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The ideal boundary of a domain in C^n

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Abstract. We shall introduce a compactification of a domain in the space C^n, based on the notion of the Bergman function. This compactification is invariant under biholomorphic mappings, and for those domains in C which can be mapped onto the unit disc it reduces to the classical Carathéodory compactification. The present paper presents a refinement of some of ideas in [8]. Our results can be used to unify and simplify some known theorems on boundary correspondence under biholomorphic mappings. In particular, we give a new proof of a theorem of Vormoor [9].

1. Classes of proportional functions. Let D be a domain in C^n. The Fréchet space of all functions which are holomorphic in D will be denoted by $H(D)$. We shall denote by $H^*(D)$ the subset of $H(D)$:

$$H^*(D) = \{f \in H(D),\ f \not\equiv 0\}.$$

The functions $f, g \in H^*(D)$ will be called *proportional* if there exists a complex constant $c \neq 0$ such that

$$f = cg.$$

This is an equivalence relation in the space $H^*(D)$. The set of all equivalence classes will be denoted by $PH(D)$

$$PH(D) = \{[f],\ f \in H^*(D)\}.$$

We shall need the following well-known fact, see [6].

THEOREM 1. *Let R be an equivalence relation in a topological space X. Let us consider the set of all equivalence classes*

$$X/R = \{[f],\ f \in X\}$$

with the standard topology. That is to say, a set $\mathfrak{A}$ is open in X/R if and only if the set

$$U = \{f,\ [f] \in \mathfrak{A}\}$$

is open in X. This is the largest topology in X/R for which the canonical mapping

$$X \ni f \to [f] \in X/R$$

is continuous. For each $A \subset X$ define $A^R \subset X$ and $A_R \subset X$ by

$$A^R = \bigcup_{f \in A} [f], \quad A_R = \bigcup_{[f] \subset A} [f].$$

Then the following conditions are equivalent:

(i) *the canonical mapping is open;*

(ii) *for each open set $A \subset X$ the set A^R is open in X;*

(iii) *for each closed set $A \subset X$ the set A_R is closed in X.*

In the particular case we are concerned with we can prove

LEMMA 1. *The space $H^*(D)$ with the relation of proportionality satisfies conditions* (i), (ii) *and* (iii) *of the previous theorem. In particular, the canonical mapping from $H^*(D)$ onto $PH(D)$ is open.*

Proof. It is enough to show (iii). Assume that A is a closed subset of $H^*(D)$, and that a sequence $f_m \in A$, $m = 1, 2, \ldots$, such that $[f_m] \subset A$ converges to $f \in H^*(D)$. We have to show that $[f] \subset A$. An element in $[f]$ can be written as cf with $c \neq 0$ and we see that

$$cf = \lim cf_m \in A,$$

since $cf_m \in A$ for each m, and A is closed in $H^*(D)$.

This yields

COROLLARY 1. *Assume that $[f] \in PH(D)$, and U_i, $i = 1, 2, \ldots$, is a basis of neighbourhoods of f in $H^*(D)$. Then the sets $\pi(U_i)$, $i = 1, 2, \ldots$, form a countable basis of neighbourhoods of $[f]$ in $PH(D)$.*

Proof. Consider a neighbourhood $\mathfrak{A}$ of $[f]$. By the definition of topology in $PH(D)$ the set $U = \{f, [f] \in \mathfrak{A}\}$ is an open set in $H^*(D)$ which contains f. By assumption, there exists i such that $f \in U_i \subset U$. It follows that $[f] \in \pi(U_i) \subset \mathfrak{A}$ and the proof is completed.

The following fact will be often useful in the sequel:

COROLLARY 2. *The sequence $[f_m] \in PH(D)$ converges to $[f] \in PH(D)$ if and only if there exist constants $c_m \neq 0$, $m = 1, 2, \ldots$, such that the sequence $c_m f_m$ converges to f in $H^*(D)$.*

Proof. The condition is sufficient, since the canonical mapping is continuous. To prove necessity, consider a basis U_i, $i = 1, 2, \ldots$, of neighbourhoods of f. For $m > M_i$ we have $[f_m] \in \pi(U_i)$, and there exist constants c_m^i such that

$$c_m^i f_m \in U_i, \quad m > M_i.$$

With no loss of generality we may assume that $U_{i+1} \subset U_i$, and $M_i < M_{i+1}$ for $i = 1, 2, \ldots$ Define

$$c_m = c_m^i \quad \text{for } M_i < m \leqslant M_{i+1}.$$

Then $c_m f_m \in U_i$ for $m > M_i$ and $c_m f_m$ converges to f.

COROLLARY 3. *Let T: $H^*(D_1) \to H^*(D_2)$ be a continuous mapping, such that for each f and g proportional in $H^*(D_1)$ the functions Tf and Tg are proportional in $H^*(D_2)$. Then the mapping P_T: $PH(D_1) \to PH(D_2)$ given by*

$$P_T[f] = [Tf]$$

is continuous.

Proof. Assume that $[f_m]$ converges to $[f]$ in $PH(D_1)$. By the previous corollary there exist constants $c_m \neq 0$ such that $c_m f_m$ converges to f in $H^*(D_1)$. It follows that $Tc_m f_m$ converges to Tf in $H^*(D_2)$. Since, by assumption

$$[Tc_m f_m] = [Tf_m],$$

it follows that

$$P_T[f_m] = [Tf_m]$$

converges to $P_T[f] = [Tf]$ in $PH(D_2)$. Hence P_T is continuous.

We need one more

LEMMA 2. *The proportionality relation in $H^*(D)$ is closed.*

Proof. Let us consider functions $f_m, g_m, m = 1, 2, \ldots$, in $H^*(D)$, such that for each m

$$f_m = c_m g_m, \quad c_m \neq 0,$$

and $\lim f_m = f$, $\lim g_m = g$ in $H^*(D)$. We have to show that f and g are proportional. By passing to a subsequence we may assume that c_m converges to a (possibly infinite) number c. The case $c = 0$ cannot occur, since it implies $f \equiv 0$. Similarly, $c = \infty$ implies that $1/c_m$ converges to 0 and $g \not\equiv 0$. Therefore c is a complex non-zero number and $f = cg$. This implies

COROLLARY 4. *The space $PH(D)$ is a Hausdorff space.*

Proof. This follows from a general theorem, since the equivalence relation is closed and the canonical mapping is open, see [6].

2. The invariant compactification. The Bergman function of a domain $D \subset C^n$ will be denoted by $K_D(z, \bar{t})$; see [1].

DEFINITION 1. Let D be a domain in C^n. A compactification of D is a homeomorphism

$$q\colon D \to X$$

onto an open dense subset in a compact Hausdorff space X. We say that compactifications $q_i\colon D \to X_i$, $i = 1, 2$, are *equivalent* if there exists a homeomorphism $w\colon X_1 \to X_2$ of X_1 onto X_2 such that the diagram

$$\begin{tikzcd} & D \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X_1 \arrow[rr, "w"] & & X_2 \end{tikzcd}$$

is commutative. This is indeed an equivalence relation in the class of all compactifications of D.

It is interesting that a rather large class of domains in C^n admits a "natural" compactification, defined in terms of the Bergman function.

Definition 2. We shall say that a domain $D \subset C^n$ admits the invariant compactification if the following conditions are satisfied:

(i) for each $t \in D$ the function

$$z \to K_D(z, \bar{t})$$

belongs to $H^*(D)$;

(ii) the mapping $p\colon D \to PH(D)$ given by

$$p(t) = [K_D(z, \bar{t})]$$

has a relatively compact image, and defines a homeomorphism of D onto an open subset of $\hat{D} = \overline{p(D)}$.

In this case the compactification

$$p\colon D \to \hat{D}$$

is called the *invariant compactification*, for the reason which will become clear later. The compact set

$$\hat{D} \setminus p(D)$$

is called the *ideal boundary* of D.

Remark 1. Condition (i) states that for every point $t \in D$ there exists a function

$$f \in L^2 H(D) = H(D) \cap L^2(D)$$

such that $f(t) \neq 0$. This condition is obviously satisfied if D can be mapped biholomorphically onto a bounded domain, but, in general, it may fail even if the space $L^2 H(D)$ is not trivial. For example, in a complete 2-circular domain $D \subset C^2$

$$D = \{z \in C^2,\ |z_2| < |z_1|^{-1},\ |z_1| < 1\}$$

the equality $K_D(z, \bar{t}) \equiv 0$ in D holds for every $t = (t_1, t_2)$ such that $t_1 = 0$.

Remark 2. The mapping p in condition (ii) is always continuous, and it is one-to-one if D can be mapped onto a bounded domain. Therefore the most essential part of this condition consists in verifying that if U is open in D, then $p(U)$ is open in $\hat{D}$, and that $\hat{D}$ is compact.

The examples of domains which admit invariant compactification are often based on the following

THEOREM 2. *Assume that a domain $D \subset C^n$ has the following properties:*

(i) *for each $t \in D$ the function*

$$z \to K_D(z, \bar{t})$$

belongs to $H^*(D)$;

(ii) *there exists a compactification of D*

$$q\colon D \to X$$

such that the mapping $p \circ q^{-1}$ extends to a one-to-one continuous mapping $w\colon X \to PH(D)$.

Then D admits invariant compactification p, and both compactifications q and p are equivalent.

Proof. Note that w maps a dense subset $q(D)$ onto $p(D)$. It follows by continuity of w that $w(X) \subset \hat{D}$. On the other hand, $w(X)$ is compact, and therefore closed in $PH(D)$. Since $p(D) \subset w(X)$, it follows that $\hat{D} \subset w(X)$. Hence $w(X) = \hat{D}$. By assumption, w is one-to-one and continuous. It is also closed, since it maps compact sets onto compact sets. It follows that w is a homeomorphism. Since q is a homeomorphism of D onto an open dense subset of X, the composition $p = w \circ q$ maps D onto an open dense subset of $\hat{D}$, and therefore

$$p\colon D \to \hat{D}$$

is a compactification of D. Finally, p and q are equivalent, since $p = w \circ q$.

3. The invariance under biholomorphic mapping. The name "invariant compactification" is justified by the following

THEOREM 3. *Consider a biholomorphic mapping*

$$h\colon D_1 \to D_2,$$

where D_1 and D_2 are domains in C^n. If D_1 admits invariant compactification, then D_2 admits invariant compactification. Furthermore, if

$$p_1\colon D_1 \to \hat{D}_1, \quad p_2\colon D_2 \to \hat{D}_2$$

are invariant compactifications of D_1 and D_2, respectively, then the mapping

$$p_2 \circ h \circ p_1^{-1}\colon p_1(D_1) \to p_2(D_2)$$

extends to a homeomorphism of $\hat{D}_1$ *onto* $\hat{D}_2$,

$$\hat{h}\colon \hat{D}_1 \to \hat{D}_2.$$

Proof. The proof is based on the following rule of transformation of the Bergman function, see [1],

$$K_{D_1}(z,\bar{t}) = K_{D_2}(h(z), \overline{h(t)}) \det h'(z) \overline{\det h'(t)},$$

where h' denotes the holomorphic Jacobi matrix of h. Consider the mapping $T\colon H^*(D_2) \to H^*(D_1)$ given by

$$Tf = (f \circ h) \det h'.$$

It is a homeomorphism, and both T and T^{-1} satisfy the assumptions of Corollary 3. It follows that P_T is a homeomorphism of $PH(D_2)$ onto $PH(D_1)$. Furthermore, by the rule transformation of the Bergman function, the following diagram

$$\begin{tikzcd} D_1 \arrow[r, "h"] \arrow[d, "p_1"'] & D_2 \arrow[d, "p_2"] \\ \hat{D}_1 & \hat{D}_2 \arrow[l, "P_T"] \end{tikzcd}$$

is commutative. It follows that $\hat{D}_1 = P_T \hat{D}_2$. Since, by assumption, p_2 is a homeomorphism of D_2 onto on open dense subset of the compact space $\hat{D}_2$, the space $\hat{D}_1$ is also compact, and $p_1 = P_T \circ p_2 \circ h$ is a homeomorphism of D_1 onto an open dense subset of $\hat{D}_1$. Therefore p_1 is the invariant compactification of D_1. Since

$$p_2 \circ h \circ p_1^{-1} = P_T^{-1},$$

the right-hand side gives the desired extension of the left-hand side to a homeomorphism of $\hat{D}_1$ onto $\hat{D}_2$. The proof is completed.

4. The case of a product domain. In this section we use a formula due to H. Bremermann [4]

$$K_{D_1 \times D_2}(z,\bar{t}) = K_{D_1}(z_1, t_1) K_{D_2}(z_2, \bar{t}_2)$$

to show that the study of the ideal boundary in the product domain $D = D_1 \times D_2$ can be reduced to an examination of both factors D_1 and D_2.

THEOREM 4. *Assume that* $D = D_1 \times D_2$. *Then* D *admits invariant compactification if and only if both* D_1 *and* D_2 *admit invariant compactification. Furthermore, if* $p_i\colon D_i \to \hat{D}_i$, $i = 1, 2$, *are invariant compactifications of* D_1 *and*

D_2, then the invariant compactification of D is equivalent to the product mapping $p_1 \times p_2$: $D \to \hat{D}_1 \times \hat{D}_2$ given by

$$(p_1 \times p_2)(t) = (p_1(t), p_2(t)).$$

Proof. The mapping S: $PH(D_1) \times PH(D_2) \to PH(D)$ given by

$$S([f_1], [f_2]) = [f_1 f_2]$$

is one-to-one. Indeed, if $g_1 \in H^*(D_1)$, $g_2 \in H^*(D_2)$ and $c \neq 0$ are such that

$$f_1 f_2 = c g_1 g_2,$$

then in view of the fact that both f_2 and g_2 vanish on closed nowhere dense sets, there exists $a_2 \in D_2$ for which $f_2(a_2) \neq 0$ and $g_2(a_2) \neq 0$. Substituting $z = (z_1, a_2)$ in the above equation we see that $[f_1] = [g_1]$. Similarly $[f_2] = [g_2]$. Hence S is one-to-one. Also S is continuous, as can be easily seen in view of Corollary 2.

We shall show that the image of S is closed in $PH(D)$ and S^{-1} is continuous. Of course, it will be enough to prove that if the sequence

$$S([f_1^m], [f_2^m]) = [f_1^m f_2^m]$$

converges to $[f]$ in $PH(D)$, then the sequence $[f_1^m]$ converges in $PH(D_1)$ and the sequence $[f_2^m]$ converges in $PH(D_2)$. In view of symmetry we may proved only the first part of the statement. By assumption, there exist constants $c_m \neq 0$, $m = 1, 2, \ldots$, such that the sequence $c_m f_1^m f_2^m$ converges in $H^*(D)$ to f:

(1) $$\lim_{m \to \infty} c_m f_1^m f_2^m = f.$$

Consider the set of all $a_2 \in D_2$ such that the function $g \in H(D_1)$ given by

$$g(z_1) = f(z_1, a_2)$$

vanishes identically. This set is closed and nowhere dense since $f \not\equiv 0$ in D. It follows that we can find $a_2 \in D_2$ such that for all $m = 1, 2, \ldots$

$$f_2^m(a_2) \neq 0,$$

and, moreover, $g \in H^*(D_1)$. Substituting $z = (z_1, a_2)$ in (1) we see that the sequence

$$c_m f_2^m(a_2) f_1^m$$

converges to g in $H^*(D_1)$. Hence $[f_1^m]$ converges to $[g]$ in $PH(D_1)$. We have proved that S^{-1} is continuous.

We shall now show that conditions (i) and (ii) in the definition of the invariant compactification are satisfied in D if and only if they are satisfied in each of the domains D_1 and D_2. This holds obviously for condition (i) in view of the Bremermann formula. Therefore we need only consider condition (ii). Note that the closed image of S contains $\hat{D}$ since it contains $p(D)$. The Bremermann formula shows that the diagram

$$\begin{tikzcd}[column sep=small] & D_1 \times D_2 = D \arrow[dl, "p_1 \times p_2"'] \arrow[dr, "p"] & \\ \hat{D}_1 \times \hat{D}_2 \arrow[rr, "S"] & & \hat{D} \end{tikzcd}$$

is commutative. It follows that $S(\hat{D}_1 \times \hat{D}_2) = \hat{D}$. Therefore p is a compactification of D if and only if $p_1 \times p_2$ is a compactification of $D_1 \times D_2$, and this is equivalent to the condition that p_i is a compactification of D_i for $i = 1, 2$. Finally, the above diagram shows that the invariant compactification p is equivalent to the compactification $p_1 \times p_2$.

5. Exceptional sets. Let D be a domain in C^n. Consider a subdomain $D_0 \subset D$ such that for each z, $t \in D_0$

$$K_{D_0}(z, \bar{t}) = K_D(z, \bar{t}).$$

In such case we say that the set $D \setminus D_0$ is negligible or "exceptional".

THEOREM 5. *If D admits invariant compactification, then D_0 also admits invariant compactification. Furthermore, if $p\colon D \to \hat{D}$ is the invariant compactification of D, then the invariant compactification of D_0 is equivalent to the compactification*

$$p\colon D_0 \to \overline{p(D_0)}.$$

Proof. We shall show that the compactification p of D_0 satisfies the assumptions of Theorem 2. Since a function $f \in H^*(D)$ does not vanish identically on D_0, condition (i) is satisfied by D_0. Also, we may consider the mapping $T\colon H^*(D) \to H^*(D_0)$ defined by

$$Tf = f_{|D_0}.$$

By Corollary 3 the mapping $P_T\colon PH(D) \to PH(D_0)$ is continuous. It is also easy to verify that P_T is one-to-one. For $t \in D_0$ consider the element

$$p_0(t) = [K_{D_0}(z, \bar{t})]$$

in $PH(D_0)$. Note that on $p(D_0)$ we have

$$p_0 \circ p^{-1} = P_T.$$

The right-hand side defines the desired extension of the left-hand side to $\overline{p(D_0)}$. It follows that condition (ii) is satisfied, and by Theorem 2 the compactification p of D_0 is equivalent to the invariant compactification of D_0.

6. Regular domains.

DEFINITION 3. A bounded domain $D \subset C^n$ is called *regular* if its Euclidean compactification

$$\mathrm{id}\colon D \to \bar{D}$$

is equivalent to the invariant compactification of D.

Regular domains are important in view of the following

THEOREM 6. *Consider a biholomorphic mapping*

$$h\colon D_1 \to D_2,$$

where D_1 and D_2 are regular domains in C^n. The mapping h extends to a homeomorphism $h\colon \bar{D}_1 \to \bar{D}_2$.

Proof. Since D_i is regular for $i = 1, 2$, the mapping p_i extends to a homeomorphism of $\bar{D}_i$ onto $\hat{D}_i$. In view of Theorem 3, the extension of h can be defined as

$$h = p_2^{-1} \circ \hat{h} \circ p_1.$$

On the other hand, we have

THEOREM 7. *Consider a biholomorphic mapping*

$$h\colon D_1 \to D_2,$$

where D_1 and D_2 are domains in C^n, and D_1 is regular. If h extends to a homeomorphism

$$h\colon \bar{D}_1 \to \bar{D}_2,$$

then D_2 is regular.

Proof. The domain D_2 is bounded since $\bar{D}_2$ is compact. We have to show that

$$p_2\colon D_2 \to \hat{D}_2$$

extends to a homeomorphism of $\bar{D}_2$ onto $\hat{D}_2$. Since

$$p_2 = \hat{h} \circ p_1 \circ h^{-1},$$

the extension is given by the right-hand side.

For product domains we can prove

THEOREM 8. *Let $D = D_1 \times D_2$. Then D is regular if and only if both D_1 and D_2 are regular.*

Proof. It is enough to consider the case when both D_1 and D_2 are bounded and admit invariant compactifications $p_i\colon D_i \to \hat{D}_i$. In view of Theorem 4, it is enough to prove that the Euclidean compactification of D is equivalent to the compactification

$$(p_1 \times p_2)\colon D \to \hat{D}_1 \times \hat{D}_2$$

if and only if, for $i = 1, 2$, p_i extends to a homeomorphism $p_i\colon \bar{D}_i \to \hat{D}_i$. The sufficiency of the condition is obvious. In order to prove necessity, assume that $p_1 \times p_2$ extends to a homeomorphism q of $\bar{D}_1 \times \bar{D}_2$ onto $\hat{D}_1 \times \hat{D}_2$. Set $q = (q_1, q_2)$. Since $q_1(t_1, t_2)$ is independent of t_2 in D_1 and continuous in $\bar{D}_1$, we have $q_1 = q_1(t_1)$ in $\bar{D}_1$ and similarly $q_2 = q_2(t_2)$ in $\bar{D}_2$. Hence $q = q_1 \times q_2$ in $\bar{D}_1 \times \bar{D}_2$. Since q is a homeomorphism of $\bar{D}_1 \times \bar{D}_2$, it follows that each q_i, $i = 1, 2$, is a homeomorphism of $\bar{D}_i$ onto $\hat{D}_i$. Obviously, q_i is a desired extension of p_i.

The proof is completed.

We now pass to the situation considered in Theorem 5.

THEOREM 9. *Let D and D_0 be domains in C^n for which the assumptions of Theorem 5 are satisfied. If D is regular, then D_0 is regular.*

Proof. By assumption, p extends to a homeomorphism of $\bar{D}$ onto D. It follows that p maps homeomorphically $\bar{D}_0$ onto $\overline{p(D_0)}$. In other words, the Euclidean compactification of D_0 is equivalent to the compactification

$$p\colon D_0 \to \overline{p(D_0)}.$$

By Theorem 5, the latter compactification is equivalent to the invariant compactification of D_0. The proof is completed.

We end this section with the following easy corollary to Theorem 2, which gives a necessary and sufficient condition for regularity.

THEOREM 10. *Let D be a bounded domain in C^n. Then D is regular if and only if $p\colon D \to PH(D)$ given by*

$$p(t) = [K_D(z, \bar{t})]$$

extends to a continuous one-to-one mapping of $\bar{D}$ into $PH(D)$.

Proof. If D is regular, then by the definition of equivalent compactifications p extends to a homeomorphism of $\bar{D}$ onto D. Conversely, if p extends to a continuous one-to-one mapping of $\bar{D}$ into $PH(D)$, then the Euclidean compactification of D satisfies the assumptions of Theorem 2. Therefore D admits invariant compactification, and this compactification is equivalent to the Euclidean compactification of D. Hence D is regular.

7. Examples.

EXAMPLE 1. Every strictly pseudoconvex domain $D \subset C^n$ with smooth boundary is regular. Indeed, from a deep result of L. Boutet de Monvel and J. Sjöstrand [3] it follows immediately that $K_D(z, \bar{t})$ is smooth as a function of $(z, t) \in \bar{D} \times \bar{D}$, with the exception of the set

$$\{(t, t), t \in bdD\},$$

and that for each $t \in bdD$ we have

$$\lim_{z \to t} K_D(z, \bar{t}) = \infty.$$

It follows that D satisfies assumptions of Theorem 10, with p: $\bar{D} \to PH(D)$ given by

$$p(t) = [K_D(z, \bar{t})], \quad t \in \bar{D}.$$

Note that for $t \in bdD$ and $t' \in \bar{D}$ such that $t' \neq t$ we have $p(t) \neq p(t')$, since $K_D(z, \bar{t})$ and $K_D(z, \bar{t}')$ are not proportional. In fact, for z close to t the first function becomes infinite, while the second approaches a finite number $K_D(t, \bar{t}')$. Note that in the case when both D_1 and D_2 are strictly pseudoconvex with smooth boundaries Theorem 6 reduces to a theorem of N. Vormoor [9].

EXAMPLE 2. Every domain $D \subset C$ bounded by a finite number of analytic Jordan curves is regular. Indeed, one can show [2] that $K_D(z, \bar{t})$ is real analytic on $\bar{D} \times \bar{D}$ with the exception of the set $\{(t, t), t \in bdD\}$, and the for each $t \in bdD$ the holomorphic function

$$z \mapsto K_D(z, \bar{t})$$

has a pole of second order at t. The proof in [2] is based on the Schwarz reflection principle, and the following identity due to M. Schiffer:

$$K_D(z, \bar{t}) = \frac{-2}{\pi} \frac{\partial^2 G_D(z, t)}{\partial z \, \partial \bar{t}},$$

where G_D denotes the Green function of D with pole at t. From the above properties of the function $K_D(z, t)$ the regularity of D follows as in Example 1. In particular, the unit disc is a regular domain. Therefore the invariant compactification of the unit disc is equivalent to the classical Carathéodory compactification. Since both compactifications are invariant under biholomorphic mappings, it follows that for each domain $D \subset C$ the Carathéodory compactification is equivalent to the invariant compactification.

EXAMPLE 3. Every domain $D \subset C$ bounded by a finite number of Jordan curves is regular. Indeed, such a domain can be mapped biholomorphically onto a domained bounded by analytic curves in such a way that the mapping function extends to a homeomorphism of closed domains. The regularity of D now follows from the previous Example and Theorem 7.

EXAMPLE 4. Every complete circular bounded domain $D \subset C^n$ such that $rD \supset \bar{D}$ for each $r > 1$ is regular.

Recall that a set $E \subset C^n$ is called *circular* (respectively *complete circular*) if for every $z \in E$ and every $w \in C$ such that $|w| = 1$ (respectively $|w| \leqslant 1$) the point wz belongs to E. A complete circular domain $D \subset C^n$ contains the origin. The Taylor series at the origin of the function $f \in H(D)$

$$f = \sum_{m=0}^{\infty} F_m, \tag{2}$$

where F_m is a homogeneous polynomial of degree m, converges uniformly in a neighbourhood of every point $z \in D$. This can be seen by considering a complex plane which passes through 0 and z. On the compact set $r^{-1}\bar{D} \subset D$ series (2) converges uniformly, and therefore also in the L^2-norm. Since homogeneous polynomials of different orders are orthogonal on the complete circular set $r^{-1}D$, it follows that

$$\|f\|^2_{r^{-1}D} = \sum_{m=0} \|F_m\|^2_{r^{-1}D}.$$

When r approaches 1, we can use the Lebesgue monotone convergence theorem for series to obtain

$$\|f\|^2_D = \sum_{m=0} \|F_m\|^2_D.$$

In the case when $f \in L^2 H(D)$ we have

$$\Big\|f - \sum_{m=0}^{M} F_m\Big\|^2_D \leqslant \Big\|f - \sum_{m=0}^{M} F_m\Big\|^2_{r^{-1}D} + (2\|f\|_{D\setminus r^{-1}D})^2.$$

It follows that series (2) converges to f in $L^2(D)$. As a consequence, the set of all homogeneous polynomials is dense in $L^2 H(D)$.

Therefore, if $P_{m,k}$, $k = 1, 2, \ldots, k_m$, is an orthonormal basis in the space of all homogeneous polynomials of degree m which belong to $L^2 H(D)$, then $P_{m,k}$, $m = 1, 2, \ldots$, $k = 1, 2, \ldots, m_k$, is a complete orthonormal system in $L^2 H(D)$, and by a general theory [1] we obtain the theorem of H. Cartan:

$$K_D(z, \bar{t}) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_m} P_{m,k}(z)\overline{P_{m,k}(t)},$$

where the series converges normally for $(z, t) \in D \times D$ and in $L^2(D)$ for each fixed $t \in D$.

After these preliminary remarks we can return to our example. For each $r > 1$ denote $D'_r = r^{-1}D$ and $D''_r = rD$ and for $(z, t) \in D'_r \times D''_r$ consider the function

$$K_r(z, \bar{t}) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_m} P_{m,k}(z)\overline{P_{m,k}(t)}.$$

The series converges normally as before, since

$$P_{m,k}(z)\overline{P_{m,k}(t)} = P_{m,k}(rz)P_{m,k}(r^{-1}t)$$

and $(rz, r^{-1}t) \in D \times D$. It follows that the series converges normally in the open set

$$\bigcup_r D'_r \times D''_r$$

to a continuous function. Since this set contains $D \times \bar{D}$, the sum of the series defines the desired extension of $K_D(z, \bar{t})$ to a continuous function on $D \times \bar{D}$. Since $K_D(0, \bar{t}) = \operatorname{vol} D^{-1}$ for all $t \in D$, it follows that $K_D(0, \bar{t}) = \operatorname{vol} D^{-1}$ for all $t \in \bar{D}$. In particular, $K_D(z, t) \in H^*(D)$, and

$$p(t) = [K_D(z, \bar{t})], \quad t \in \bar{D},$$

is a continuous mapping. To see that p is one-to-one, we shall prove that $K_D(z, \bar{t}) \neq K_D(z, \bar{t}')$ if $t' \neq t$.

We already know that a series for $K_D(z, \bar{t})$ converges uniformly for $z \in 2^{-1}D$, and therefore in $L^2(2^{-1}D)$. Since $P_{m,k}$ form a complete orthogonal system not only in D, but also in $2^{-1}D$, the series represe[illegible] a Fourier development of $K_D(z, \bar{t})$ in terms of an orthonormal system in $2^{-1}D$. Therefore, if $K_D(z, \bar{t}) \equiv K_D(z, \bar{t}')$, then for each m, k

$$P_{m,k}(t) = P_{m,k}(t').$$

The linear functions z_i, $i = 1, 2, \ldots, n$, can be expressed in terms of linear polynomials $P_{1,1}, \ldots, P_{1,n}$. It follows that $t'_i = t_i$ for $i = 1, 2, \ldots, n$ and therefore $t' = t$.

By Theorem 10 the domain D is regular. In particular, every complete n-circular domain in C^n, and every classical domain [5] is regular.

EXAMPLE 5. The Hartogs triangle

$$D = \{z \in C^2, |z_1| < |z_2| < 1\}$$

is mapped biholomorphically by

$$(z_1, z_2) \overset{h}{\mapsto} (z_1/z_2, z_2)$$

onto

$$h(D) = \{|w_1| < 1\} \times \{0 < |w_2| < 1\}.$$

In view of Theorem 8, $h(D)$ is regular, since the second factor is regular by Theorem 9. It follows that D admits invariant compactification. Nevertheless, D is not regular, since h does not extend to a homeomorphism of closed domains.

In general, it is not easy to decide whether a domain $D \subset C^n$ is regular. In particular, it would be interesting to settle the question of regularity for analytic polyhedra, or weakly pseudoconvex domains. On the other hand, it would be interesting to find an example of a bounded domain $D \subset C^n$ which does not admit invariant compactification. It should be noted that Theorem 6 is useful also in the study of the problem of extending a biholomorphic mapping to a diffeomorphism of closed domains, see [7].

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