ANNALES
POLONICI MATHEMATICI
119.3 (2017)

Probabilistic approximation of partly filled-in composite Julia sets

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Abstract. We study properties of the metric space of pluriregular sets and of contractions on that space induced by finite families of proper polynomial mappings of several complex variables. In particular, we show that closed balls in the space of pluriregular sets do not have to be compact and we give a simple proof of applicability of the so-called chaos game in the case of composite Julia sets. Part of the construction of those sets also leads to a computationally viable approximation by simpler sets based on Monte-Carlo simulation.

1. Introduction. Ever since the appearance in 1981 of the seminal paper of Hutchinson [Hu] introducing iterated function systems (IFS), numerous studies, modifications, applications and generalizations have appeared in the literature. A pivotal role in generating great interest in the computational realizations of the underlying mathematics was played by the discovery of the *chaos game* (see [BD] and [Ba]) allowing visualization of the attractors of iterated function systems with the help of probabilistic methods. A modern and general version of the definition of IFS can incorporate probabilistic aspects as part of the concept (see e.g. [BHS]), and thus brings together both deterministic and probabilistic points of view.

The field of discrete dynamical systems, including the complex dynamics and study of Julia sets, has enjoyed decades of unprecedented popularity. As a consequence the list of authors, starting perhaps with B. Mandelbrot, whose research contributions are significant, would be very long indeed. Since this paper pertains to pluricomplex analysis, and more specifically to the metric space of pluriregular sets, we will only refer here to works directly

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2010 *Mathematics Subject Classification*: Primary 32H50, 32U35, 37F10; Secondary 28A80, 65C05.

Key words and phrases: composite Julia sets, pluricomplex Green functions, iterated function system, the chaos game, complex dynamics, Monte-Carlo simulation.

Received 26 September 2016; revised 8 August 2017.

DOI: 10.4064/ap4100-8-2017

influencing our investigations. Even in complex analysis, particularly in several variables, the study of complex dynamics and Julia sets usually goes in quite different direction from the topics researched in this article (see e.g. [Fo], [Sm], [ABBDSS]). In one complex variable, perhaps the closest in spirit to our work are the articles on the Julia sets generated by semigroups of holomorphic mappings (see [Su1], [Su2], [SS]). It should also be noted that the methodologies employed in the literature in the study of IFS and that of Julia sets have been in most cases significantly different, particularly in higher-dimensional settings. However, a solid link does exist, as shown in [Kl2], [Kl3] and in a range of articles on analytic multifunctions: [Kl4], [Kl5], [KK1], [KK2], [KK3].

The main research objective of this paper is to better understand the metric space $\mathcal{R}$ of pluriregular polynomially convex compact sets in $\mathbb{C}^N$, as many properties of this metric space remain still unknown. We will look at some general properties of $\mathcal{R}$ and we will also study, in some detail, composite Julia sets. There are at least two good reasons for doing so in this context. Firstly, the composite Julia sets constitute one of the very few general classes of pluriregular sets which are reasonably well understood. Secondly, arising from the somewhat modified iterated function systems, they permit the use of probabilistic methods.

The organization of the paper is as follows. In Section 2 we recall the definition of the metric space of pluriregular sets, and we show some of its topological properties in Theorem 1. In particular, we prove that the space of pluriregular sets is separable but not proper, meaning that closed balls do not have to be compact. In Section 3 we describe some characteristics of proper polynomial mappings in $\mathbb{C}^N$ relevant to this study. These concern the growth rate of such maps at infinity and the invariance property of pluricomplex Green functions. Section 4 looks at the anatomy of composite Julia sets, combining new notation with results from earlier articles in a way suitable for the problem at hand.

The objective of the penultimate Section 5 is to state and prove Theorem 2 pertaining to properties of families of contractions on the space of pluriregular sets. One of the conclusions of Theorem 2 is a version of the chaos game. A few comments are in order here. Basically, it is shown that the classic chaos game applies in the space of pluriregular sets to IFS generated by finite families of proper polynomial mappings. This can be demonstrated in many ways using the general theory, e.g. it can be deduced from [BHS]. Here however we present an argument based only on the relevant concepts from pluricomplex analysis and relying essentially on the geometry of metric spaces and density of ordinary Julia sets. Alternatively, elementary proofs of the classical case proposed by [Cr] and [Ma] could be adapted to our setting. In fact, similarly to [Cr] and [Ma] our argument works because of finiteness of the diameter of the attractor of the IFS considered and the compliance of Bernoulli processes with the Strong Law of Large Numbers. Since Theorem 1 asserts that the space of pluriregular sets is not proper, it means in particular that the main result of [BV] does not apply to the case analyzed in Theorem 2. In fact, the geometry of the pseudometric in the last conclusion of Theorem 2 is largely unknown, and it is difficult to say how to deal with polynomial convexity in this context. In particular, in contrast to the conventional case, the statement cannot be directly translated into computer visualization.

To circumnavigate these obstacles on the road to computational viability, the last section of the paper contains Theorem 3 describing the Monte-Carlo approach to plotting images of partly filled-in composite Julia sets. It uses monotonicity of sequences of sets approximating composite Julia sets to deduce convergence with respect to the usual Hausdorff distance, leading to a range of examples in which the theorem is applied to visualize such sets in the complex plane.

Finally, a few words about notation. If A, B are non-empty sets, then B^A will denote the set of all mappings from A to B. If $\mathcal{S}$ is a set of non-empty subsets of a set W, then we will use the symbol $\bigcup \mathcal{S}$ to denote $\bigcup_{A\in\mathcal{S}} A \subset W$. If f is a bounded complex-valued function defined on a set A, then $\|f\|_A = \sup\{|f(a)| : a \in A\}$. Let (X, d) be a metric space. Given a set $A \subset X$, the symbol $\operatorname{int}(A)$ will stand for the interior of A. The set of all non-empty compact subsets of X will be denoted by $\kappa(X)$. The open ball with center a and radius r will be denoted by $B_d(a, r)$. The distance of a point from a set, the diameter of a set and the Hausdorff distance between two compact sets will be denoted by dist_d, diam_d and χ_d, respectively. By the *ε-dilation* of a set $E \subset X$, where $\varepsilon > 0$, we will mean the set $\{x \in X : \operatorname{dist}_d(x, E) \le \varepsilon\}$. When working with the usual Euclidean metric in $\mathbb{C}^N$ we will use similar notation but without the subscript d.

2. The space of pluriregular sets.

We need to recall a few concepts from pluripotential theory (see e.g. [Kl1] for background and references). Given a set $E \subset \mathbb{C}^N$, the *pluricomplex Green function* of E is defined as

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^N),\ u \le 0 \text{ on } E\}, \qquad z \in \mathbb{C}^N,$$

where $\mathcal{L}(\mathbb{C}^N)$ denotes the family of all plurisubharmonic functions $u : \mathbb{C}^N \to [-\infty, \infty)$ with at most logarithmic growth at infinity:

$$\sup\{u(z) - \log(1 + \|z\|) : z \in \mathbb{C}^N\} < \infty.$$

If $E \subset \mathbb{C}^N$ is a compact set, then E and its polynomially convex hull $\widehat{E}$ have the same pluricomplex Green function. Recall that

$$\widehat{E} = \bigcap_p \{z \in \mathbb{C}^N : |p(z)| \le \|p\|_E\}, \tag{1}$$

where the intersection is taken over the set of all complex polynomials $p : \mathbb{C}^N \to \mathbb{C}$. The zero set of V_E is equal to the polynomially convex hull of E (see e.g. [Kl1]). A compact set E is said to be *pluriregular* if V_E is continuous. Pluriregular sets are also called *L-regular* in older literature.

Let $\mathcal{R}$ denote the family of all compact, polynomially convex and pluriregular subsets of $\mathbb{C}^N$. This family of sets can be turned into a complete metric space (see [Kl2]) when it is furnished with the metric

$$(2) \qquad \Gamma(E,F) = \max\{\|V_E\|_F, \|V_F\|_E\} = \|V_E - V_F\|_{\mathbb{C}^N}, \qquad E, F \in \mathcal{R}.$$

The same formula defines a pseudometric on a larger family of pluriregular compact subsets of $\mathbb{C}^N$ which are not necessarily polynomially convex. We will refer to the metric space $(\mathcal{R}, \Gamma)$ as the *space of pluriregular sets*. Some topological properties of this space have been studied in [Kl2], [Si2] and sporadically in other papers, but in general it is fair to say that the topology of $\mathcal{R}$ is not well understood. We will need the following.

THEOREM 1. *The space $(\mathcal{R}, \Gamma)$ has the following properties:*

(a) *If $\mathcal{K} \subset \mathcal{R}$ is compact, then $\bigcup \mathcal{K}$ is compact in $\mathbb{C}^N$.*

(b) *If $\mathcal{A}, \mathcal{B} \subset \mathcal{R}$ are non-empty and compact, then*

$$\Gamma\Big(\bigcup \mathcal{A}, \bigcup \mathcal{B}\Big) \le \chi_\Gamma(\mathcal{A}, \mathcal{B}).$$

(c) *For every $m \in \mathbb{N}$, the mapping*

$$\mathcal{R}^m \ni (C_1, \ldots, C_m) \mapsto \widehat{\bigcup_{j=1}^{m} C_j} \in \mathcal{R}$$

is continuous, where the hat denotes the operation of taking the polynomially convex hull.

(d) *The space $(\mathcal{R}, \Gamma)$ is separable, but is not proper, i.e. closed balls do not have to be compact.*

Proof of Theorem 1(a). In view of [Si2, Theorem 3.3], the set $\overline{\bigcup \mathcal{K}}$ is bounded. We only have to show that it is closed. Take $a \in \overline{\bigcup \mathcal{K}}$. There exists a sequence of points $(a_n) \subset \bigcup \mathcal{K}$ convergent to a. Also, for every $n \in \mathbb{N}$, one can find $K_n \in \mathcal{K}$ such that $a_n \in K_n$. Then there exists a subsequence K_{n_k} which is Γ-convergent to a set $K \in \mathcal{K}$ as $k \to \infty$. Hence the corresponding pluricomplex Green functions converge uniformly in $\mathbb{C}^N$, and in particular on the set $\{a, a_1, a_2, \ldots\}$. In view of continuity of V_K and the uniform convergence, if $\varepsilon > 0$, there exists m such that for all $p, k \ge m$,

$$|V_K(a) - V_{K_{n_p}}(a_{n_k})| \le |V_K(a) - V_K(a_{n_k})| + |V_K(a_{n_k}) - V_{K_{n_p}}(a_{n_k})| < \varepsilon.$$

In particular we can take $p = k$, and so we conclude that $V_K(a) = 0$, which means that $a \in K$. ∎

Proof of Theorem 1(b). Fix $\varepsilon > 0$ such that $\chi_\Gamma(\mathcal{A}, \mathcal{B}) < \varepsilon$. Then

$$\mathcal{B} \subset \bigcup_{A \in \mathcal{A}} B_\Gamma(A, \varepsilon),$$

and so each set $B \in \mathcal{B}$ is an element of $B_\Gamma(A, \varepsilon)$ for some $A \in \mathcal{A}$, which may depend on B. Hence

$$V_{\bigcup \mathcal{A}} \le V_A < \varepsilon \quad \text{on } B,$$

and consequently $V_{\bigcup \mathcal{A}} < \varepsilon$ on $\bigcup \mathcal{B}$. Similarly $V_{\bigcup \mathcal{B}} < \varepsilon$ on $\bigcup \mathcal{A}$, as required. ∎

Proof of Theorem 1(c). This is so, as for any $A_j, B_j \in \mathcal{R}$ we have the estimate

$$\Gamma(A_1 \cup \cdots \cup A_m, B_1 \cup \cdots \cup B_m) \le \max\{\Gamma(A_j, B_j) : j = 1, \ldots, m\},$$

according to [Kl2]. ∎

Proof of Theorem 1(d). In view of Mazurek's property (see [Si1, Proposition 5.11]), if $E \in \mathcal{R}$ and $\varepsilon > 0$, then $E(\varepsilon) = \{z \in \mathbb{C}^N : V_E(z) \le \varepsilon\} \in \mathcal{R}$ and $V_{E(\varepsilon)} = (V_E - \varepsilon)^+$. Moreover, if F is a finite union of closed balls such that $E \subset F \subset E(\varepsilon)$, then $\widehat{F} \in \mathcal{R}$ (see (1)) and $\Gamma(E, F) \le \Gamma(E, E(\varepsilon)) = \varepsilon$ (see also [Kl2, Corollary 2]). Consequently, polynomially convex hulls of finite unions of closed balls with rational radii and centers with rational coordinates form a countable dense subset of $(\mathcal{R}, \Gamma)$. In fact, the proofs of [Kl4, Theorem 3] and [KK1, Theorem 5.1] also imply that composite Julia sets, which we will study later on, generated by finite families of quadratic polynomial mappings with rational coefficients provide a countable dense subset of $(\mathcal{R}, \Gamma)$ as well.

For the second part of (d), we consider first $N = 1$. It was shown by Siciak (see [Si2, Example 3.6 and Theorem 3.3]) that if the numbers $\varepsilon_n \in (0, 1/n)$ are small enough, then the sequence of sets

$$E_n = \bigcup_{j=0}^{n-1} [j/n, j/n + \varepsilon_n] \cup [1, 2] \in \mathcal{R}, \qquad n \in \mathbb{N},$$

has the property that

$$\limsup_{n \to \infty} \Gamma(E_n, [0, 2]) > 0,$$

but at the same time $\lim_{n\to\infty} \chi(E_n, [0, 2]) = 0$ since $\chi(E_n, [0, 2]) \le 1/n$. Hence, because of [Si2, Theorem 3.3 and Proposition 3.5], the sequence E_n is not relatively compact in the metric space $(\mathcal{R}, \Gamma)$. Since $\Gamma(E_n, [0, 2]) \le \Gamma([1, 2], [0, 2])$, the closed ball in $\mathcal{R}$ with center at $[0, 2]$ and radius $\Gamma([1, 2], [0, 2])$ is not compact. If $N > 1$, we consider the Nth Cartesian powers of the sets E_n, $[1, 2]$ and $[0, 2]$. Because of the product property of the pluricomplex Green functions (see [Si1, Proposition 5.9] or [Kl1, Theorem 5.1.8]),

$$\limsup_{n\to\infty} \mathit{\Gamma}(E_n^N, [0,2]^N) = \limsup_{n\to\infty} \mathit{\Gamma}(E_n, [0,2]) > 0.$$

Also, since $\chi(E_n^N, [0,2]^N) \to 0$ as $n \to \infty$, [Si2, Theorem 3.3 and Proposition 3.5] lead to the same conclusion as in the one-dimensional case. ■

3. Proper polynomial mappings and their exponents. We will denote the vector space of all complex polynomial mappings $P : \mathbb{C}^N \to \mathbb{C}^N$ by $\mathcal{P}(\mathbb{C}^N)$. The *Łojasiewicz exponent at infinity* of the mapping $P \in \mathcal{P}(\mathbb{C}^N)$ is the real number defined as

$$\mathscr{L}_\infty(P) = \sup\left\{\delta \in \mathbb{R} : \liminf_{\|z\|\to\infty} \frac{\|P(z)\|}{\|z\|^\delta} > 0\right\}.$$

Obviously the choice of any particular norm is irrelevant, but we will use the Euclidean norm throughout the paper. The supremum is achieved, and if $R > 0$ is large enough, then for some $M > 0$,

$$\|P(z)\| \ge M\|z\|^{\mathscr{L}_\infty(P)}, \qquad \|z\| \ge R.$$

The mapping P is proper if and only if $\mathscr{L}_\infty(P) > 0$. If $\deg(P) = d$, we say that P is *regular* if $\inf\{\|P_d(z)\| : \|z\| = 1\} > 0$, where P_d is the homogeneous part of P of degree d. If this is the case, then $\mathscr{L}_\infty(P) = d$. Interestingly, as shown in [Pł], any positive rational number is the Łojasiewicz exponent at infinity of a proper polynomial mapping. Various estimates for $\mathscr{L}_\infty(P)$ can be found in [CKT], [Kr], [RS]. The task of finding the value of the Łojasiewicz exponent at infinity of a given P is often made easier by a result from [CK] according to which it is enough to check the lower bound for the growth of P on the zero set of the product of the components of P.

Let $P : \mathbb{C}^N \to \mathbb{C}^N$ be a proper polynomial mapping. Then

$$\mathscr{L}_\infty(P)V_{P^{-1}(E)} \le V_E \circ P \le \deg(P)V_{P^{-1}(E)} \tag{3}$$

for any $E \subset \mathbb{C}^N$ (see [Kl1, Theorem 5.3.1]). In particular, if $E \in \mathcal{R}$, then also $P^{-1}(E) \in \mathcal{R}$. Moreover, the function defined by

$$A_{\{P\}} : \mathcal{R} \ni E \mapsto P^{-1}(E) \in \mathcal{R} \tag{4}$$

satisfies the Lipschitz condition with the constant $1/\mathscr{L}_\infty(P)$ (see [Kl2]). Indeed, by (3), for any $E, F \in \mathcal{R}$, we have the estimate

$$\mathscr{L}_\infty(P)\|V_{P^{-1}(E)}\|_{P^{-1}(F)} \le \|V_E\|_F,$$

and a similar one when E and F swap places. Hence the first equality in the definition (2) of $\mathit{\Gamma}$ yields the stated Lipschitz condition.

If $R > 0$, the symbol $\mathbb{B}_R$ will always denote the closed Euclidean ball with center at the origin and radius R. An *escape radius* for $P \in \mathcal{P}(\mathbb{C}^N)$, if it exists, is a number $R > 0$ such that if $z \in \mathbb{C}^N \setminus \mathbb{B}_R$, then $\lim_{n\to\infty} \|z_n\| = \infty$, where $z_0 = z$ and $z_n = P(z_{n-1})$ for $n \geq 1$. If $\mathscr{L}_\infty(P) > 1$ and $\delta \in (1, \mathscr{L}_\infty(P)]$, then an escape radius $R > 1$ can be found such that (see [Kl3])

$$\inf\left\{\frac{\|P(z)\|}{\|z\|^\delta} : \|z\| \geq R\right\} > R^{1-\delta}. \tag{5}$$

We will then say that the escape radius is *δ-adjusted.* In particular, in this case

$$\operatorname{int}(\mathbb{B}_R) \supset P^{-1}(\mathbb{B}_R) = \overline{\operatorname{int}(P^{-1}(\mathbb{B}_R))} \tag{6}$$

as P, being a proper holomorphic mapping, is both continuous and open. Furthermore

$$\Gamma(P^{-1}(\mathbb{B}_R), \mathbb{B}_R) = \|V_{P^{-1}(\mathbb{B}_R)}\|_{\partial\mathbb{B}_R} \leq \frac{1}{\delta}\log^+ \frac{\|P\|_{\partial\mathbb{B}_R}}{R} \leq \frac{\|P\|_{\partial\mathbb{B}_R}}{R\,\delta}. \tag{7}$$

The last number can be easily estimated numerically, e.g. using basic Monte-Carlo optimization.

4. Composite Julia sets and their internal structure. For any finite subset $\mathcal{F} = \{P_1, \ldots, P_k\} \subset \mathcal{P}(\mathbb{C}^N)$ such that $\delta = \min\{\mathscr{L}_\infty(P) : P \in \mathcal{F}\} > 1$ and any set $K \in \mathcal{R}$ we generalize in two ways the mapping $A_{\{P\}}$ from (4) by defining

$$A_\mathcal{F}(K) = \bigcup_{P \in \mathcal{F}} P^{-1}(K) \quad \text{and} \quad \mathcal{H}_\mathcal{F}(K) = \widehat{A_\mathcal{F}(K)}.$$

Here, just as before, the hat denotes the operation of taking the polynomially convex hull of the set in question. It was shown in [Kl2] that the mapping

$$\mathcal{H}_\mathcal{F} : \mathcal{R} \ni E \mapsto \mathcal{H}_\mathcal{F}(E) \in \mathcal{R}$$

is a contraction with contraction ratio $1/\delta$. Thus by Banach's fixed point theorem, $\mathcal{H}_\mathcal{F}$ has a unique fixed point $\mathbb{J}[P_1, \ldots, P_k]$, which is called the *filled-in composite Julia set* generated by $P_1, \ldots, P_k$ (see [Kl2], [Kl3]). The mappings $A_{\{P_1\}}, \ldots, A_{\{P_k\}}$ form an iterated function system on the metric space $(\mathcal{R}, \Gamma)$ whose attractor will be denoted by $\mathcal{S}[P_1, \ldots, P_k]$, or simply $\mathcal{S}$ if the generating mappings are agreed on in a specific context. In particular, $\mathcal{S}$ is a compact subset of $\mathcal{R}$ (see e.g. [Hu]).

Finite and infinite iterations which can use different maps at each step require a precise labeling system. Suppose that $k \geq 2$ is an integer. We define the space Σ_k of *full addresses* as the set of all functions $\sigma : \mathbb{N} \to \{1, \ldots, k\}$ and furnish it with the metric

$$d(\sigma, \tau) = \sum_{j=1}^{\infty} \frac{|\sigma(j) - \tau(j)|}{k^j}, \qquad \sigma, \tau \in \Sigma_k. \tag{8}$$

By a *partial address of length* m we mean any function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, k\}$. With any full address $\sigma \in \Sigma_k$ and $m \in \mathbb{N}$ we associate two partial addresses of length m:

$$\sigma|m = \sigma|_{\{1,\dots,m\}} \quad \text{and} \quad \operatorname{rev}(\sigma|m)(j) = (\sigma|m)(m+1-j), \quad j = 1,\dots,m.$$

It is easy to see that if $d(\sigma,\tau) < k^{-m}$, then $\sigma|m = \tau|m$. Also, if $\sigma|m = \tau|m$, then $d(\sigma,\tau) \le k^{-m}$. Moreover, it can be shown that the metric space (Σ_k, d) is compact, with a base for its topology given by sets of all full addresses sharing a fixed partial address.

Suppose that the mappings $P_1,\dots,P_k$ with $\mathscr{L}_\infty(P_j) > 1$ are chosen. Let

$$\delta \in (1, \min\{\mathscr{L}_\infty(P_j) : j = 1,\dots,k\}] \tag{9}$$

and assume that a common δ-adjusted escape radius R for $P_1,\dots,P_k$ is fixed. Given a point $z \in \mathbb{C}^N$ and an address $\sigma \in \Sigma_k$ we define a *σ-orbit* of z (with respect to $P_1,\dots,P_k$) as the infinite sequence $z_0, z_1, \dots$, where $z_0 = z$ and

$$z_n = P_{\sigma(n)}(z_{n-1}), \qquad n \in \mathbb{N}.$$

In the case of a partial address σ of length m, we can similarly define a *truncated σ-orbit* of the point z (of length $m+1$).

Define $\mathbb{J}_{\mathrm{tr}}[P_1,\dots,P_k]$ as the set of all $z \in \mathbb{C}^N$ such that for each $m \in \mathbb{N}$ there exists $\sigma \in \Sigma_k$ such that the $(\sigma|m)$-orbit of z is contained in $\mathbb{B}_R$. It can be shown that the set $\mathbb{J}_{\mathrm{tr}}[P_1,\dots,P_k]$ is compact (see the proof of [KK1, Theorem 4.6] or [KK2, Lemma 5.1]) and that $\mathbb{J}[P_1,\dots,P_k]$ is the polynomially convex hull of $\mathbb{J}_{\mathrm{tr}}[P_1,\dots,P_k]$ (see [KK1]).

If $\sigma \in \Sigma_k$ and $E \in \mathcal{R}$, then the following set is an element of $\mathcal{R}$ which is independent of the choice of E (see [Kl3]):

$$S_\sigma = \lim_{m\to\infty} (P_{\sigma(m)} \circ \dots \circ P_{\sigma(1)})^{-1}(E),$$

where the limit is taken with respect to the metric Γ. The set S_σ can be described as the set of all points $z \in \mathbb{C}^N$ whose σ-orbits are bounded, or as the filled-in Julia set of the non-autonomous dynamical system $\{P_{\sigma(m)} : m \in \mathbb{N}\}$. It turns out that the set

$$\mathcal{S} = \{S_\sigma : \sigma \in \Sigma_k\} \subset \mathcal{R}$$

is compact, being the attractor of the iterated function system

$$\{A_{\{P_1\}}, \dots, A_{\{P_k\}}\}$$

(see [Kl3]), and hence by Theorem 1(a) the set

$$\mathbb{J}_{\mathrm{tr}}[P_1,\dots,P_k] = \bigcup_{\sigma\in\Sigma_k} S_\sigma = \bigcup \mathcal{S}$$

is compact in $\mathbb{C}^N$ (see also [KK2]). Since the union $\bigcup \mathcal{S}$ does not have to be polynomially convex (see [Kl3] or [Ko]), the set $\mathbb{J}_{\mathrm{tr}}[P_1,\dots,P_k]$ can be described as a *partly filled-in composite Julia set.* We refer to the set S_σ as the *σ-constituent* of $\mathbb{J}_{\mathrm{tr}}[P_1,\dots,P_k]$.

For any $m \in \mathbb{N}$, we will define an *m-outline* as any set of the form

$$S[\sigma] = (P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)})^{-1}(\mathbb{B}_R),$$

where $\sigma : \{1, \dots, m\} \to \{1, \dots, k\}$ is a partial address. If $\sigma \in \Sigma_k$, then

$$S[\sigma|m] = (P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)})^{-1}(\mathbb{B}_R),$$
$$S[\mathrm{rev}(\sigma|m)] = (P_{\sigma(1)} \circ \cdots \circ P_{\sigma(m)})^{-1}(\mathbb{B}_R).$$

Note that for a given $\sigma \in \Sigma_k$, the m-outlines $S[\sigma|m]$ have the following properties:

- For all m,

$$\mathrm{int}(S[\sigma|m]) \supset S[\sigma|m+1] = \overline{\mathrm{int}(S[\sigma|m+1])}. \tag{10}$$

- If $E \in \mathcal{R}$, then (see [Kl3])

$$\Gamma\big(S_\sigma, (P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)})^{-1}(E)\big) \le \frac{\max_{1\le j\le k} \Gamma(P_j^{-1}(E), E)}{\delta^{m-1}(\delta - 1)}. \tag{11}$$

  In particular, if $\Delta = \max_{1\le j\le k} \Gamma(P_j^{-1}(\mathbb{B}_R), \mathbb{B}_R)$, then

$$\Gamma(S_\sigma, S[\sigma|m]) \le \frac{\Delta}{\delta^{m-1}(\delta - 1)}. \tag{12}$$

- In particular,

$$S_\sigma = \lim_{m\to\infty} S[\sigma|m] = \bigcap_{m\ge m_0} S[\sigma|m], \qquad m_0 \in \mathbb{N}, \tag{13}$$

  where the limit makes sense with respect to both the metric Γ in $\mathcal{R}$ (see [Kl3]) and the ordinary Hausdorff distance between compact sets in $\mathbb{C}^N$ associated with the Euclidean metric. (This is so because of monotonicity and compactness: the intersection of a non-increasing sequence of compact sets is also the Hausdorff limit of this sequence.)

Due to the way in which Σ_k is metrized (see (8)), it is easy to notice that the mapping $\Sigma_k \ni \sigma \mapsto S_\sigma \in \mathcal{R}$ is uniformly continuous. Indeed, given $\varepsilon > 0$, if $j \in \mathbb{N}$ is chosen so that

$$\frac{\Delta}{\delta^{j-1}(\delta - 1)} < \frac{\varepsilon}{2},$$

and if $d(\sigma, \tau) < k^{-j}$ for $\sigma, \tau \in \Sigma_k$, then $\Gamma(S_\sigma, S_\tau) < \varepsilon$ in view of (12) and the triangle inequality. Since one of our objectives is to approximate partly filled-in Julia sets with simpler sets, it will be useful to have the following notation at hand. We will denote the set of all m-outlines by $\mathcal{S}_m$. Obviously

$$\mathcal{S}_m = \{S[\sigma|m] : \sigma \in \Sigma_k\} \subset \mathcal{R}$$

and the number of elements of $\mathcal{S}_m$ is at most k^m. If $\varepsilon > 0$, take

$$m = \left\lceil \frac{\log \frac{\delta\Delta}{\varepsilon(\delta-1)}}{\log \delta} \right\rceil,$$

where $\lceil\cdot\rceil$ denotes the ceiling function. Then $\Gamma(S_\sigma, S[\sigma|m]) \le \varepsilon$ and also $\chi_\Gamma(\mathcal{S}, \mathcal{S}_m) \le \varepsilon$. Again, because of monotonicity and compactness, we can see that

$$\lim_{m\to\infty} \bigcup \mathcal{S}_m = \bigcup \mathcal{S}$$

with respect to the ordinary Hausdorff distance on the set $\kappa(\mathbb{C}^N)$. To complete our terminology we will call the set $\bigcup \mathcal{S}_m \in \kappa(\mathbb{C}^N)$ the *cumulative m-outline* of the partly filled-in composite Julia set $\bigcup \mathcal{S}$. Note also that because of (10) and the fact that the families $\mathcal{S}_m$ are finite, we have

$$\operatorname{int}\Bigl(\bigcup \mathcal{S}_m\Bigr) \supset \bigcup \mathcal{S}_{m+1} = \overline{\operatorname{int}\Bigl(\bigcup \mathcal{S}_{m+1}\Bigr)}. \tag{14}$$

5. IFSs on the space of pluriregular sets. The objective of this section is to prove the following result listing a number of properties of mappings induced by polynomial endomorphisms of $\mathbb{C}^N$ and acting on the space of pluriregular sets.

THEOREM 2. *Consider polynomial mappings $P_1, \ldots, P_k : \mathbb{C}^N \to \mathbb{C}^N$ with Łojasiewicz exponents greater than* 1.

(a) *If $\mathcal{C} \subset \mathcal{R}$ is bounded and a sequence of partial addresses*

$$\tau^m : \{1, \ldots, m\} \to \{1, \ldots, k\},$$

with $m \in \mathbb{N}$, is given, then any dilation of $\mathcal{S}$ in $\mathcal{R}$ contains almost all of the sets

$$\mathcal{C}_m = \{(P_{\tau^m(1)} \circ \cdots \circ P_{\tau^m(m)})^{-1}(C) : C \in \mathcal{C}\}, \qquad m \in \mathbb{N}.$$

(b) *Let $\mathcal{U} \subset \mathcal{R}$ be an open set such that $\mathcal{U} \cap \mathcal{S} \neq \emptyset$ and let $n \in \mathbb{N}$. There exists a partial address θ of length $m \ge n$ and $\varepsilon > 0$ such that the image of the ε-dilation of $\mathcal{S}$ via the mapping*

$$\mathcal{R} \ni F \mapsto (P_{\theta(1)} \circ \cdots \circ P_{\theta(m)})^{-1}(F) \in \mathcal{R}$$

is contained in $\mathcal{U}$.

(c) *Let an address $\tau \in \Sigma_k$ be generated according to a set of probabilities $p_1, \ldots, p_k > 0$ such that $p_1 + \cdots + p_k = 1$, that is, the values $\tau(j)$ of τ are selected at random, independently of each other, so that $\mathbb{P}[\tau(j) = i] = p_i$ for $j \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$. Then for any $E \in \mathcal{R}$,*

$$\lim_{m\to\infty} \Gamma\Bigl(\mathbb{J}_{\mathrm{tr}}[P_1, \ldots, P_k], \bigcup \overline{\mathcal{E}}_m\Bigr) = 0 \tag{15}$$

with probability 1, *where*

$$\mathcal{E}_m = \{(P_{\tau(1)} \circ \dots \circ P_{\tau(n)})^{-1}(E) : n \geq m\} \tag{16}$$

and the closure of $\mathcal{E}_m$ *is taken in* $\mathcal{R}$.

Note that the set $\bigcup \overline{\mathcal{E}}_m$ appearing in (15) is compact because of Theorem 1(a), since in view of Theorem 2(a) all accumulation points of $\mathcal{E}_m$ are in the compact subset $\mathcal{S}$ of $\mathcal{R}$. Moreover, the sets in (15) are pluriregular but not necessarily polynomially convex, and so we use the fact that the pseudometric Γ is well defined for such sets too.

It should also be noted that the main theorem obtained in [BV] does not apply in Theorem 2(c), as our metric space is not proper according to Theorem 1(d).

Proof of Theorem 2(a). Fix $C_0 \in \mathcal{C}$. By the triangle inequality,

$$\begin{aligned}\Delta &= \max_{1\leq i\leq k} \sup_{C\in\mathcal{C}} \Gamma(C, P_i^{-1}(C)) \\ &\leq \operatorname{diam}_\Gamma(\mathcal{C}) + \max_{1\leq i\leq k} \Gamma(C_0, P_i^{-1}(C_0)) + \delta^{-1} \operatorname{diam}_\Gamma(\mathcal{C}) < \infty.\end{aligned}$$

For each $m \in \mathbb{N}$ we choose $\sigma^m \in \Sigma_k$ such that $\tau^m = \operatorname{rev}(\sigma^m|m)$. According to (11), if $C \in \mathcal{C}$, then

$$\begin{aligned}&\operatorname{dist}_\Gamma\big((P_{\sigma^m(m)} \circ \dots \circ P_{\sigma^m(1)})^{-1}(C), \mathcal{S}\big) \\ &\qquad \leq \Gamma\big((P_{\sigma^m(m)} \circ \dots \circ P_{\sigma^m(1)})^{-1}(C), S_{\sigma^m}\big) \leq \frac{\Delta}{\delta^{m-1}(\delta-1)},\end{aligned}$$

where δ is as in (9). This yields the desired property. ■

Proof of Theorem 2(b). As noted earlier, the set $\mathcal{S}$ is compact in $\mathcal{R}$. Take $\varepsilon > 0$ and $E \in \mathcal{S}$ such that $B_\Gamma(E, 2\varepsilon) \subset \mathcal{U}$. Let $n \in \mathbb{N}$. Without loss of generality we may suppose that

$$\delta^{-n} \operatorname{diam}_\Gamma(\mathcal{S}) < \varepsilon/2.$$

Now, $E = S_\sigma$ for some $\sigma \in \Sigma_k$. As was shown in [Kl3], density of periodic sequences in Σ_k implies that

$$\lim_{m\to\infty} \Gamma(S_\sigma, \mathbb{J}[P_{\sigma(m)} \circ \dots \circ P_{\sigma(1)}]) = 0.$$

Therefore, we can choose $m \geq n$ such that

$$\Gamma(S_\sigma, \mathbb{J}[P_{\sigma(m)} \circ \dots \circ P_{\sigma(1)}]) < \varepsilon/2. \tag{17}$$

Define $\theta = \operatorname{rev}(\sigma|m)$. Let F belong to the ε-dilation of $\mathcal{S}$ in $\mathcal{R}$, that is, $\operatorname{dist}_\Gamma(F, \mathcal{S}) \leq \varepsilon$. There exists $G \in \mathcal{S}$ such that $\Gamma(F, G) \leq \varepsilon$. Thus

$$\begin{aligned}&\Gamma\big((P_{\theta(1)} \circ \dots \circ P_{\theta(m)})^{-1}(F), (P_{\theta(1)} \circ \dots \circ P_{\theta(m)})^{-1}(G)\big) \\ &\qquad \leq \delta^{-m}\Gamma(F, G) < \varepsilon.\end{aligned} \tag{18}$$

Moreover

$$
\begin{aligned}
(19)\quad & \Gamma\big(\mathbb{J}[P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)}], (P_{\theta(1)} \circ \cdots \circ P_{\theta(m)})^{-1}(G)\big) \\
&= \Gamma\big((P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)})^{-1}(\mathbb{J}[P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)}]), (P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)})^{-1}(G)\big) \\
&\le \delta^{-m}\Gamma(\mathbb{J}[P_{\sigma(m)} \circ \cdots \circ P_{\sigma(1)}], G) \le \delta^{-m} \operatorname{diam}_\Gamma(\mathcal{S}) < \varepsilon/2.
\end{aligned}
$$

Combining (17)–(19) and using the triangle inequality, we see that

$$\Gamma\big(E, (P_{\theta(1)} \circ \cdots \circ P_{\theta(m)})^{-1}(F)\big) < 2\varepsilon,$$

as required. ■

Proof of Theorem 2(c). In view of Theorem 1(b), it suffices to prove that with probability 1,

$$\lim_{m\to\infty} \chi_\Gamma(\mathcal{S}, \overline{\mathcal{E}_m}) = 0.$$

Take $\varepsilon > 0$. In view of Theorem 2(a), if m is sufficiently large, then the ε-dilation of $\mathcal{S}$ contains $\mathcal{E}_m$, and hence also $\overline{\mathcal{E}_m}$. In order to show that for sufficiently large m, the ε-dilation of $\mathcal{E}_m$ contains $\mathcal{S}$, it is enough to prove that any point from an $\varepsilon/2$-dense finite subset of $\mathcal{S}$ is within $\varepsilon/2$-distance from a point of $\mathcal{E}_m$. So let $A \in \mathcal{S}$ be an element of a fixed $\varepsilon/2$-dense finite subset of $\mathcal{S}$. By Theorem 2(b), there exists a partial address θ of length ℓ and $\alpha \in (0, \varepsilon/2)$ such that the image of the α-dilation of $\mathcal{S}$ via the mapping

$$F \mapsto (P_{\theta(1)} \circ \cdots \circ P_{\theta(\ell)})^{-1}(F)$$

is a subset of $B_\Gamma(A, \varepsilon/2)$. Using Theorem 2(a) again if necessary, we can increase m so that the α-dilation of $\mathcal{S}$ contains $\mathcal{E}_m$. In view of the Strong Law of Large Numbers applied to the Bernoulli processes, we can conclude that with probability 1 for some $n \ge m$ we have $\theta(j) = \tau(n+j)$ for $j = 1, \ldots, \ell$. Consequently,

$$
\begin{aligned}
&(P_{\tau(1)} \circ \cdots \circ P_{\tau(n+\ell)})^{-1}(E) \\
&\qquad = (P_{\theta(1)} \circ \cdots \circ P_{\theta(\ell)})^{-1}\big((P_{\tau(1)} \circ \cdots \circ P_{\tau(n)})^{-1}(E)\big) \in B_\Gamma(A, \varepsilon/2),
\end{aligned}
$$

as required. ■

6. Direct Monte-Carlo approximation of Julia sets. If one wants to generate an image of a partly filled-in composite Julia set using a computer, then the last conclusion of Theorem 2 is not applicable directly, as the geometry of the Γ-convergence is quite complicated and not well understood. What is needed is an approximation with respect to the Hausdorff distance associated with the underlying Euclidean metric. Since the cumulative outlines of a partly filled-in Julia set form a decreasing sequence (see (10), (13) and (14)), they converge to that Julia set both with respect to Γ and in the ordinary Hausdorff distance in $\kappa(\mathbb{C}^N)$. This means that we only need efficient approximation of the cumulative outlines. To this end we will examine in more detail the number of iterations needed for orbits to exceed in magnitude a radius of escape.

Let $P_1, \ldots, P_k : \mathbb{C}^N \to \mathbb{C}^N$ be polynomial mappings with Łojasiewicz exponents greater than 1, and let R be a common escape radius for these mappings, adjusted to the minimum of their exponents. Given $m \in \mathbb{N}$ we define the *escape time function* for the cumulative m-outline of our Julia set by the formula

$$\mathbb{t}_m(z) = \sum_{i=1}^{m} \mathbb{1}_{\bigcup \mathcal{S}_i}(z), \qquad z \in \mathbb{B}_R.$$

For any $j \in \{1, \ldots, m\}$ this function's jth superlevel set $\{z \in \mathbb{B}_R : \mathbb{t}_m(z) \geq j\}$ is exactly the cumulative j-outline $\bigcup \mathcal{S}_j$ of the Julia set $\mathbb{J}_{\mathrm{tr}}[P_1, \ldots, P_k]$. For the purpose of approximation we need to have a version of the escape time function which does not take into account all the orbits of points in $\mathbb{B}_R$ of length m, but only specific selection of such orbits. Let

$$\emptyset \neq \mathit{\Lambda} \subset \{1, \ldots, k\}^{\{1,\ldots,m\}}.$$

For $z \in \mathbb{B}_R$ we define

$$\mathbb{t}_m^{\mathit{\Lambda}}(z) = \begin{cases} 0 \quad \text{if } \|P_{\lambda(1)}(z)\| > R \text{ for all } \lambda \in \mathit{\Lambda}, \\ \max\{j \leq m : \|(P_{\lambda(j)} \circ \cdots \circ P_{\lambda(1)})(z)\| \leq R \text{ for some } \lambda \in \mathit{\Lambda}\} \\ \quad \text{otherwise.} \end{cases}$$

One could describe $\mathbb{t}_m^{\mathit{\Lambda}}$ as being a *partial escape time function.* Obviously $\mathbb{t}_m^{\mathit{\Lambda}} \leq \mathbb{t}_m$ for any choice of $\mathit{\Lambda}$, and if $\mathit{\Lambda} = \{1, \ldots, k\}^{\{1,\ldots,m\}}$, then $\mathbb{t}_m^{\mathit{\Lambda}} \equiv \mathbb{t}_m$.

THEOREM 3. *Assume that $P_1, \ldots, P_k : \mathbb{C}^N \to \mathbb{C}^N$ are polynomial mappings with Łojasiewicz exponents greater than* 1, *and that R is a common escape radius for these polynomials, adjusted to the minimum of their exponents. Given $m \in \mathbb{N}$, the escape time function $\mathbb{t}_m$ for the cumulative m-outline of the partly filled-in composite Julia set $\mathbb{J}_{\mathrm{tr}}[P_1, \ldots, P_k]$ can be approximated as follows. Let $D \subset \mathbb{C}^N$ be a domain and let μ be a Borel probability measure whose support is $\bar{D}$. Let ν be a strictly positive probability mass function on the set $\{1, \ldots k\}^{\{1,\ldots,m\}}$ of partial addresses. Fix $L, M \in \mathbb{N}$. Consider independent random variables $Z \sim \mu$ and $\lambda^1, \ldots, \lambda^L \sim \nu$* (*with probability distributions μ and ν, respectively*)*. Then producing a sequence of samples*

$$\big\{(z_i, t_i) \in \bar{D} \times \{0, \ldots, m\} : i = 1, \ldots, M\big\} \tag{20}$$

of the random variable

$$(Z, \mathbb{t}_m^{\mathit{\Lambda}}(Z)), \qquad \textit{where } \mathit{\Lambda} = \{\lambda^1, \ldots, \lambda^L\},$$

and letting $M \to \infty$, we approximate the graph of the escape time function $\mathbb{t}_m$ in the following sense. For any $j \in \{1, \ldots, m\}$, the discrete superlevel *sets*

$$\{z_i \in \bar{D} : t_i \geq j,\ i = 1, \dots, M\}$$

converge in the usual Hausdorff metric in $\kappa(\mathbb{C}^N)$, *with probability* 1, *to the corresponding superlevel sets* $\{z \in \bar{D} : \mathbb{t}_m(z) \geq j\}$ *of the escape time functions, which coincide with the matching cumulative* j*-outlines of the Julia set considered.*

Proof. Fix $j \in \{1, \dots, m\}$. It suffices to consider the case $L = 1$.

Let $V \subset \mathbb{C}^N$ be a non-empty open set such that

$$V \cap \bar{D} \cap \bigcup \mathcal{S}_j \neq \emptyset.$$

Then $\mu(V) > 0$, because of the definition of the support of a measure. We can consider the event of selecting a point $z \in V$ and an address $\lambda \in \{1, \dots, k\}^{\{1,\dots,m\}}$ such that $\mathbb{t}_m^{\{\lambda\}}(z) \geq j$ as a success in a Bernoulli trial. Then the probability p of success satisfies the inequality

$$p \geq \mu(V) \cdot \min\bigl\{\nu(\eta) : \eta \in \{1, \dots, k\}^{\{1,\dots,m\}}\bigr\} > 0.$$

Let $s(M)$ denote the number of successes in M trials. Then in view of Bernoulli's Law of Large Numbers, $s(M)/M \to p$ with probability 1, and thus also $s(M) \to \infty$ with probability 1.

Now, choose $\varepsilon > 0$. Let $w_1, \dots, w_\ell$ be an $\varepsilon/2$-dense subset of the compact set $\bar{D} \cap \bigcup \mathcal{S}_j$. If we let each of the balls $B(w_1, \varepsilon/2), \dots, B(w_\ell, \varepsilon/2)$ play the role of the set V above, we can select, with probability 1, points $z_1, \dots, z_M$ forming an ε-dense subset of $\bar{D} \cap \bigcup \mathcal{S}_j$.

This way we can produce an increasing sequence of finite sets approximating the target set with respect to the usual Hausdorff distance in $\kappa(\mathbb{C}^N)$ with probability 1. ■

A corresponding deterministic approach can be described as follows. Let $\{z_i : i \in \mathbb{N}\}$ be a dense sequence in $\mathbb{B}_R$. Let $M \in \mathbb{N}$ be fixed—basically M represents the resolution of the discretization of the escape time function one wants to obtain. In order to determine the set

$$\{(z_i, \mathbb{t}_m(z_i)) : i = 1, \dots, M\}$$

one would have to test the behavior of up to k^m orbits of length m for each point z_i. Only if for a particular tested point z_i a full-length orbit is detected, one could stop further tests for this point and move to z_{i+1}. Otherwise the full number of tests must be carried out. So, for example, if $k = 2$, $m = 20$, $M = 10^4$, the number of necessary tests could exceed 10^{10}.

A mixed approach is also possible. One possibility is to use a deterministic sequence z_i combined with a random choice of orbits of length m for each z_i. Alternatively, one can first select at random L elements of $\{1, \dots, k\}^{\{1,\dots,m\}}$ and then test the L corresponding orbits of length m for each z_i. This last approach with a relatively small L can be very fast, but as a result we only get approximation of L constituents within the cumulative m-outline.

In the following examples we use the method from Theorem 3 to plot some partly filled-in Julia sets in the complex plane, by testing $M = 500000$ points drawn at random from the uniform distribution. The support is either the relevant rectangle in which the picture is generated, or a part of it if fragments of the plotted set are congruent. The length of truncated orbits was set at $m = 10$ and we tested $L = 10$ randomly selected orbits for each point, assuming uniform probability distribution on $\{1, \ldots, k\}^{\{1,\ldots,10\}}$. As far as the number k of polynomials is concerned, we have $k = 2$ in the first two examples and $k = 4$ in the last two. The figures associated with each example show the resulting approximations of the m-outlines of the relevant Julia set. The first three Julia sets were considered in [Kl3] and the last one in [Ko]. Note that the Julia sets in Examples 3 and 4 are obviously not polynomially convex, justifying visually why the term *partly filled-in* is used in connection with composite Julia sets. The issue of polynomial convexity is one of the main difficulties in trying to understand the geometry of the metric space $(\mathcal{R}, \Gamma)$.

Example 1. Consider the polynomials $P_1(z) = z^2 - 2$ and $P_2(z) = z^2$. Then the composite Julia set $\mathbb{J}_{\mathrm{tr}}[P_1, P_2]$ generated by these two functions is shown in Figure 1. Note that $\mathbb{J}[P_1] = [-2, 2]$ and $\mathbb{J}[P_2] = \bar{D}(0, 1)$.

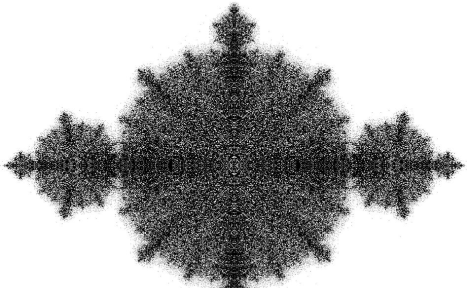

Fig. 1. $\mathbb{J}_{\mathrm{tr}}[P_1, P_2]$

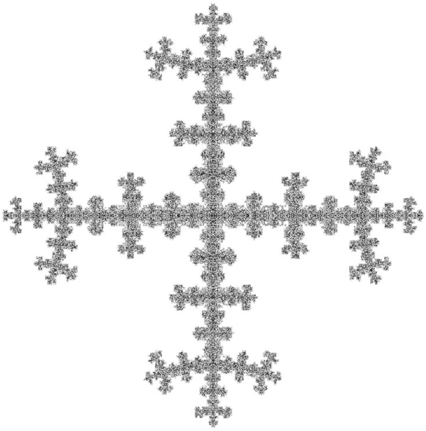

Fig. 2. $\mathbb{J}_{\mathrm{tr}}[P_3, P_4]$

Example 2. Let $P_3(z) = z^2 - 2$ and $P_4(z) = -i(z^2 + 2)$. The composite Julia set $\mathbb{J}_{\mathrm{tr}}[P_3, P_4]$ corresponding to these polynomials is represented in Figure 2. The individual Julia sets of the polynomials P_3 and P_4, are respectively the interval $[-2, 2]$ and the line segment joining $-2i$ and $2i$.

Example 3. Let $Q_j(z) = -|a_j|^2 a_j^{-2}(z - a_j)^3 + a_j$, where z is a complex number and $j = 1, 2, 3, 4$. Let $a_j = i^{j-1}(1 + i)$. Note that $\mathbb{J}[Q_j] = \bar{D}(a_j, 1)$ for $j = 1, 2, 3, 4$, which accounts for the four largest circular shapes that can be seen in Figure 3 within the partly filled-in composite Julia set $\mathbb{J}_{\mathrm{tr}}[Q_1, Q_2, Q_3, Q_4]$.

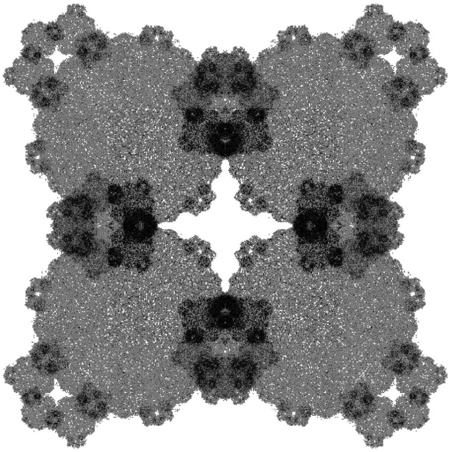

Fig. 3. $\mathbb{J}_{\mathrm{tr}}[Q_1, Q_2, Q_3, Q_4]$

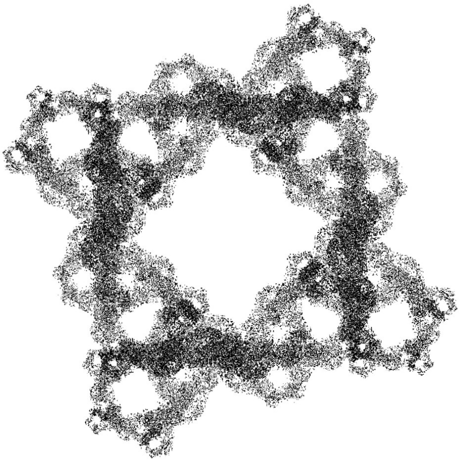

Fig. 4. $\mathbb{J}_{\mathrm{tr}}[R_1, R_2, R_3, R_4]$

Example 4. Let

$$R_j(z) = -\frac{2i}{a_j}(z - a_j)^2 - ia_j + a_j,$$

where $j = 1, 2, 3, 4$. Let $a_j = 2i^{j-1}$. Figure 4 shows the partly filled-in composite Julia set $\mathbb{J}_{\mathrm{tr}}[R_1, R_2, R_3, R_4]$. In this case the union of the ordinary Julia sets generated individually by the four polynomials is the boundary of the square with vertices at the points $2 \pm 2i$, $-2 \pm 2i$.

Acknowledgements. The authors would like to thank Örjan Stenflo and the anonymous referees for helpful comments.

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