $g_{\alpha\beta}$ to lower indices, as in classical tensor analysis.

The curvature forms are defined by

$$\Omega_\alpha^\beta = d\omega_\alpha^\beta - \sum_\gamma \omega_\alpha^\gamma \wedge \omega_\gamma^\beta, \tag{8}$$

$$\Omega_{\alpha\beta} = \sum g_{\beta\gamma}\, \Omega_\alpha^\gamma. \tag{9}$$

It can be proved that

$$\Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0. \tag{10}$$

The pseudo-Riemannian metric (3a) is said to be of *constant curvature c* if

$$\Omega_{\alpha\beta} = -c\omega_\alpha \wedge \omega_\beta, \tag{11}$$

where

$$\omega_\alpha = \sum g_{\alpha\beta}\, \omega^\beta. \tag{12}$$

In applications it will be advantageous to use frames, where $g_{\alpha\beta} = \text{const}$, such as orthonormal frames in the Riemannian case. Then (6) becomes

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0. \tag{13}$$

3. Surfaces in three-dimensional manifolds. Let M be a three-dimensional pseudo-Riemannian manifold and

$$f: S \to M \tag{14}$$

be an immersed surface. In a neighbourhood on S we take frames $xe_1 e_2 e_3$, so that e_1, e_2 are tangent vectors to S at $x \in S$. Restricted to these frames we have

$$\omega^3 = 0, \tag{15}$$

and the induced pseudo-Riemannian metric on S is

$$I = g_{11}(\omega^1)^2 + 2g_{12}\omega^1\omega^2 + g_{22}(\omega^2)^2, \tag{16}$$

when I stands for first fundamental form. The surface S is said to be *spacelike* (resp. *timelike*) at x if I is definite (resp. indefinite).

By exteriorly differentiating (15) and using (5), we get

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0. \tag{17}$$

It follows that

$$\omega_i^3 = \sum h_{ik}\omega^k, \quad 1 \leqslant i,\ k \leqslant 2, \tag{18}$$

where

$$h_{12} = h_{21}. \tag{19}$$

From now on we use orthonormal frames, so that

$$\begin{aligned} g_{\alpha\beta} &= 0, \quad \alpha \neq \beta,\ 1 \leqslant \alpha,\ \beta \leqslant 3, \\ g_{\alpha\alpha} &= \pm 1. \end{aligned} \tag{20}$$

By (13) the matrix

$$(\omega_{\alpha\beta}) = \begin{pmatrix} g_{11}\omega_1^1 & g_{22}\omega_1^2 & g_{33}\omega_1^3 \\ g_{11}\omega_2^1 & g_{22}\omega_2^2 & g_{33}\omega_2^3 \\ g_{11}\omega_3^1 & g_{22}\omega_3^2 & g_{33}\omega_3^3 \end{pmatrix} \tag{21}$$

is anti-symmetric. This implies in particular

$$\omega_1^1 = \omega_2^2 = \omega_3^3 = 0. \tag{22}$$

The second fundamental form is defined by

$$\begin{aligned} II &= -(dx, De_3) = -(g_{11}\omega^1\omega_3^1 + g_{22}\omega^2\omega_3^2) \\ &= g_{33}(\omega^1\omega_1^3 + \omega^2\omega_2^3) \\ &= g_{33}\{h_{11}(\omega^1)^2 + 2h_{12}\omega^1\omega^2 + h_{22}(\omega^2)^2\}. \end{aligned} \tag{23}$$

The curves defined by

$$II = 0 \tag{24}$$

are the asymptotic curves.

The principal directions of S are determined by the equations

(25) $$g_{33}(h_{11}\omega^1+h_{12}\omega^2) = \lambda g_{11}\omega^1, \quad g_{33}(h_{12}\omega^1+h_{22}\omega^2) = \lambda g_{22}\omega^2,$$

where λ is the corresponding principal curvature. It follows that the principal curvatures are the roots of the equation

(26) $$\begin{vmatrix} g_{33}h_{11}-\lambda g_{11} & g_{33}h_{12} \\ g_{33}h_{12} & g_{33}h_{22}-\lambda g_{22} \end{vmatrix} = 0.$$

The Gaussian curvature of S is the product of the principal curvatures and is given by

(27) $$K = \frac{1}{g_{11}g_{22}}(h_{11}h_{22}-h_{12}^2).$$

Put

(28) $$\varepsilon = g_{11}g_{22} = \pm 1,$$

so that

(29) $$K = \varepsilon(h_{11}h_{22}-h_{12}^2).$$

The surface S is spacelike or timelike according as $\varepsilon = +1$ or -1.

4. Surfaces of constant Gaussian curvature. From now on we suppose M to be of constant curvature and the surface S to be of constant Gaussian curvature

(30) $$K = -\varepsilon b^2, \quad b = \text{const} \neq 0,$$

i.e., of constant negative or positive Gaussian curvature, according as S is spacelike or timelike. In both cases the asymptotic directions are real and distinct.

We choose frames so that e_1, e_2 are along the principal directions, i.e.,

(31) $$h_{12} = 0.$$

Then the asymptotic directions are defined by

(32) $$h_{11}(\omega^1)^2+h_{22}(\omega^2)^2 = 0.$$

The tangent plane T_x has a definite or indefinite metric according as S is spacelike or timelike. Let 2ψ be the angle between the asymptotic directions relative to that metric. Since $h_{11}h_{22} = -b^2$, we have the two cases:

Case 1. S is spacelike ($\varepsilon = 1$). Then we have

(33) $$h_{11} = b\cot\psi, \quad h_{22} = -b\tan\psi.$$

Case 2. S is timelike ($\varepsilon = -1$). We have

(34) $$h_{11} = b\coth\psi, \quad h_{22} = -b\tanh\psi.$$

Over S we have defined a field of orthonormal frames $xe_1e_2e_3$ such that e_3 is along the normal and e_1, e_2 along the principal directions at x. We wish to show that there are local coordinates, to be called *Tchebycheff coordinates*, relative to which the connection forms of this field of frames take a simple form. At this stage we relate the frames to local coordinates.

Restricted to the field of frames described above we have, since M is of constant curvature,

(35) $$\Omega_{13} = \Omega_{23} = 0,$$

whence

(36) $$\Omega_1^3 = \Omega_2^3 = 0.$$

Equations (5) and (8) give respectively

(37) $$\begin{aligned} d\omega^1 &= \omega^2 \wedge \omega_2^1, & d\omega^2 &= \omega^1 \wedge \omega_1^2, \\ d\omega_1^3 &= \omega_1^2 \wedge \omega_2^3, & d\omega_2^3 &= \omega_2^1 \wedge \omega_1^3. \end{aligned}$$

On the other hand, the anti-symmetry of the matrix (21) gives

(38) $$\omega_2^1 = -\varepsilon\omega_1^2.$$

Hence we can write

(39) $$d\omega^1 = -\varepsilon\omega^2 \wedge \omega_1^2, \quad d\omega_2^3 = -\varepsilon\omega_1^2 \wedge \omega_1^3.$$

Equation (8) also gives

(40) $$d\omega_1^2 = \omega_1^3 \wedge \omega_3^2 + \Omega_1^2 = -g_{22}g_{33}\,\omega_1^3 \wedge \omega_2^3 + \Omega_1^2,$$

where

(41) $$\Omega_1^2 = g_{22}\,\Omega_{12} = -g_{22}\,c\omega_1 \wedge \omega_2 = -g_{11}\,c\omega^1 \wedge \omega^2.$$

Hence the above equation can be written

(42) $$d\omega_1^2 = (g_{22}g_{33}b^2 - g_{11}c)\omega^1 \wedge \omega^2.$$

Let u, v be local coordinates such that

(43) $$\omega^1 = Adu, \quad \omega^2 = Cdv.$$

Geometrically this means using the lines of curvature (= integral curves of principal directions) as parametric curves. Then

$$d\omega^1 = -A_v du \wedge dv = -\varepsilon C dv \wedge \omega_1^2, \quad d\omega^2 = C_u du \wedge dv = Adu \wedge \omega_1^2.$$

It follows that

(44) $$\omega_1^2 = -\varepsilon \frac{A_v}{C} du + \frac{C_u}{A} dv.$$

By (18) we have

(45) $$\omega_1^3 = h_{11} A du, \quad \omega_2^3 = h_{22} C dv.$$

Substituting (44), (45) into (37), (39), we get

(46) $$(Ah_{11})_v = \varepsilon h_{22} A_v, \quad (Ch_{22})_u = \varepsilon h_{11} C_u.$$

We now consider the two cases separately:

Case 1. $\varepsilon = +1$. Equation (46) gives

$$(A \cot \psi)_v = -\tan \psi A_v, \quad (C \tan \psi)_u = -\cot \psi C_u.$$

The first equation can be written

$$\left(\log \frac{A}{\sin \psi}\right)_v = 0,$$

so that $A/\sin \psi$ is a function of u only. Absorbing this function into u, we can suppose

$$A = \sin \psi.$$

Similarly, we can choose v such that

$$C = \cos \psi.$$

These (u, v)-coordinates are called the *Tchebycheff coordinates*. By (44), we have

$$\omega_1^2 = -\psi_v du - \psi_u dv,$$

and (42) gives

(47) $$\psi_{uu} - \psi_{vv} = (-g_{22} g_{33} b^2 + g_{11} c) \sin \psi \cos \psi.$$

By choosing

$$c = 0, \quad b = 1, \quad g_{\alpha\alpha} = 1, \quad u = t, \quad v = x,$$

and considering 2ψ as the dependent variable, this reduces to (1).

Case 2. $\varepsilon = -1$. Equation (46) gives

$$(A \coth \psi)_v = +\tanh \psi A_v, \quad (C \tanh \psi)_u = \coth \psi C_u.$$

Exactly the same manipulations as in Case 1 show that we can choose u, v, so that

$$A = \sinh \psi, \quad C = \cosh \psi.$$

By (44) we have then

$$\omega_1^2 = \psi_v du + \psi_u dv.$$

It follows from (42) that

(48) $$\psi_{uu} - \psi_{vv} = (+g_{22} g_{33} b^2 - g_{11} c) \sinh \psi \cosh \psi.$$

This is essentially the SHGE, the expression in the parenthesis being a constant.

We summarize the results in the following theorem:

In a three-dimensional pseudo-Riemannian manifold of constant curvature consider a spacelike (resp. timelike) surface of constant negative (resp. positive) Gaussian curvature. Then the asymptotic directions are everywhere real and distinct, and the function 2ψ *(= the angle between the asymptotic directions) satisfies, relative to the Tchebycheff coordinates, a SGE (resp. SHGE).*

5. SL (2; *R*). An important example of a three-dimensional pseudo-Riemannian manifold of constant curvature is given by the special linear group in two real variables:

$$\mathrm{SL}\,(2; R) = \left\{ X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \middle| \, xt - yz = 1 \right\}, \tag{49}$$

provided with the biinvariant metric. The latter is defined by

$$ds^2 = \tfrac{1}{2}\,\mathrm{Tr}\,(dXX^{-1}\,dXX^{-1}), \tag{50}$$

and is Lorentzian. This metric has curvature -1, according to the definition of Section 2.

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