COLLOQUIUM MATHEMATICUM

VOL. XLIV 1981 FASC. 1

*ALMOST PERIODIC COMPACTIFICATIONS OF DIRECT AND SEMIDIRECT PRODUCTS**

BY

PAUL MILNES (SHEFFIELD, ENGLAND)

0. PRELIMINARIES

For a semitopological semigroup S let $C(S)$ denote the C^*-algebra of continuous, bounded, complex-valued functions on S. In the dual space $C(S)^*$, the set βS of multiplicative linear functionals on $C(S)$ is contained, which, when furnished with the relative weak* topology, is just the Stone-Čech compactification of S if S is completely regular. The *left translate* $L_s f$ of $f \in C(S)$ by $s \in S$ is defined by $L_s f(t) = f(st)$ for all $t \in S$; the *right translate* $R_s f$ is defined analogously. A function $f \in C(S)$ is called (*weakly*) *almost periodic* provided $\{L_s f\colon s \in S\}$ is (weakly) relatively compact in $C(S)$. We quote

THEOREM 0.1. *The following assertions about a function $f \in C(S)$ are equivalent*:

(i) *f is weakly almost periodic.*

(ii) *$\{R_s f\colon s \in S\}$ is weakly relatively compact in $C(S)$.*

(iii) *$\{L_s f\colon s \in S\}$ (or $\{R_s f\colon s \in S\}$) is relatively $\sigma(C(S), \beta S)$-compact in $C(S)$.*

(iv) Grothendieck's criterion. *Whenever $\{s_n\}$ and $\{t_m\}$ are sequences in S such that the limits*

$$A = \lim_m \lim_n f(s_n t_m) \quad \textit{and} \quad B = \lim_n \lim_m f(s_n t_m)$$

both exist, then $A = B$.

The reader is referred to [1] or [3] for a proof of this theorem and a thorough treatment of (weakly) almost periodic functions in general.

* Research supported in part by NRC grant A7857.

(We note that assertions (i) and (ii) remain equivalent if the word "weakly" is omitted from them.)

The set of all (weakly) almost periodic functions on S is denoted by $AP(S)$ $(W(S))$, and forms a left and right translation invariant C^*-subalgebra of $C(S)$ whose spectrum S^a (S^w) is a compact topological (semitopological) semigroup called the *(weakly) almost periodic compactification* of S; $a(\omega)$ denotes the canonical continuous homomorphism of S into S^a (S^w).

We also note that each almost periodic function is weakly almost periodic and, when S is a locally compact group G (and somewhat more generally [9], Theorem 8), each function $f \in W(G)$ is (both left and right) *uniformly continuous*, i.e., given $\varepsilon > 0$ there is a neighbourhood V of $e \in G$ such that $|f(s)-f(t)| \leqslant \varepsilon$ whenever $s^{-1}t \in V$ or $st^{-1} \in V$.

Another fact we need about $W(G)$ (G a group) is that $W(G)$ always admits a unique invariant mean $m \in W(G)^*$, $m(f) = m(L_s f) = m(R_s f)$ for all $s \in G$, $f \in W(G)$; and $W(G)$ splits into a direct sum $AP(G) \oplus W_0(G)$, $W_0(G)$ consisting of those functions $f \in W(G)$ for which $m(|f|) = 0$.

LEMMA 0.1. *Let G_1 and G_2 be topological groups and let φ be a continuous homomorphism of G_1 onto G_2. Then the adjoint φ^* is an isometry of $AP(G_2)$ into $AP(G_1)$, of $W(G_2)$ into $W(G_1)$, and of $W_0(G_2)$ into $W_0(G_1)$.*

Finally, if G is locally compact with left Haar measure μ and admits an F-sequence (which it will if it is amenable and σ-compact [6]), i.e., a sequence $\{U_n\}$ of compact subsets of G such that

$$\bigcup_1^\infty U_n = G$$

and, in particular,

$$\mu(sU_n \cap U_n)/\mu(U_n) \to 1, \qquad s \in G,$$

then

$$\mu(U_n)^{-1} \int_{U_n} f(s)\, d\mu(s) \to m(f), \qquad f \in W(G).$$

1. DIRECT PRODUCTS

Let $S = T_1 \times T_2$ be a direct product of semitopological semigroups; to avoid trivialities we always assume that at least one of T_1 and T_2 is not compact. It follows directly that every function f in $AP(T_1)$ extends to a function h in $AP(S)$ by the formula

$$h(s,t) = f(s), \qquad s \in T_1, t \in T_2$$

(see Lemma 0.1). A similar statement holds for functions in $AP(T_2)$. It follows from general theory that the C^*-subalgebra $C_a(S)$ of $C(S)$ generated by all functions of these two kinds is canonically isomorphic (via $((a_{T_1}\times a_{T_2})^*)^{-1}$) to $C(T_1^a\times T_2^a)$. If $C_a(S)=AP(S)$, then S^a is canonically isomorphic to $T_1^a\times T_2^a$; we write $S^a=T_1^a\times T_2^a$.

The following result, generalizing Corollary 4.4 of deLeeuw and Glicksberg [5], was proved in [2].

THEOREM 1.1. *Let $S=T_1\times T_2$ be a direct product of semitopological semigroups, where T_1 has a right identity and T_2 has a left identity. Then $S^a=T_1^a\times T_2^a$.*

Remark. An example was given in [2] to show that the hypothesis concerning the identities is necessary: if $T_1=[0,1]$ and $T_2=R$ are both given left-zero multiplication ($st=s$ for all s and t), then S is also a left-zero semigroup, $AP(S)=C(S)$ and $S^a\neq T_1^a\times T_2^a$. (It is interesting to note that if one adjoins a discrete identity u to T_2 in this example, then Theorem 1.1 implies $\big(T_1\times(\{u\}\cup T_2)\big)^a=T_1\times(\{u\}\cup T_2)^a$.)

Things are more complicated for the weakly almost periodic functions. We begin by noting that the analogue of the first paragraph of this section, with AP, a's and α's replaced by W, w's and ω's, respectively, holds. To state the next theorem, which involves ideas in [5] and [10], and a part of which was proved in [2], we need some notation. For $f\in C(T_1\times T_2)$, define $f_s\in C(T_2)$ and $f^t\in C(T_1)$ by

$$f_s(t)=f(s,t)=f^t(s),\qquad s\in T_1,\ t\in T_2,$$

and put

$$A_f=\{f_s: s\in T_1\}\quad\text{and}\quad B_f=\{f^t: t\in T_2\}.$$

THEOREM 1.2. *Let $S=T_1\times T_2$ be a direct product of semitopological semigroups, where T_1 has a right identity and T_2 has a left identity. Then the following assertions about S are equivalent:*

(i) $S^w=T_1^w\times T_2^w$.

(ii) $W(S)=C_w(S)$.

(iii) *Whenever $f\in W(S)$ and $\{s_\beta\}\subset T_1$, $\{t_\nu\}\subset T_2$ are nets such that $\omega(s_\beta)\to x\in T_1^w$ and $\omega(t_\nu)\to y\in T_2^w$, then the joint limit $\lim\limits_{\beta,\nu} f(s_\beta,t_\nu)$ exists.*

(iv) *For every $f\in W(S)$, one of A_f and B_f is relatively compact (in the norm topology).*

It was shown in [2] that the direct product $S=T\times G$ satisfies $S^w=T^w\times G^w=T^w\times G$ if G is a compact topological group and T is a semitopological semigroup with right identity; it was also shown that, for the left group $S=[0,1]\times R$ ($(x,s)(y,t)=(x,s+t)$ for all $x,y\in[0,1]$, $s,t\in R$),

$$S^w\neq[0,1]^w\times R^w=[0,1]\times R^w.$$

It is known that $(G\times G)^w \neq G^w\times G^w$ if G is Z or R (see [5] and [2]); the next result is a mild generalization and shows that $(G\times G)^w$ will seldom equal $G^w\times G^w$ (if G is not compact). See, however, Theorem 2.5 in Section 2.

THEOREM 1.3. *$W_0(G\times G)\setminus C_w(G\times G)$ contains an isometric copy of l^∞ if $G=R$ or Z, or if G admits a continuous homomorphism onto R or Z.*

Proof. We first show that if $f\in W(Z)\setminus C_0(Z)$, then the function h on $Z\times Z$, defined by $h(m,n)=f(n)$ if $m=n$, and by $h(m,n)=0$ otherwise, is in $W_0(Z\times Z)\setminus C_w(Z\times Z)$. Consider a sequence $\{L_{(m_j,n_j)}h\}=\{h_j\}$ of left translates of h; we must show that a subsequence of this sequence converges weakly in $C(Z\times Z)$, and we may assume for a start that one of the following cases holds:

1. $m_j-n_j=k$ a fixed constant independent of j,
2. $|m_j-n_j|\to\infty$ as $j\to\infty$.

In case 1, if we assume as well that $L_{n_j}f\to f_0$ weakly in $C(Z)$ and define $h_0\in C(Z\times Z)$ by $h_0(m,n)=f_0(n)$ if $m=n$, and by $h_0(m,n)=0$ otherwise, then $h_j\to L_{(k,0)}h_0$ weakly in $C(Z\times Z)$.

In case 2, suppose that $\nu\in\beta(Z\times Z)$ and $\{(p_k,q_k)\}\subset Z\times Z$ is such that

$$\lim_{k\to\infty} h_j(p_k,q_k)=\nu(h_j)\quad\text{for all } j.$$

Then

$$\lim_j \nu(h_j)=0;$$

hence $h_j\to 0$ weakly in $C(Z\times Z)$, and $h\in W_0(Z\times Z)$. Now, since $f\notin C_0(Z)$, there are a net $\{n_\beta\}\subset Z$ and a $p\in Z^w\setminus\omega(Z)$ such that $f(n_\beta)\to p(f)\neq 0$; hence

$$\lim_\beta h(n_\beta,n_\beta)=p(f),\qquad \lim_\gamma h(n_\beta,n_\gamma)=0\quad\text{for all }\beta$$

and Theorem 1.2 implies that $\|h-h'\|_\infty\geqslant|p(f)|$ for all $h'\in C_w(Z\times Z)$.

The proof for $Z\times Z$ is concluded by referring to Theorem 4.6 in [4], where it is shown that, for many groups G, including abelian ones and hence Z, $W_0(G)\setminus C_0(G)$ contains an isometric copy of l^∞.

The proof for $R\times R$ can be conducted along similar lines by considering functions $h\in C(R\times R)$,

$$h(s,t)=\max\{1-|s-t|,0\}f\big((s+t)/2\big),\qquad f\in W(R),$$

and the last assertion of the theorem follows from Lemma 0.1.

2. SEMIDIRECT PRODUCTS

A topological group G is a *semidirect product* if it contains a closed normal subgroup N and another closed subgroup K such that $N\cap K=\{e\}$, $G=NK$, and G is homeomorphic to the product space $K\times N$. The group G

is the direct product of N and K if and only if K is also normal. Also, the quotient group G/N is always isomorphic to K, and thus every function f in $AP(K)$ ($W(K)$) extends to a function h in $AP(G)$ ($W(G)$) by the formula $h(st) = f(t)$, $s \in N$, $t \in K$ (Lemma 0.1). However, in the examples which follow, only the constant functions on N extend to functions in $AP(G)$ and only linear combinations of constant functions and functions in $C_0(N)$ extend to functions in $W(G)$. We remark that for the first group N is compact, for the second K is compact and the last one has neither N nor K compact. (Of course, if both N and K are compact, then so is G and none of the pathologies under consideration here will occur.) Also, each of these groups has inequivalent left and right uniform structures, which seems to be at the heart of the matter here: for these groups, many (left and right) uniformly continuous functions (including all non-trivial almost periodic functions) on N do not extend to functions uniformly continuous on G. Thus the (left and right) uniformly continuous functions on a topological group are not the class of functions uniformly continuous with respect to any uniformity that behaves "properly" with respect to the taking of subgroups. By contrast, the left uniformly continuous functions on G are precisely those functions uniformly continuous with respect to the left uniformity of G, and the left uniformity of the topological group N is the same as the uniformity which N gets by virtue of being a subspace of G furnished with its left uniformity; so, it follows from a general theorem of Katětov [8] (see [7] for the particular case required here) that every bounded left uniformly continuous function on N extends to a left uniformly continuous function on G.

2.1. The semidirect product $R\times R\times T$. Here R is the set of additive real numbers, T is the circle group and the product in $G = R\times R\times T$ is defined by

$$(x, y, \exp[i\theta])(x', y', \exp[i\theta']) = \big(x+x', y+y', \exp[i(\theta+\theta'+xy')]\big).$$

LEMMA 2.1. *Let $f \in W(G)$. Then*

$$\lim_{x^2+y^2\to\infty} \max\{|f(x, y, w)-f(x, y, w')|: w, w' \in T\} = 0.$$

Proof. For each $\delta > 0$, define W_δ by

$$W_\delta = \{(x, y, w): |x| \leqslant \delta,\ |y| \leqslant \delta,\ |w-1| \leqslant \delta\}.$$

Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there is a $\delta = \delta(\varepsilon) > 0$ such that $|f(x, y, w)-f(x', y', w')| \leqslant \varepsilon$ whenever

$$(x', y', w')(x, y, w)^{-1} = \big(x'-x, y'-y, \exp[iy(x-x')]w'/w\big) \in W_\delta$$

or

$$(x, y, w)^{-1}(x', y', w') = \big(x'-x, y'-y, \exp[ix(y-y')]\, w'/w\big) \in W_\delta.$$

Suppose that $|x| \geqslant 2\pi/\delta$, $w, w' \in T$ and $\delta' > 0$ is such that $\delta' \leqslant \delta$ and $\exp[i\delta' x] = w/w'$. Then

$$|f(x, y, w) - f(x, y, w')|$$
$$\leqslant |f(x, y, w) - f(x, y+\delta', w)| + |f(x, y+\delta', w) - f(x, y, w')| \leqslant 2\varepsilon.$$

A similar calculation can be done if $|y| \geqslant 2\pi/\delta$.

THEOREM 2.1. *$AP(G) \simeq AP(R\times R)$; hence $G^a \simeq (R\times R)^a$.*

Proof. The isomorphism between $AP(G)$ and $AP(R\times R)$ intended here takes a function $h \in AP(R\times R)$ and extends it to a function $f \in C(G)$ by the formula $f(x, y, w) = h(x, y)$. It follows from the remarks at the beginning of this section that all functions obtained in this way are in $AP(G)$; and, if $f \in AP(G)$ and $w, w' \in T$, then the function

$$(x, y) \to f(x, y, w) - f(x, y, w')$$

is in $AP(R\times R)$ and in $C_0(R\times R)$ (Lemma 2.1); hence this function is zero and f is obtained as above.

THEOREM 2.2. *$W(G) \simeq W(R\times R) + C_0(G)$.*

Proof. Note that the isomorphism intended here is like that of Theorem 2.1. If $f \in W(G)$, then $h(x, y) = f(x, y, 1)$ defines a function $h \in W(R\times R)$, and Lemma 2.1 shows that g, defined by $g(x, y, w) = f(x, y, w) - f(x, y, 1)$, is in $C_0(G)$.

Remarks. (a) The sum in Theorem 2.2 is not direct as

$$W(R\times R) \cap C_0(G) \neq \{0\}.$$

(b) G^w can be obtained by adjoining the points of $(R\times R)^w \setminus (R\times R)$ to G and defining a neighbourhood of such a point p to be a set of the form

$$\big(V \setminus \omega(R\times R)\big) \cup \{(x, y, w) \in G\colon \omega(x, y) \in V\},$$

where V is a neighbourhood of p in $(R\times R)^w$.

2.2. The semidirect product $T\times C$. Here C is the usual additive group of complex numbers, T is the multiplicative group $\{w \in C\colon |w| = 1\}$, and the multiplication in $T\times C = G$ is given by

$$(w, z)(w', z') = (ww', wz' + z).$$

(G is the Euclidean group of the plane.)

For $0 < \delta < \pi$, let $W_\delta = \{(w, z)\colon |\arg w| \leqslant \delta, |z| \leqslant \delta\}$, where $\arg w$ is chosen in the interval $(-\pi, \pi]$ for $w \in T$. It is easy to check that the product of normalized Lebesgue measure on T and of Lebesgue measure on C gives a left and right Haar measure $\mu = \mu_G$ on G and $\mu(W_\delta) = \delta^3$; thus G is unimodular. A function $h \in C(G)$ is uniformly continuous if, for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $|h(u)-h(v)| < \varepsilon$ whenever $u = (a, b)$ and $v = (w, z)$ satisfy

$$uv^{-1} = (a/w, b-az/w) \in W_\delta \quad \text{or} \quad u^{-1}v = \big(w/a, (z-b)/a\big) \in W_\delta.$$

Also, if $\delta = \delta(\varepsilon)$ is as above and $(w_0, z_0) \in G$ is given, then

(a) $|h(w, z)-h(w_0, z_0)| \leqslant 2\varepsilon$ if $|z| = |z_0|$, $|\arg z/z_0| \leqslant \delta$ and

$$\arg(w_0 z/z_0)-\delta \leqslant \arg w \leqslant \arg(w_0 z/z_0)+\delta,$$

and, in particular,

(b) $|h(w_0, z)-h(w_0, z_0)| \leqslant 2\varepsilon$ if $|z| = |z_0|$, $|\arg z/z_0| \leqslant \delta$.

We note that, in fact, $|h(w, z)-h(w_0, z_0)| \leqslant 2\varepsilon$ on a neighbourhood of (w_0, z_0) whose Haar measure is greater than $8|z_0|\delta^2/\pi$ if $|z_0| \geqslant \delta$.

LEMMA 2.2. *Suppose that a function f on C is the restriction to C $(= \{1\}\times C)$ of $h \in AP(G)$. Then f is a constant function.*

Proof. Since h is uniformly continuous, let $\delta = \delta(\varepsilon) > 0$ be as above. It follows that if $x \geqslant 1/\delta$ and $|y| \leqslant (\delta x)^{1/2}$, then

$$\begin{aligned}|f(x)-f(x+iy)| &= |h(1, x)-h(1, x+iy)| \\ &\leqslant |h(1, x)-h(e^{iy/x}, xe^{iy/x})|+|h(e^{iy/x}, xe^{iy/x})-h(1, x+iy)| \leqslant 2\varepsilon.\end{aligned}$$

To see, for example, that the second absolute value in the line above is less than ε, note that $(1, x+iy)^{-1}(e^{iy/x}, xe^{iy/x})$ is in W_δ since $|\arg e^{iy/x}| = |y/x| \leqslant \delta$ and both $x\big(1-\cos(y/x)\big)$ and $|y-x\sin(y/x)|$ are not greater than $x(y^2/2x^2) \leqslant \delta/2$.

Much as in the last part of the proof of Theorem 2.1, we consider, for fixed $y, y' \in R$, the function

$$x \to f(x+iy)-f(x+iy')$$

and conclude that f is constant on lines $x = \text{const}$.

By treating $R_i f$ in similar fashion, we see that f is constant on lines $y = \text{const}$ as well, which completes the proof.

THEOREM 2.3. $AP(G) = C(T)$; *hence* $G^a \simeq T$.

This follows from Lemma 2.2 and remarks made at the beginning of this section.

The next result is due to Chou [4]. We give a proof of it here, since the ideas involved will also be used to prove Theorem 2.5.

THEOREM 2.4. $W(G) = AP(G)\oplus C_0(G)$, *i.e.*, $W_0(G) = C_0(G)$.

Proof. We take an $f \in W_0(G)$, $f \geqslant 0$, and derive a contradiction from the assumption that $f \notin C_0(G)$. Thus, we assume the existence of a $\gamma > 0$ and a sequence $\{u_n\} = \{(w_n, r_n \exp[i\theta_n])\} \subset G$ with $r_n \to \infty$ and $f(u_n) > \gamma$ for all n. Without loss, we may assume that $w_n \to w_0$ and $\exp[i\theta_n] \to w'$. Applying $R_{(-iw_0/w', 0)}L_{(w', 0)}$ to f, we may assume that $w_0 = 1$ and $w' = i$.

Finally, using the consequence of uniform continuity (b), we may assume that $f(1, ir_n) > \gamma$ for all n.

We now want to show that, for any fixed y, $\mu_R(A_m)/2m \to 0$ as $m \to \infty$, where

$$A_m = \{x \in R\colon -m \leqslant x \leqslant m, f(1, x+iy) \geqslant \gamma/2\}$$

and μ_R is Lebesgue measure. For, if not, then there are a sequence $\{m_j\}$ and a $\nu > 0$ such that $\mu_R(A_{m_j})/2m_j \geqslant \nu$ for all j. It is easy to verify that the sequence $\{U_j\}$, where

$$U_j = \{(w, z)\colon w \in T, |z| \leqslant m_j\},$$

is an F-sequence for G; in particular,

$$\mu(uU_j \cap U_j)/\mu(U_j) \to 1, \qquad u \in G.$$

Hence

$$d_j = \mu(U_j)^{-1} \int_{U_j} L_{(1,iy)} f(w, z)\, d\mu(w, z) \to 0,$$

the value of the invariant mean at f. However, if $\delta > 0$ is such that

$$|L_{(1,iy)}f(u) - L_{(1,iy)}f(v)| \leqslant \gamma/8$$

whenever $uv^{-1} \in W_\delta$ or $u^{-1}v \in W_\delta$ and $L_{(1,iy)}f(1, x) \geqslant \gamma/2$ for $x \in [x_1, x_2]$, where $0 \leqslant x_1 < x_2$, then the consequence of uniform continuity (a) implies that $L_{(1,iy)}f \geqslant \gamma/4$ on

$$\{(w, z)\colon x_1 \leqslant |z| \leqslant x_2, |\arg z| \leqslant \delta, \arg z - \delta \leqslant \arg w \leqslant \arg z + \delta\},$$

a set whose Haar measure is $\delta^2(x_2^2 - x_1^2)/\pi$. It follows that, for each j, $L_{(1,iy)}f \geqslant \gamma/4$ on a set whose Haar measure is at least twice that of

$$\{(w, z)\colon 0 \leqslant |z| \leqslant \nu m_j, |\arg z| \leqslant \delta, \arg z - \delta \leqslant \arg w \leqslant \arg z + \delta\},$$

i.e., at least $2\delta^2\nu^2 m_j^2/\pi$; since $\mu(U_j) = \pi m_j^2$, this implies $d_j \geqslant \gamma\delta^2\nu^2/2\pi^2$ and $d_j \not\to 0$, a contradiction.

The next conclusion we want to draw is that, given $k > 0$, we can find an x_k such that

$$f(1, x_k + ir_n) \leqslant \gamma/2, \qquad 1 \leqslant n \leqslant k.$$

But this follows from what was proved in the previous paragraph.

We are now ready to prove that $f \notin W(G)$, the desired contradiction. Note that $(1, x_k + ir_n) = (1, x_k)(1, ir_n)$, and suppose that the limits

$$\lim_k \lim_n f(1, x_k + ir_n) \quad \text{and} \quad \lim_n \lim_k f(1, x_k + ir_n)$$

both exist (as we may, since we can take subsequences if necessary).

Then the second limit is not greater than $\gamma/2$, while the argument in the proof of Lemma 2.2 shows that the first limit is not less than γ, and f violates Grothendieck's criterion for weak almost periodicity.

COROLLARY 2.1. *G^w is homeomorphic to $T \times C_p$, where C_p is the one-point compactification of C.*

We are now ready to prove

THEOREM 2.5. *For the semidirect product $G = T \times C$, let H be the direct product $G \times G$. Then $H^w = G^w \times G^w$.*

Proof. Since $AP(H) = C_a(H) \subset C_w(H)$ (which follows from Theorem 1.1), we will be done if we can show that any $f \in W_0(H)$, $f \geqslant 0$, satisfies the condition of (iii) or the condition of (iv) of Theorem 1.2.

If $f \in C_0(H)$, then f satisfies the condition of (iii) of Theorem 1.2; and, if for each $\varepsilon > 0$ there is an $M = M(\delta)$ such that $|f(u, v)| < \varepsilon$ for all $v \in G$ and $u = (w, z) \in G$ with $|z| \geqslant M$, then f satisfies (iv) of Theorem 1.2. Indeed, A_f is relatively compact in $C(G)$. After an argument similar to this last one, we are to the point where we will be done when we show that assuming the existence of a $\gamma > 0$ and of a sequence

$$\{(u_n, v_n)\} = \{((w_n, r_n \exp[i\theta_n]); (w_n', r_n' \exp[i\theta_n']))\} \subset H,$$

with $f(u_n, v_n) \geqslant \gamma$ for all n, $r_n \to \infty$ and $r_n' \to \infty$, leads to a contradiction.

So, we make these assumptions and, following the proof of Theorem 2.4, we may further assume that

$$(u_n, v_n) = ((1, ir_n); (1, ir_n')) \quad \text{for all } n$$

and we can show that, for any fixed $y, y' \in R$, $\mu_{R\times R}(B_m)/4m^2 \to 0$ as $m \to \infty$, where

$$B_m = \{(x, x') \in R \times R: |x| \leqslant m, |x'| \leqslant m, f((1, x+iy); (1, x'+iy')) \geqslant \gamma/2\}.$$

Then, still following the proof of Theorem 2.4, we can produce sequences in H to show that f violates Grothendieck's criterion for weak almost periodicity, which is the desired contradiction.

2.3. The semidirect product $R^+ \times R$. Here R^+ is the multiplicative group of positive real numbers and the product in $G = R^+ \times R$ is

$$(x, y)(x', y') = (xx', xy' + y).$$

(G is the affine group of the line.)

LEMMA 2.3. *Let $f \in WAP(G)$. Then, for each $x > 0$, $\lim_{|y|\to\infty} f(x, y)$ exists.*

Proof. We may assume that $x = 1$. (If $x \neq 1$, consider $R_{(x,0)}f$.) If $\lim_{|y|\to\infty} f(1, y)$ does not exist, then there exist sequences $\{y_n\}$ and $\{z_m\}$ with $|y_n| \to \infty$, $|z_m| \to \infty$, and

$$\lim_n f(1, y_n) = a \neq b = \lim_m f(1, z_m). \tag{1}$$

Let $d = |a-b|$. We may assume that

$$|f(1, y_n)-a| \leqslant d/6 \quad \text{and} \quad |f(1, z_m)-b| \leqslant d/6 \quad \text{for all } m \text{ and } n.$$

Since f is uniformly continuous, there is a $\delta > 0$ such that, if $u = (a, b)$ and $v = (x, y)$ satisfy

$$u^{-1}v = (x/a, (y-b)/a) \in V_\delta \quad \text{or} \quad uv^{-1} = (a/x, b-ya/x) \in V_\delta,$$

where

$$V_\delta = \{(x', y'): |x'-1| \leqslant \delta, |y'| \leqslant \delta\},$$

then $|f(u)-f(v)| \leqslant d/12$. It follows that, if $|y-y_n| \leqslant \delta|y_n|$, then

$$|f(1, y)-f(1, y_n)| \leqslant |f(1, y)-f(y/y_n, y)| + |f(y/y_n, y)-f(1, y_n)| \leqslant d/6.$$

Note that $\delta|y_n| \to \infty$. Similarly, $|f(1, y)-f(1, z_m)| \leqslant d/6$ if $|y-z_m| \leqslant \delta|z_m|$ and $\delta|z_m| \to \infty$.

Consider the double sequence

$$\{f((1, y_n)(1, z_m))\}_{m,n=1}^{\infty} = \{f(1, y_n+z_m)\}_{m,n=1}^{\infty}.$$

Since we can take subsequences if necessary, we may assume that the limits

$$\lim_m \lim_n f(1, y_n+z_m) = a_0 \quad \text{and} \quad \lim_n \lim_m f(1, y_n+z_m) = b_0$$

exist. Then the calculation at the end of the last paragraph and (1) show that $|a-a_0| \leqslant d/3$ and $|b-b_0| \leqslant d/3$; hence $a_0 \neq b_0$ and f violates Grothendieck's criterion for weak almost periodicity, which is a contradiction.

Suppose that $f \in WAP(G)$ and, in view of Lemma 2.3, consider the function $h \in C(G)$ defined by

$$h(x, y) = \lim_{n\to\infty} f(x, n), \quad (x, y) \in G.$$

Then h is constant on lines $x = \text{const}$ and, since it is the pointwise limit of $\{L_{(1,n)}f\}$, is in $WAP(G)$; hence it corresponds to a function in $WAP(R^+)$. Thus $g = f-h \in WAP(G)$ and, by uniform continuity,

$$\lim_{|y|\to\infty} g(x, y) = 0$$

uniformly on compact subsets of R^+.

LEMMA 2.4. *The function $g = f-h$ defined above is in $C_0(G)$.*

Proof. If $g \notin C_0(G)$, there are a sequence $\{(x_n, y_n)\}$ with $x_n^2+y_n^2 \to \infty$ and a $\gamma > 0$ such that $|g(x_n, y_n)| \geqslant \gamma$ for all n. We may assume that $\gamma = 1$; also, by a remark above and considering $\check{g}$, defined by $\check{g}(u) = g(u^{-1})$ for $u \in G$, we may assume that $x_n \to \infty$. Consider now the sequence $\{L_{(x_n,y_n)}g\}$.

Since $g \in WAP(G)$ and we can take a subsequence if necessary, we may assume that $\{L_{(x_n,y_n)}g(1,y)\}$ converges to a limit $k(y)$ for all $y \in R$ and that the limit function k is in $WAP(R)$. We have $|k(0)| \geqslant 1$ and there exists a $\lambda > 0$ such that $|k(y)| \geqslant 2/3$ for $|y| \leqslant \lambda$. Now g is uniformly continuous; so, in particular, if $\varepsilon > 0$, there is a $\delta > 0$ such that $|g(x,y)-g(x,y')| \leqslant \varepsilon$ whenever $|y'-y|/x \leqslant \delta$. Hence, if $|y-z| \leqslant \delta$ or else

$$|(x_n y + y_n) - (x_n z + y_n)|/x_n \leqslant \delta \quad \text{for all } n,$$

then, putting $L_{(x_n,y_n)g(1,y)} = g(x_n, x_n y + y_n) = g_n(y)$, we have $|g_n(y)-g_n(z)| \leqslant \varepsilon$, i.e., the sequence $\{g_n\}$ is (uniformly) equicontinuous and its pointwise convergence to k on the interval $|y| \leqslant \lambda$ is actually uniform. Therefore, there is an n_0 such that $n \geqslant n_0$ implies

$$|g_n(y) - k(y)| = |g(x_n, x_n y + y_n) - k(y)| \leqslant 1/3 \quad \text{for } |y| \leqslant \lambda,$$

i.e., $|g(x_n, y+y_n)| \geqslant 1/3$ for $|y| \leqslant \lambda x_n$. Let $\{z_m\} \subset R$ be a sequence, $z_m \to \infty$ and consider the double sequence

$$\{g((1,z_m)(x_n,y_n))\}_{m,n=1}^{\infty} = \{g(x_n, z_m+y_n)\}.$$

Taking subsequences if necessary, we may assume that both limits

$$\lim_m \lim_n g(x_n, z_m+y_n) \quad \text{and} \quad \lim_n \lim_m g(x_n, z_m+y_n)$$

exist. It follows that the second limit is 0, while (since $\lambda x_n \to \infty$) the first one has magnitude greater than or equal to 1/3. Thus g violates Grothendieck's criterion for weak almost periodicity, which is a contradiction.

THEOREM 2.6. *We have $AP(G) \simeq AP(R^+)$, whence $G^a \simeq (R^+)^a$. Moreover,*

$$W(G) \simeq W(R^+) \oplus C_0(G).$$

Remarks. 1. The first part of Theorem 2.6 is equivalent to the well-known result that every irreducible finite-dimensional representation of the group G is of the form $(x,y) \to x^{ix_0}$ for some $x_0 \in R$.

2. G^w can be viewed as follows. G is homeomorphic in an obvious way to

$$\{(x,y,z): x > 0,\ y^2+(z-1)^2 = 1,\ y = 0 \text{ only if } z = 0\} = H \subset R^3.$$

G^w is obtained by adjoining the points $\{(x,0,2): x > 0\} = K$ and the points of $(R^+)^w \setminus \omega(R^+)$; neighbourhoods of a point p of this latter kind are sets of the form

$$\big(V \setminus \omega(R^+)\big) \cup \{(x,y,z) \subset H \cup K: \omega(x) \in V\},$$

where V is a neighbourhood of p in $(R^+)^w$.

Added in proof (July 28, 1981). Two papers of special relevance have appeared since the present paper was accepted for publication.

1. *Minimally weakly almost periodic groups,* by C. Chou, Journal of Functional Analysis 36 (1980), p. 1-17. Here the author calls a locally compact group G *minimally weakly almost periodic* (m.w.a.p., for short) if $W(G) = AP(G) \oplus C_0(G)$ and proves, among other things, that $T \times C$, the Euclidean group of the plane, is essentially the only noncompact, connected, solvable, m.w.a.p. group.

2. *Weakly almost periodic functions on semisimple Lie groups,* by W. A. Veech, Monatshefte für Mathematik 88 (1979), p. 55-68. Here it is proved that all simple analytic groups with finite center are m.w.a.p., and that, if $G = G_1 \times G_2 \times \ldots \times G_n$ is a product of such groups, then $G^w = G_1^w \times G_2^w \times \ldots \times G_n^w$ (the notation being that of the present paper).

REFERENCES

[1] J. F. Berglund and K. H. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions,* Berlin 1967.

[2] J. F. Berglund and P. Milnes, *Algebras of functions on semitopological left-groups,* Transactions of the American Mathematical Society 222 (1976), p. 157-178.

[3] R. B. Burckel, *Weakly almost periodic functions on semigroups,* New York 1970.

[4] C. Chou, *Weakly almost periodic functions and almost convergent functions on a group,* Transactions of the American Mathematical Society 206 (1975), p. 175-200.

[5] K. deLeeuw and I. Glicksberg, *Almost periodic functions on semigroups,* Acta Mathematica 105 (1961), p. 99-140.

[6] W. R. Emerson and F. P. Greenleaf, *Covering properties and Følner conditions for locally compact groups,* Mathematische Zeitschrift 102 (1967), p. 370-384.

[7] L. T. Gardner and P. Milnes, *On the extension of uniformly continuous functions,* Canadian Mathematical Bulletin 18 (1975), p. 143-145.

[8] M. Katětov, *On real-valued functions in topological spaces,* Fundamenta Mathematicae 38 (1951), p. 85-91, and correction, ibidem 40 (1953), p. 203-205.

[9] P. Milnes, *Extension of continuous functions on topological semigroups,* Pacific Journal of Mathematics 58 (1975), p. 553-562.

[10] V. Pták, *An extension theorem for separately continuous functions and its application to functional analysis,* Czechoslovak Mathematical Journal 89 (1964), p. 562-581.

UNIVERSITY OF WESTERN ONTARIO
LONDON, CANADA

Reçu par la Rédaction le 4. 10. 1977;
en version modifiée le 26. 1. 1978