ANNALES
POLONICI MATHEMATICI
119.3 (2017)

Hyperbolicity and Vitali properties of unbounded domains in Banach spaces

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Abstract. Let Ω be an unbounded domain in a Banach space. In this work, we wish to impose *local conditions* on the boundary points of Ω (including the point at infinity) that guarantee hyperbolicity and complete hyperbolicity of Ω. We also search for local boundary conditions so that Vitali properties hold true for Ω. These properties might be considered as analogues of the usual taut property in the finite-dimensional case.

1. Introduction. Given a domain (an open connected subset) Ω in a Banach space, we say that Ω is (Kobayashi) *hyperbolic* if the Kobayashi pseudo-distance is a distance and defines the topology of the domain. Moreover, if every Cauchy sequence with respect to the Kobayashi pseudo-distance in Ω is convergent then Ω is called *complete hyperbolic*. On the other hand, we introduce in [KDK] (see also [HQ$^+$]) some properties which are intermediate between hyperbolicity and complete hyperbolicity. Loosely speaking, Ω is said to have the Vitali property if every sequence of holomorphic mappings from a domain A in some Banach space into Ω that converges pointwise only on a sufficiently large subset of A must converge locally uniformly. Furthermore, Ω is said to have the weak (resp. strong) Vitali property (WVP) (resp. (SVP)) depending on the nature of the locus where the pointwise convergence holds. We must say that the above Vitali properties are inspired from a classical theorem due to Vitali which asserts that a locally uniformly bounded sequence of scalar holomorphic functions from a domain D in $\mathbb{C}$ is locally uniformly convergent if it is pointwise convergent on a set having an accumulation point in D. Unfortunately, we do not know if SVP and WVP

2010 *Mathematics Subject Classification*: Primary 32F45, 32Q45; Secondary 32H02, 32A07, 32K05.
Key words and phrases: Lempert functions, Kobayashi pseudo-distance, hyperbolic domains, complete hyperbolic domains, Vitali properties.
Received 7 December 2016; revised 16 August 2017.

DOI: 10.4064/ap4146-8-2017

are really different properties. However, in [KDK] we have been able to show the following implications:

$$\text{complete hyperbolic} \Rightarrow \text{SVP} \Rightarrow \text{WVP} \Rightarrow \text{hyperbolic}.$$

Moreover, it is shown in [KDK, Theorem 3.14] that for domains in $\mathbb{C}^n$, SVP and WVP agree and coincide with the property of tautness. Recall that a domain $\Omega \subset \mathbb{C}^n$ is called *taut* if each sequence $\{f_j\}$ of holomorphic maps from the unit disk $\Delta \subset \mathbb{C}$ into Ω contains a subsequence which is either convergent or compactly divergent. It was observed in [HQ$^+$] (see also [KDK]) that this notion of tautness does not admit a natural generalization to the infinite-dimensional case. Indeed, consider the sequence

$$f_j(\lambda) := (0, \ldots, 0, \lambda/2, 0, \ldots, 0, \ldots), \qquad \lambda \in \Delta.$$

Then $\{f_j\}$ is a sequence of holomorphic mappings from Δ into the unit ball of c_0 that contains no subsequence which is either convergent or compactly divergent. This means that the unit ball of c_0 does not have the expected tautness property. According to [KDK] (see also [HQ$^+$]) the Vitali properties seem to be appropriate analogues of tautness in infinite-dimensional spaces.

A common theme in complex analysis is to decide whether a domain has some property if locally near each boundary point it has. In the finite-dimensional case, we know that a *bounded* domain is taut or complete hyperbolic if and only if it is locally taut or locally complete hyperbolic, respectively. The aim of this note is to extend these results to the case of *unbounded* domains in Banach spaces.

Here is a brief outline of our work. In the next section, we review basic elements of complex analysis in Banach spaces pertaining to our work. We recall, among other things, the Lempert functions and the Kobayashi pseudo-distance. Their construction is completely analogous to the finite-dimensional case. Nevertheless, these concepts, as in [NP], allow us to introduce the key notions of k'- and t-boundary points.

Our main results are explained in Section 3. Theorem 3.1 characterizes hyperbolicity of unbounded domains in Banach spaces in terms of the behavior at infinity of the Lempert function. In the finite-dimensional case, this result is exactly [NP, Proposition 3.1]. However, the proof given there does not extend directly to our case since it requires local compactness of the ambient space. The same applies to our next main result, Theorem 3.5, which in the same spirit as [NP, Proposition 3.6] relates complete hyperbolicity of a domain Ω and the k'-property of its boundary points. Moreover, even in the finite-dimensional case, our proof is perhaps simpler and more direct than the original proof in [NP]. Namely, we rely essentially on a comparison principle for Kobayashi pseudo-distances on domains in Banach spaces (Lemma 3.6). We continue our investigation by proving in Theorem 3.8 that

SVP is indeed a *local* property for domains having ∞ as a t-point. This is again an infinite-dimensional version of [NP, Proposition 3.9] in which tautness is replaced by SVP. The final result, Proposition 3.11, allows us to cook up more examples of domains having SVP by using holomorphic transformations between domains in Banach spaces. This type of result, in the finite-dimensional case, is only discussed in [K, Proposition 5.1.8]. We also provide explicit applications of the above mentioned results.

2. Preliminaries.

We first introduce some standard notation. Given open connected subsets A, Ω of Banach spaces, we write $\operatorname{Hol}(A, \Omega)$ for the space of holomorphic mappings from A into Ω. This space is equipped with the topology of local uniform convergence. We also let $\Delta(0, r) := \{z \in \mathbb{C} : |z| < r\}$ and $\Delta := \Delta(0, 1)$ for short. More generally, $\mathbb{B}(a, r)$ is the open ball with center a and radius $r > 0$ in some Banach space.

Next, we move to the construction of the Lempert function, the first fundamental concept in our work. To this end, we require the following easy fact whose proof is left to the reader.

Lemma 2.1. *Let $\alpha, \beta \in \Delta$ be distinct and $\alpha', \beta' \in E$. Then there exist $\lambda_1, \lambda_2 \in E$ such that for $h(t) = \lambda_1 t + \lambda_2$ $(t \in \mathbb{C})$ we have*

$$h(\alpha) = \alpha', \qquad h(\beta) = \beta', \qquad \|h\|_\Delta \le \|\alpha'\| + 2\left\|\frac{\alpha' - \beta'}{\alpha - \beta}\right\|.$$

The following result clarifies the arguments in [JP, Remark 3.1.1].

Lemma 2.2. *Let Ω be a domain in a Banach space E and $z, w \in \Omega$. Then there exists $f \in \operatorname{Hol}(\Delta, \Omega)$ such that $z, w \in f(\Delta)$.*

Proof. First, since Ω is a connected open set, we can choose a continuous map $\gamma : [0, 1] \to \Omega$ such that $\gamma([0, 1]) \Subset \Omega, \gamma(0) = z, \gamma(1) = w$. Choose $\varepsilon > 0$ so small that

$$\gamma([0, 1]) + \overline{\mathbb{B}}(0, 4\varepsilon) \subset \Omega.$$

Next, we consider the sequence of E-valued Bernstein polynomials

$$P_n(t) := \sum_{k=0}^{n} \gamma\left(\frac{k}{n}\right)\binom{n}{k} t^k (1 - t)^{n-k}, \qquad t \in [0, 1].$$

As in the case of $E = \mathbb{R}$ (see [DL, p. 6]), we can show that P_n converges uniformly to γ on $[0, 1]$. In particular, there exists a polynomial $P : \mathbb{C} \to E$ such that $\|P - \gamma\|_{[0,1]} < \varepsilon$. By Lemma 2.1, there exists an E-valued polynomial h of degree 1 in $\mathbb{C}$ that satisfies

(a) $h(0) = z - P(0)$ and $h(1) = w - P(1)$.

(b) $\|h\|_\Delta \le \|z - P(0)\| + \|(z - P(0)) - (w - P(1))\| < 3\varepsilon$.

Set

$$g(t) := P(t) + h(t), \qquad t \in \Delta.$$

It follows from (a) and (b) that $g(0) = z$, $g(1) = w$ and

$$\|g - \gamma\|_{[0,1]} < 4\varepsilon.$$

Therefore $g([0,1]) \Subset \Omega$, by the choice of ε. Finally, we let U be a bounded simply connected domain in $\mathbb{C}$ such that $[0,1] \subset U$ and $g(U) \subset \Omega$. Let $\varphi : \Delta \to U$ be a biholomorphic mapping and $f := g \circ \varphi \in \mathrm{Hol}(\Delta, E)$. Then $z, w \in f(\Delta) = g(U) \Subset \Omega$. ■

In virtue of the above result, it makes sense to define the *Lempert function* for an arbitrary domain Ω as follows:

$$\begin{aligned}\ell_\Omega(z,w) :=\ & \inf\{|\lambda| : \exists f \in \mathrm{Hol}(\Delta, \Omega),\ f(\lambda) = z,\ f(0) = w\}\\ =\ & \inf\{|\lambda| : \exists f \in \mathrm{Hol}(\Delta, \Omega),\ f(0) = z,\ f(\lambda) = w\}.\end{aligned}$$

A useful feature of the Lempert functions is the following decreasing property: If $F : \Omega \to \Omega'$ is a holomorphic mapping between domains in Banach spaces then

$$\ell_{\Omega'}(F(a), F(b)) \le \ell_\Omega(a,b), \qquad \forall a, b \in \Omega.$$

This follows immediately from the definition. A less obvious property is provided by the next result whose proof is the same as the one in the finite-dimensional case [JP, Proposition 3.1.13].

LEMMA 2.3. *The function $\ell_\Omega(z,w)$ is upper semicontinuous on $\Omega \times \Omega$.*

Following [NP], we will use Lempert functions to analyze boundary points (possibly at infinity) of a domain Ω in a Banach space E. More precisely, we say that $a \in \partial\Omega \cup \{\infty\}$ is a (global) *t-point* of Ω if

$$\lim_{z\to a,\, w\to b} \ell_\Omega(z,w) = 1, \qquad \forall b \in \Omega.$$

It is quite easy to check that ∞ is necessarily a t-point for an unbounded *taut* domain $\Omega \subset \mathbb{C}^n$.

We now turn to the construction of the Kobayashi pseudo-distance on a domain Ω in a Banach space. For $\lambda \in \Delta$ we let

$$p(\lambda) := \frac{1}{2}\log\frac{1+|\lambda|}{1-|\lambda|}$$

be the Poincaré distance from 0 to λ. As in the finite-dimensional case [JP, p. 73], the Lempert function can now be used to define the *Kobayashi pseudo-distance* on Ω as follows. For $z, w \in \Omega$ we let

$$k_\Omega(z,w) := \inf\Big\{\sum_{i=1}^{n} p(\ell_\Omega(z_{i-1}, z_i)) : z_0 = z, z_1, \ldots, z_{n-1} \in \Omega,\ z_n = w\Big\}.$$

It is easy to see that k_Ω defined as above coincides with the original construction (see [K, p. 50] or [FV, p. 81]) using holomorphic chains.

By the same proof as in the finite-dimensional case [K, Proposition 3.1.7] we can show that k_Ω is decreasing under holomorphic maps, i.e. if $f : \Omega \to \Omega'$ is a holomorphic mapping between domains in Banach spaces then

$$k_{\Omega'}(f(z), f(w)) \leq k_\Omega(z, w), \quad \forall z, w \in \Omega.$$

Then Ω is called *hyperbolic* if k_Ω is a distance and defines the topology of Ω. Notice that, in contrast to the finite-dimensional case, k_Ω may be a distance but may not define the topology of Ω [FV, p. 93]. Furthermore, Ω is said to be *complete hyperbolic* if every k_Ω-Cauchy sequence in Ω is convergent. Up to now, there have been only very few results about hyperbolicity and complete hyperbolicity in (infinite-dimensional) Banach spaces. For instance, according to [D, Proposition 6.9] (see also [HQ$^+$, Proposition 3.6]), every bounded convex domain Ω in a Banach space is complete hyperbolic. Hence, all connected open subsets of Ω are hyperbolic. In particular, each bounded connected open subset of a Banach space is hyperbolic. Hyperbolic and complete hyperbolic domains in Banach spaces have also been used in [TS] to study extension phenomena for holomorphic maps between infinite-dimensional Banach spaces.

The next ingredient needed in our work is the concept of plurisubharmonic functions on Banach spaces. We say that $\varphi : \Omega \to [-\infty, \infty)$, where Ω is a domain in a Banach space, is *plurisubharmonic* if u is upper semicontinuous on Ω and the restriction of u to the intersection of Ω with each complex line in E is subharmonic. Notice that we allow the function $u \equiv -\infty$ to be plurisubharmonic.

Using the Kobayashi pseudo-distance and plurisubharmonic functions, we obtain a sufficient condition for a boundary point to be a t-point. This slightly generalizes [NP, Proposition 3.4].

PROPOSITION 2.4. *Let Ω be a domain in a Banach space E and $a \in \partial\Omega \cup \{\infty\}$. Assume that there exists $u \in \mathrm{PSH}(D)$, $u < 0$, satisfying the following conditions:*

(a) *u is a barrier at a, i.e., $\lim_{z\to a} u(z) = 0$.*
(b) *If $\{x_n\} \subset \Omega$ and $u(x_n) \to 0$ then $\{x_n\}$ does not converge in k_Ω.*

Then a is a t-point of Ω.

Note that if Ω is hyperbolic, condition (b) is trivially satisfied.

Proof of Proposition 2.4. Assume that a is not a t-point. Then there exist $\{f_j\} \subset \mathrm{Hol}(\Delta, \Omega)$ and $\{\lambda_j\} \subset \Delta$ with $f_j(0) \to b \in \Omega$, $f_j(\lambda_j) \to a$ and $\lambda_j \to \lambda_0 \in \Delta$. Set

$$v(\lambda) := \sup_{j\geq 1} (u \circ f_j)(\lambda), \quad \forall \lambda \in \Delta.$$

Then v^*, the upper regularization of v, is ≤ 0 and subharmonic on Δ. Moreover, by assumption (a) we obtain

$$v(\lambda_j) \geq (u \circ f_j)(\lambda_j) \to 0 \quad \text{as } j \to \infty.$$

It follows that $v^*(\lambda_0) = 0$. The maximum principle now implies that $v^* \equiv 0$ on Δ. Since $v = v^*$ almost everywhere on Δ, we may choose $\{\beta_n\} \subset \Delta$ with $\beta_n \to 0$ and $v(\beta_n) = 0$. So for each $n \geq 1$ we can choose j_n such that

$$(u \circ f_{j_n})(\beta_n) > -1/n.$$

Set $x_n := f_{j_n}(\beta_n)$. We have $u(x_n) \to 0$ as $n \to \infty$. By the triangle inequality and the decreasing property of the Kobayashi pseudo-distance we get

$$\begin{aligned} k_\Omega(x_n, b) &\leq k_\Omega(x_n, f_{j_n}(0)) + k_\Omega(f_{j_n}(0), b) \\ &\leq k_\Delta(\beta_n, 0) + k_\Omega(f_{j_n}(0), b), \quad \forall n \geq 1. \end{aligned}$$

By letting $n \to \infty$ we have $k_\Omega(x_n, b) \to 0$. This contradicts (b). ■

The following notions are natural generalizations of the corresponding ones in the finite-dimensional case [NP, p. 611]. They will play a central role in our investigation of complete hyperbolic domains in Banach spaces.

DEFINITION 2.5. Let Ω be a domain in a Banach space E and $a \in \partial\Omega \cup \{\infty\}$. We say that a is a *k'-point* of Ω if no k_Ω-Cauchy sequence in Ω converges to a. The point a is called a *local k'-point* of Ω if it admits a neighborhood U such that a is a k'-point for each connected component of $U \cap \Omega$.

REMARK. (i) If a is a k'-point for each connected component of $U \cap \Omega$ then by the decreasing property of the Kobayashi pseudo-distance it is also a k'-point for each connected component of $V \cap \Omega$, where V is an open neighborhood of a such that $\overline{V} \subset U$.

(ii) The relationship between t-points and k'-points is, however, not clear to us.

To check whether a boundary point of a domain in Banach space is a k'-point (resp. local k'-point) we rely on the following result.

PROPOSITION 2.6. *Let Ω and Ω' be domains in Banach spaces and $\varphi : \Omega \to \Omega'$ be a holomorphic mapping. Let $a \in \partial\Omega \cup \{\infty\}$ be a boundary point having the following property: for every sequence $\{a_n\} \subset \Omega$ with $a_n \to a$, there exists a subsequence $\{a_{n_k}\}$ such that $\varphi(a_{n_k})$ converges to some $a' \in \partial\Omega' \cup \{\infty\}$ which is a k'-point of Ω'. Then a is a k'-point of Ω. Moreover, if $\lim_{z\to a} \varphi(z) = a' \in \partial\Omega' \cup \{\infty\}$ and a' is a local k'-point of Ω' then a is also a local k'-point of Ω.*

Proof. The first assertion follows immediately from the decreasing property of the Kobayashi pseudo-distance. For the second one, we let $V \subset F$ be an open neighborhood of a' such that a' is a k'-point for each connected component of $V \cap \Omega'$. By continuity of φ at a, we can find an open neighborhood $U \subset E$ of a such that $\varphi(U \cap \Omega) \subset V \cap \Omega'$. Fix a connected component U' of $U \cap \Omega$. We claim that a is a k'-point of U'. For this, it suffices to take a connected component V' of $V \cap \Omega'$ such that $\varphi(U') \subset V'$. By assumption, a' is a k'-point of V'. Thus the first part of the proposition shows that a is k'-point of U'. ■

Finally, we recall the notions of Vitali properties investigated in [KDK]. Before giving the precise definitions, we introduce the following notation: given a subset S of a domain A in a Banach space, we let

$$S^u := \{z \in A \cap \overline{S} : \text{for every connected neighborhood } U \text{ of } z \text{ and}\\ \text{every } f \in \mathrm{Hol}(U, \mathbb{C}),\ f|_{U \cap S} = 0 \Rightarrow f|_U = 0\}.$$

Definition 2.7. Let Ω be a domain in a Banach space. Then we say that Ω has the *weak Vitali property* (WVP for short) if a sequence $\{f_j\}$ of holomorphic mappings from a connected open set A in a Banach space into Ω is convergent in $\mathrm{Hol}(A, \Omega)$ provided that $Z_{\{f_j\}} \cap Z^u_{\{f_j\}} \neq \emptyset$. Here $Z_{\{f_j\}}$ is the collection of points $x \in A$ such that $f_j(x)$ is convergent to some point in Ω. Moreover, Ω is said to have the *strong Vitali property* (SVP for short) if $\{f_j\}$ is convergent in $\mathrm{Hol}(A, \Omega)$ as long as $Z^u_{\{f_j\}} \neq \emptyset$.

In [KDK], these properties are formulated in the broader context of Banach analytic manifolds. For domains in Banach spaces, Definition 2.7 coincides with Definition 2.1 in [KDK] (note that Definition 2.1 in [KDK] should be changed to match the above Definition 2.7). At the end of this paper, we will prove that the two Vitali properties agree at least in the category of unbounded domains having ∞ as a k'-point.

3. Main results.

The first result generalizes [NP, Proposition 3.1] to unbounded domains in Banach spaces.

Theorem 3.1. *Let Ω be an unbounded domain in a complex Banach space. Then the following statements are equivalent:*

- (a) *Ω is hyperbolic.*
- (b) $\liminf_{z \to \infty,\, w \to b} \ell_\Omega(z, w) > 0$ *for all $b \in \Omega$.*

Since Montel's theorem is not valid for Banach-valued holomorphic functions, the proof given in [NP, Proposition 3.1] does not directly apply in our case. Instead, we employ ideas from [KDK, Theorem 3.1] that relate hyperbolicity and the Vitali property of domains in Banach spaces. The lemma below is an analogue of an earlier result due to Kiernan in the finite-dimensional case [K, Lemma 5.1.4].

Lemma 3.2 ([KDK, Lemma 3.2]). *Let Y be a connected open subset of a complex Banach space and $x \in Y$. Let U, V, W be open subsets of Y such that $x \in V$, $\overline{V} \subset U$, $\overline{U} \cap \overline{W} = \emptyset$ and U is hyperbolic. Assume that there exists $\delta \in (0,1)$ such that for every $f \in \mathrm{Hol}(\Delta, Y)$ with $f(0) \in V$ we have $f(\Delta(0,\delta)) \subset U$. Then $k_Y(x, W) := \inf_{y \in W} k_Y(x,y) > 0$.*

We also need the following variant of Vitali's convergence theorem for holomorphic vector-valued functions.

Lemma 3.3 ([AN, Theorem 2.1]). *Let X be a Banach space and Ω an open connected subset of $\mathbb{C}$. Let $\{f_j\}$ be a sequence in $\mathrm{Hol}(\Omega, X)$ such that:*

(a) *$\{f_j\}$ is locally uniformly bounded on Ω.*
(b) *$Z_{\{f_j\}}$ has an accumulation point in Ω.*

Then $\{f_j\}$ converges in $\mathrm{Hol}(\Omega, X)$.

In [KDK, Lemma 3.5] we provide a generalization of the above lemma to the case where Ω is an open connected subset of a Banach space.

Proof of Theorem 3.1. (a)⇒(b). Assume that there exists $b \in \Omega$ such that

$$\liminf_{z\to\infty,\, w\to b} \ell_\Omega(z,w) = 0.$$

Then there exists a sequence $\{f_j\} \subset \mathrm{Hol}(\Delta, \Omega)$ such that $f_j(0) \to \infty$ and $f_j(x_j) \to b$ for some $x_j \in \Delta$ with $x_j \to 0$. Since Ω is hyperbolic, by the decreasing property of the Kobayashi pseudo-distance we obtain

$$k_\Omega(f_j(0), f_j(x_j)) \le k_\Delta(0, x_j) \to 0 \quad \text{as } j \to \infty.$$

It follows that $f_j(0) \to b$ as $j \to \infty$. This is impossible.

(b)⇒(a). First we note that if $x_n \to x$ in the original topology of Ω then $k_\Omega(x_n, x) \to 0$ as $n \to \infty$. Conversely, assume that there exists a sequence $\{x_n\} \subset \Omega$ such that $k_\Omega(x_n, x) \to 0$ as $n \to \infty$ but $x_n \nrightarrow x$. Take a bounded open neighborhood U of x and an open neighborhood W of the sequence $\{x_n\}$ such that $\overline{U} \cap \overline{W} = \emptyset$. Then U is hyperbolic. Choose a sequence of open sets V_n such that $U \supset V_n \downarrow x$. Define a sequence $\{\delta_n\}_{n\ge 0}$ by $\delta_0 = 1/2$, $\delta_1 = 1/3$ and

$$0 < \delta_{n+1} < \min\left\{\frac{1}{n}, r_n := \delta_n \prod_{j=0}^{n-1} \frac{\delta_j - \delta_n}{1 - \delta_j \delta_n}\right\}, \qquad \forall n \ge 1.$$

It follows that $r_{n+1} < \delta_{n+1} < r_n$. In particular, $r_n \downarrow 0$. Using Lemma 3.2, we obtain a sequence $\{f_n\} \subset \mathrm{Hol}(\Delta, \Omega)$ and points $a_n \in \Delta(0, r_n)$ such that

$$f_n(0) \in V_n, \qquad f_n(a_n) \notin U, \qquad \forall n \ge 1.$$

For each $n \geq 1$, define

$$\theta_n(\lambda) := \frac{a_n}{r_n}\lambda \prod_{j=0}^{n-1} \frac{\delta_j - \lambda}{1 - \delta_j \lambda}, \qquad \forall \lambda \in \Delta.$$

Then $\theta_n \in \mathrm{Hol}(\Delta, \Delta)$. Moreover,

$$\theta_n(0) = \theta_n(\delta_j) = 0, \quad \forall 0 \leq j \leq n-1; \qquad \theta_n(\delta_n) = a_n.$$

Finally, for each $n \geq 1$ we define

$$g_n := f_n \circ \theta_n \in \mathrm{Hol}(\Delta, \Omega).$$

Then

$$g_n(\delta_j) = g_n(0) = f_n(0), \qquad \forall n \geq 1, \forall 0 \leq j \leq n-1.$$

This implies that

$$\lim_{n\to\infty} g_n(0) = \lim_{n\to\infty} g_n(\delta_j) = \lim_{n\to\infty} f_n(0) = x, \qquad \forall j \geq 0,$$
$$g_n(\delta_n) = f_n(a_n) \notin U.$$

By the hypothesis, we have

$$r := \liminf_{z\to\infty,\, w\to x} \ell_\Omega(z, w) > 0.$$

We claim that $\{g_n\}$ is locally bounded on $r\Delta$. Indeed, otherwise there exist $\lambda_n \in r\Delta$ with $\lambda_n \to \lambda_0 \in r\Delta$ such that $z_n = g_n(\lambda_n) \to \infty$. This implies

$$\liminf_{n\to\infty} \ell_\Omega(z_n, g_n(0)) \leq |\lambda_0| < r.$$

Since $g_n(0) \to x$, we obtain a contradiction to the choice of r.

Now we use Lemma 3.3 to infer that $g_n \to g$ in $\mathrm{Hol}(r\Delta, \Omega)$. This is impossible, because

$$a_n \to 0 \quad \text{and} \quad g_n(a_n) \notin U \ni x, \quad \forall n \geq 1.$$

The proof is complete. ■

REMARK. If ∞ is a k'-point of Ω then Ω is hyperbolic. Indeed, if this is false then there exist $a_n \to \infty$ and $b_n \to b \in \Omega$ such that $\ell_\Omega(a_n, b_n) \to 0$. The triangle inequality now yields

$$k_\Omega(a_n, b) \leq k_\Omega(a_n, b_n) + k_\Omega(b_n, b) \leq p(\ell_\Omega(a_n, b_n)) + k_\Omega(b_n, b) \to 0 \text{ as } n \to \infty.$$

Hence $a_n \to \infty$ is a k_Ω-Cauchy sequence, which is absurd.

Let E, F be Banach spaces, Ω a domain in E and $h : \Omega \times F \to \mathbb{R}$ a non-negative upper semicontinuous function satisfying

$$h(z, \lambda w) = |\lambda| h(z, w), \qquad \forall \lambda \in \Delta.$$

Define

$$\Omega_h := \{(z, w) \in \Omega \times F : h(z, w) < 1\}.$$

Then Ω_h is a Hartogs domain over Ω with (balanced) fibers in F. Using Theorem 3.1, we are able to characterize hyperbolicity of Ω_h in terms of Ω and h.

Corollary 3.4. *Ω_h is hyperbolic if only if Ω is hyperbolic and*

$$\inf_{K\times\partial\mathbb{B}} h(z,w) > 0, \qquad \forall K \text{ compact in } \Omega,$$

where $\mathbb{B} = \{w \in F : \|w\| < 1\}$.

Proof. We follow [NP, proof of Proposition 4.2]. First, assume that Ω_h is hyperbolic. Since $\{\Omega\} \times \{0\} \subset \Omega_h$, we infer that Ω is hyperbolic as well. Now, suppose that there exists $K \Subset \Omega$ such that

$$\inf_{K\times\partial\mathbb{B}} h(z,w) = 0.$$

Then there exist $a_j \in \Omega_h$ with $a_j = (a'_j, a''_j) \in K \times \partial\mathbb{B}$ such that $h(a_j) \to 0$. We may assume that $a'_j \to a'$. For each $j \ge 1$, we define

$$f_j(t) = \left(a'_j, \frac{ta''_j}{h(a_j)}\right) \in \mathrm{Hol}(\Delta, \Omega_h).$$

It is easy to check that

$$f_j(0) = (a'_j, 0) \to a^* := (a', 0) \in K \times \{0\} \subset \Omega_h,$$

and for every $\delta \in (0,1)$ we have $f_j(t) \to \infty$ for all $t \in \Delta \setminus \Delta(0,\delta)$. Thus, for $\delta \in (0,1)$, by the decreasing property of the Lempert functions we obtain

$$\ell_\Omega(f_j(\delta), f_j(0)) \le \ell_\Delta(\delta, 0) = \delta.$$

This implies that

$$\liminf_{z\to\infty,\, w\to a^*} \ell_{\Omega_h}(z,w) = 0,$$

which contradicts Theorem 3.1.

Conversely, assume that Ω_h is not hyperbolic. By Theorem 3.1, there exist $\lambda_j \in \Delta$ and $f_j \in \mathrm{Hol}(\Delta, \Omega_h)$ with $\lambda_j \to 0$, $f_j(\lambda_j) \to \infty$ and $f_j(0) \to a \in \Omega_h$. Write $a = (a', a'')$ with $a' \in \Omega$, $a'' \in F$ and $f_j = (f'_j, f''_j)$. Then $f'_j(\lambda_j) \to a'$ by hyperbolicity of Ω. Set $K := \{f'_j(\lambda_j)\} \cup \{a'\}$. Then K is a compact subset of Ω. Moreover, $f''_j(\lambda_j) \to \infty$ and we have

$$\begin{aligned} 1 > h(f_j(\lambda_j)) &= \|f''_j(\lambda_j)\| h\left(f'_j(\lambda_j), \frac{f''_j(\lambda_j)}{\|f''_j(\lambda_j)\|}\right) \\ &\ge \|f''_j(\lambda_j)\| \inf_{K\times\partial\mathbb{B}} h(z,w) \to \infty, \end{aligned}$$

which is clearly absurd. ∎

Our next main result characterizes complete hyperbolicity of unbounded domains in Banach spaces.

THEOREM 3.5. *Let Ω be an unbounded domain in a Banach space such that ∞ is a k'-point of Ω. Then the following assertions are equivalent:*

- (a) *Ω is complete hyperbolic;*
- (b) *any finite boundary point p of Ω admits an open neighbourhood U of p such that each connected component of $U \cap \Omega$ is complete hyperbolic;*
- (c) *any finite boundary point of Ω is a local k'-point;*
- (d) *any finite boundary point of Ω is a k'-point.*

Before going into the proof, a few remarks are in order.

REMARK. (i) The main thrust of the theorem is the implication (c)⇒(a) that characterizes complete hyperbolicity of Ω in terms of *local* k'-finite boundary points.

(ii) The implication (d)⇒(a) is false if ∞ is only supposed to be a *local* k'-point. For a trivial counterexample, we may take $\Omega = \mathbb{C}$.

(iii) In the finite-dimensional case, the above theorem is partially contained in [NP, Proposition 3.6]. The proof there does not, however, directly generalize to our case since it requires two essential ingredients: the local compactness of $\mathbb{C}^n$ and the representation of the Kobayashi pseudo-distance in terms of the Kobayashi–Royden infinitesimal pseudo-metric. See for example the implications (iv)⇒(i) and (iii)⇒(iv) in [NP] which correspond to our (d)⇒(a) and (c)⇒(d), respectively.

(iv) If $p \in \partial\Omega$ satisfies the condition given in (b) then there exists $r_0 > 0$ such that each connected component of $\mathbb{B}(p,r) \cap \Omega$ is complete hyperbolic for $0 < r < r_0$. This follows from the decreasing property of the Kobayashi pseudo-distance and the fact that $\mathbb{B}(p,r)$ is complete hyperbolic.

For the proof of Theorem 3.5, we first introduce the following notation. Let Ω be a domain in a Banach space and $a \in \Omega$. For each $\delta > 0$, we denote by $U_\Omega(a,\delta)$ the Kobayashi "ball" $\{x \in \Omega : k_\Omega(a,x) < \delta\}$. The needed technical result is the following comparison principle.

LEMMA 3.6. *Let Ω, a be as above and δ, ε be positive numbers. Then*

- (a) *$U_\Omega(a,\delta)$ is connected.*
- (b) *There exists a constant $C > 1$ such that for every $\varepsilon' > 0$, each pair $p, q \in U_\Omega(a,\delta)$ can be joined by a chain of holomorphic disks lying in $U_\Omega(a, 3\delta+\varepsilon)$ with length not exceeding $C(k_\Omega(p,q)+\varepsilon')$. In particular,*

$$k_{U_\Omega(a,3\delta+\varepsilon)}(p,q) \le Ck_\Omega(p,q), \qquad \forall p,q \in U_\Omega(a,\delta).$$

Proof. (a) Fix $z \in U_\Omega(a,\delta)$. Since $k_\Omega(a,z) < \delta$, we may find a chain of points $z_0 = a, z_1, \dots, z_{n-1}, z_n = z$ lying in Ω, holomorphic maps $f_1, \dots, f_n \in \mathrm{Hol}(\Delta, \Omega)$ and points $a_1, \dots, a_n$ of Δ such that

$$f_i(0) = z_{i-1}, \quad f_i(a_i) = z_i, \quad 1 \le i \le n,$$

and

$$p(0,a_1)+\cdots+p(0,a_n)<\delta.$$

Fix $1\le i\le n$ and $\xi\in\Delta$ with $|\xi|\le|a_i|$. Then by the triangle inequality and the decreasing property for the Kobayashi pseudo-distance we obtain

$$\begin{aligned}k_\Omega(a,f_i(\xi))&\le k_\Omega(a,z_1)+\cdots+k_\Omega(z_{i-1},f_i(\xi))\\&\le p(0,a_1)+\cdots+p(0,a_{i-1})+p(0,\xi)<\delta.\end{aligned}$$

Thus $f_i(\xi)\in U_\Omega(a,\delta)$. This implies that a and z can be joined by a continuous curve lying in $U_\Omega(a,\delta)$. Hence $U_\Omega(a,\delta)$ is connected.

(b) The proof follows [K, proof of Proposition 3.1.19]. The key idea is to express the Kobayashi pseudo-distance as the infimum of the lengths of holomorphic chains and then apply the decreasing property of the Kobayashi pseudo-distance. We do not repeat the details here. ■

Proof of Theorem 3.5. (a)⇒(b)⇒(c) are obvious.

(c)⇒(d). Assume for the sake of contradiction that there exists a k_Ω-Cauchy sequence $\{z_n\}\subset\Omega$ such that $z_n\to a\in\partial\Omega$. For $n\ge1$ we set

$$\rho_n:=\sup_{m,l\ge n}k_\Omega(z_m,z_l).$$

Then $\rho_n\downarrow0$. For each $r>0$ consider the open set $\Omega_r:=\mathbb{B}(a,r)\cap\Omega$. We claim that there exists $N\ge1$ such that

$$U_\Omega(z_N,4\rho_N)\subset\Omega_N.$$

Indeed, if the claim is false then for each $n\ge1$, there exists $w_n\in\Omega$ such that

$$\|w_n-a\|\ge n,\qquad k_\Omega(z_n,w_n)<4\rho_n.$$

By the triangle inequality, $\{w_n\}$ is a k_Ω-Cauchy sequence. Since $w_n\to\infty$, we obtain a contradiction to the assumption that ∞ is a k'-point of Ω.

By Lemma 3.6(a), there exists a connected component Ω'_N of Ω_N that includes $U_\Omega(z_N,4\rho_N)$. Moreover, we can find a constant $C>1$ such that for $m,l\ge N$,

$$k_{\Omega'_N}(z_m,z_l)\le k_{U_\Omega(z_N,4\rho_N)}(z_m,z_l)\le Ck_\Omega(z_m,z_l).$$

This implies that $\{z_n\}_{n\ge N}\subset\Omega'_N$ is a $k_{\Omega'_N}$-Cauchy sequence. Now we choose $r\in(0,N)$ such that a is a k'-point for each connected component of Ω_r. For $n\ge1$, we set

$$\eta_n:=\sup_{m,l\ge n}k_{\Omega'_N}(z_m,z_l).$$

Then $\eta_n\downarrow0$.

We now prove that there exists $N'\ge N$ such that

$$U_{\Omega'_N}(z_{N'},4\eta_{N'})\subset\Omega_r.$$

Assume otherwise; then for each $n \geq N$, there exists $w_n \in \Omega'_N \setminus \mathbb{B}(a,r)$ such that

$$k_{\mathbb{B}(a,N)}(z_n, w_n) \leq k_{\Omega'_N}(z_n, w_n) \leq 4\eta_n \to 0 \quad \text{as } n \to \infty.$$

This contradicts hyperbolicity of $\mathbb{B}(a,N)$.

Next, we let Ω'_r be the connected component of Ω_r that contains $U_{\Omega'_N}(z_{N'}, 4\eta_{N'})$. By Lemma 3.6(b), there exists a constant $C' > 1$ such that for $m, l \geq N'$,

$$k_{\Omega'_r}(z_m, z_l) \leq k_{U_{\Omega'_N}(z_{N'}, 4\eta_{N'})}(z_m, z_l) \leq C' k_{\Omega'_N}(z_m, z_l).$$

This implies that $\{z_n\}_{n \geq N'} \subset \Omega'_r$ is a $k_{\Omega'_r}$-Cauchy sequence, giving a contradiction.

(d)⇒(a). Let $\{z_n\}$ be a k_Ω-Cauchy sequence of Ω. We must show that $\{z_n\}$ is convergent to some point in Ω in the original topology of E. For $n \geq 1$, we set

$$\rho'_n := \sup_{m,l \geq n} k_\Omega(z_m, z_l) < \infty.$$

Then $\rho'_n \downarrow 0$. We also let

$$U_\Omega(z_n, 4\rho'_n) := \{z \in \Omega : k_\Omega(z_n, z) < 4\rho'_n\}.$$

We first claim that there exists N'' such that $U_\Omega(z_{N''}, 4\rho'_{N''})$ is bounded. If the claim is false, then there exists a sequence $\{w_n\} \subset \Omega$ such that

$$\|w_n\| > n, \quad k_\Omega(z_n, w_n) < 4\rho'_n.$$

Using the triangle inequality we deduce that $w_n \to \infty$ is also a k_Ω-Cauchy sequence. This violates the assumption that ∞ is a k'-point of Ω.

Now, we apply Lemma 3.6(b) to find a constant $C'' > 1$ such that

$$k_{U_\Omega(z_{N''}, 4\rho'_{N''})}(p,q) \leq C'' k_\Omega(p,q), \quad \forall p, q \in U_\Omega(z_{N''}, \rho'_{N''}).$$

Hence $\{z_n\}_{n \geq N''}$ is a $k_{U_\Omega(z_{N''}, 4\rho'_{N''})}$-Cauchy sequence. Choose $R > 0$ so large that

$$U_\Omega(z_{N''}, 4\rho'_{N''}) \subset \mathbb{B}(0,R).$$

Observe that $\mathbb{B}(0,R)$ is complete hyperbolic, so $z_n \to p \in \overline{\Omega}$ (in the original topology of E). Since every (finite) boundary point of Ω is a k'-point, we have, in fact, $p \in \Omega$. Hence Ω is k_Ω-complete. ■

The above theorem yields the following partial generalization of [G, Theorem 1].

Corollary 3.7. *Let Ω be an unbounded domain in a Banach space E. Then Ω is complete hyperbolic if:*

(a) *Every $a \in \partial\Omega$ admits a local peak holomorphic functions, i.e., there exists a neighborhood U of a and a holomorphic function h_a such that $|h_a| < 1$ on $U \cap \Omega$ and $\lim_{z \to a} |h_a(z)| = 1$.*

(b) *There exists a peak holomorphic function at* ∞ *for* Ω, *i.e., a holomorphic function* h *on* Ω *such that* $|h| < 1$ *on* Ω *and* $|h(z)| \to 1$ *as* $z \to \infty$.

Proof. We first show that ∞ is a k'-point for Ω. Indeed, by the assumption, we see that h maps Ω into Δ. Moreover, from each sequence $\{a_n\} \subset \Omega$ with $a_n \to \infty$ we may select a subsequence $\{a_{n_k}\}$ such that $h(a_{n_k}) \to a' \in \partial\Delta$. Since a' is a k'-point of Δ, by Proposition 2.6 we infer that a is a k'-point of Ω. By the same reasoning, every finite boundary point of Ω is also a k'-point. Thus, we may apply Theorem 3.5 to complete the proof. ∎

Remark. (i) We do not know if the above result remains true if there exists only a *local* peak holomorphic function at ∞ for Ω, i.e., there exists $R > 0$ and a holomorphic function h on $\Omega_R := \Omega \setminus \mathbb{B}(0,R)$ such that $|h| < 1$ on Ω_R and $|h(z)| \to 1$ as $z \to \infty$.

(ii) Let $\mathbb{B}^\infty$ be the unit ball in ℓ^∞ and $\{a_n\}_{n\ge1}$ a sequence of positive numbers such that $\sum_{n\ge1} a_n < \infty$. Consider the open set

$$\Omega := \Big\{ z = (z_0, z_1, \ldots) \in \mathbb{C} \times \mathbb{B}^\infty \subset \ell^\infty : \rho(z) := \Im z_0 + \sum_{n\ge1} a_n |z_n|^2 < 0 \Big\}.$$

It is easy to check that ρ is a convex function on ℓ^∞. This implies that Ω is convex. Thus, every $a \in \partial\Omega$ admits a local peak holomorphic function. Let

$$f(z_0, z') = \frac{z_0 + i}{z_0 - i}, \qquad (z_0, z') \in \ell^\infty,\ z_0 \neq i.$$

Then

$$|f(z_0, z')|^2 - 1 = \left| \frac{z_0 + i}{z_0 - i} \right|^2 - 1 = \frac{4\Im z_0}{|z_0 - i|^2}, \qquad \forall z_0 \neq i.$$

Since $\Im z_0 < 0$ on Ω, it follows that $|f| < 1$ on Ω. Moreover, by easy estimates we also obtain $|f(z)| \to 1$ as $z \to \infty, z \in \Omega$. Hence f is a peak holomorphic function at ∞ for Ω. By Theorem 3.5 we conclude that Ω is complete hyperbolic.

The next result roughly says that SVP and local SVP are equivalent for domains having ∞ as a t-point.

Theorem 3.8. *Let* Ω *be an unbounded domain in* E *such that* ∞ *is a* t*-point of* Ω. *Then the following statements are equivalent:*

(a) Ω *has SVP.*

(b) *For every* $p \in \partial\Omega$, *there exists a neighborhood* U *of* p *such that* $U \cap \Omega$ *has SVP.*

Remark. (i) Since SVP coincides with tautness in the class of domains in $\mathbb{C}^n$ (see [KDK, Theorem 3.14 and following remarks]), Theorem 3.8 essentially generalizes [NP, Proposition 3.2] (see also [G, Proposition 2]).

(ii) If $p \in \partial\Omega$ satisfies the condition in (b) then $\mathbb{B}(p,r) \cap \Omega$ has SVP for every $r > 0$ such that $\overline{\mathbb{B}}(p,r) \subset U$. To see this, we fix a connected open subset A of a Banach space and a sequence $\{f_j\}_{j\geq 1} \subset \mathrm{Hol}(A, \mathbb{B}(p,r) \cap \Omega)$ such that $Z^u_{\{f_j\}} \neq \emptyset$. Then f_j converges to a holomorphic map f in $\mathrm{Hol}(A, U \cap \Omega)$. We must show that there exists no point $\xi \in A$ such that $f_j(\xi)$ converges to $\eta \in \partial\mathbb{B}(p,r)$. Assume otherwise; then we let $\varphi : E \to \mathbb{C}$ be a complex linear map such that $\Re\varphi(x) < c := \Re\varphi(\eta)$ for every $x \in \overline{\mathbb{B}}(p,r) \setminus \{\eta\}$. Then $\psi := \Re(\varphi \circ f)$ is a plurisubharmonic function on A which attains its maximum at $\xi \in A$. Thus $\psi|_A \equiv c$. Hence $f(A) \subset \partial\mathbb{B}(p,r)$. This contradicts the fact that $Z^u_{\{f_j\}} \neq \emptyset$.

Proof of Theorem 3.8. (a)⇒(b) is trivial.

(b)⇒(a). Let A be a connected open subset of a Banach space and $\{f_j\}_{j\geq 1} \subset \mathrm{Hol}(A, \Omega)$ be a sequence such that $Z^u_{\{f_j\}} \neq \emptyset$. We first claim that $\{f_j\}$ is locally uniformly bounded on A. Indeed, if this is not so then there are $A \ni z_j \to z_0 \in A$ such that $f_j(z_j) \to \infty$. Fix $\xi \in Z_{\{f_j\}}$. Then $f_j(\xi) \to \eta \in \Omega$. Further, by the decreasing property of the Lempert functions we obtain

$$\ell_\Omega(f_j(z_j), f_j(\xi)) \leq \ell_A(z_j, \xi).$$

Since ∞ is a t-point of Ω, by letting $j \to \infty$ and taking limsup we obtain

$$1 = \limsup_{j\to\infty} \ell_\Omega(f_j(z_j), f_j(\xi)) \leq \limsup_{j\to\infty} \ell_A(z_j, \xi) \leq \ell_A(z_0, \xi) < 1,$$

which is absurd. Here the second inequality follows from upper semicontinuity of ℓ_A (Lemma 2.2).

Thus we have shown that $\{f_j\}$ is locally uniformly bounded on A. Now, we use [KDK, Lemma 3.5] to get $f_j \to f$ in $\mathrm{Hol}(A, E)$. Set $A' := f^{-1}(\Omega)$. Then obviously $A' \neq \emptyset$ and A' is open. Suppose that $A' \neq A$. Then we can find $z_0 \in A \cap \partial A'$. By (b), we may choose a neighborhood V of $f(z_0)$ such that $V \cap \Omega$ has SVP. Since $f_j \to f$ in $\mathrm{Hol}(A, E)$, there exists a neighborhood U of z_0 and $j_0 \geq 1$ such that

$$f_j(U) \subset V \cap \Omega, \quad \forall j \geq j_0.$$

It follows that $f_j \to f$ in $\mathrm{Hol}(A, V \cap \Omega)$, because $V \cap \Omega$ has SVP and $A' \cap U \subset Z^u_{\{f_j|_U\}}$. Hence $A' = A$, a contradiction.

Summing up, we have proved our claim that $\{f_j\}$ is convergent in $\mathrm{Hol}(A, \Omega)$. ■

As an illustration of the above theorem we have the following equivalence between Vitali properties on certain classes of unbounded domains in Banach spaces.

COROLLARY 3.9. *Let Ω be an unbounded domain in a Banach space such that ∞ is a t-point. Then Ω has SVP if only if Ω has WVP.*

Proof. Assume Ω has WVP. Fix $a \in \partial\Omega$ and a ball $\mathbb{B}$ around a. We claim that $\Omega \cap \mathbb{B}$ has SVP. For this, we let A be a connected open subset of a Banach space and $\{f_j\}$ be a sequence in $\mathrm{Hol}(A, \Omega\cap\mathbb{B})$ such that $Z^u_{\{f_j\}} \neq \emptyset$. By Lemma 3.3, $\{f_j\}$ is convergent to $f \in \mathrm{Hol}(A, E)$. Set $U := f^{-1}(\Omega \cap \mathbb{B})$. Then U is open and $U = Z_{\{f_j\}} \neq \emptyset$. It follows that $\emptyset \neq U = Z_{\{f_j\}} \cap Z^u_{\{f_j\}}$. Hence, as Ω has WVP, $\{f_j\}$ is convergent to $f \in \mathrm{Hol}(A, \Omega)$. It remains to check that $f(A) \cap \partial\mathbb{B} = \emptyset$.

Assume otherwise; then there exists $\xi \in A$ such that $f_j(\xi) \to \partial\mathbb{B}$. Fix $a \in Z_{\{f_j\}}$. By the decreasing property of the Kobayashi pseudo-distance we obtain

$$\sup_{j\geq 1} k_{\mathbb{B}}(f_j(\xi), f_j(a)) \leq k_A(\xi, a) < \infty.$$

We arrive at a contradiction, since $\mathbb{B}$ is complete hyperbolic. ■

REMARK. In light of the above result, the following question is of interest: *Let Ω be an unbounded domain in a Banach space E having SVP. Is ∞ necessarily a t-point of Ω?* Observe that the answer is affirmative if $E = \mathbb{C}^n$, since SVP is equivalent to tautness in this case.

The result below provides explicit examples of unbounded domains in Banach spaces having SVP.

COROLLARY 3.10. *Let Ω be an unbounded domain in a Banach space E such that ∞ is a t-point. Then Ω has SVP if one of the following holds:*

(i) *Every (finite) boundary point of Ω is a t-point;*
(ii) *Every (finite) boundary point of Ω admits a barrier.*

Proof. Assume that (i) holds. Fix $p \in \partial\Omega$ and $r > 0$. We claim that $U := \mathbb{B}(p, r) \cap \Omega$ has SVP. Indeed, let A be a connected open subset of a Banach space and $\{f_j\} \subset \mathrm{Hol}(A, U)$ with $Z^u_{\{f_j\}} \neq \emptyset$. Since $\mathbb{B}(p, r)$ is bounded in E, by Lemma 3.3 we have $f_j \to f$ in $\mathrm{Hol}(A, E)$. Now we suppose that there exists $\lambda \in A$ such that

$$q := f(\lambda) = \lim_{j\to\infty} f_j(\lambda) \in \partial U.$$

Since by Proposition 2.4 every boundary point of $\mathbb{B}(p, r)$ is a t-point of U, and since every boundary point of Ω is a t-point of Ω by assumption, we conclude that q is a t-point of U. Next, we choose $\lambda' \in A \cap Z_{\{f_j\}}$. Since $f_j(\lambda')$ converges to some point in Ω, by the decreasing property of the Lempert functions we obtain

$$1 = \sup_{j\geq 1} \ell_U\big(f_j(\lambda), f_j(\lambda')\big) \leq \ell_A(\lambda, \lambda') < 1,$$

which is absurd. It follows that $f(A) \subset U$. Hence $f_j \to f$ in $\mathrm{Hol}(A, U)$. This is exactly our claim. It remains to apply Theorem 3.8 to conclude that Ω has SVP.

Finally, suppose that (ii) is true. Then we first apply Theorem 3.1 to see that Ω is hyperbolic. Next, we use Proposition 2.4 to find that every finite boundary point of Ω is a t-point. By (i), Ω has SVP. ■

REMARK. Let E be a Banach space, $\mathbb{B}$ the unit ball in E and $u \in \mathrm{PSH}(\mathbb{B}) \cap \mathcal{C}(\overline{\mathbb{B}})$ such that $u \geq 0$ on $\mathbb{B}$. Set $F := \mathbb{C} \times E$. Consider the unbounded domain

$$\Omega := \{z = (z_0, z') \in F : z' \in \mathbb{B},\ v(z) := \Im z_0 + u(z') < 0\}.$$

Since Ω is contained in the product $\{\Im z_0 < 0\} \times \mathbb{B}$ of hyperbolic domains in $\mathbb{C}$ and in E respectively, we infer that Ω is hyperbolic as well.

We now claim that every $a \in \partial\Omega \cup \{\infty\}$ admits a barrier. Indeed, for $a \in \partial\Omega$, we may pick $\max\{v(z), \|z'\|^2 - 1\}$ as a barrier at a. For $a = \infty$, by the same reasoning as in the remark that follows Corollary 3.7 we see that

$$\varphi(z) := \left|\frac{z_0 + i}{z_0 - i}\right|^2 - 1$$

is a plurisubharmonic barrier at ∞. Hence we may apply Corollary 3.10(ii) to conclude that Ω is an unbounded domain having SVP.

PROPOSITION 3.11. *Let $\theta : X \to Y$ be a holomorphic map between connected open subsets of Banach spaces. Assume that Y has SVP* (*resp. WVP*) *and for every $y \in Y$ there exists a connected hyperbolic neighborhood U of y such that $\theta^{-1}(U)$ has SVP* (*resp. WVP*). *Then X has SVP* (*resp. WVP*).

Proof. For ease of exposition we only deal with the case where Y and all the fibers $\theta^{-1}(U)$ have SVP; the other case can be treated analogously. Fix a connected open subset A in a Banach space and $\{f_j\} \subset \mathrm{Hol}(A, X)$ with $Z^u_{\{f_j\}} \neq \emptyset$. Set $g_j := \theta \circ f_j$. Since $Z^u_{\{g_j\}} \supset Z^u_{\{f_j\}} \neq \emptyset$ and Y has SVP, we have $g_j \to g$ in $\mathrm{Hol}(A, Y)$. Take $\lambda_0 \in Z^u_{\{f_j\}}$ and $U \ni g(\lambda_0)$ such that $\theta^{-1}(U)$ has SVP. Since U is hyperbolic and $g_j \to g$ in $\mathrm{Hol}(A, Y)$, there exists a neighborhood $V \ni \lambda_0$ and $j_0 \geq 1$ such that

$$g_j(V) \subset U, \quad \forall j \geq j_0.$$

It follows that

$$f_j(V) \subset \theta^{-1}(U), \quad \forall j \geq j_0.$$

Since $Z^u_{\{f_j|_V\}} \ni \lambda_0$, it follows that $\{f_j|_V\}$ converges in $\mathrm{Hol}(V, X)$. Set

$$\Omega = \bigcup \{V \subset A : V \text{ is open and } \{f_j|_V\} \text{ is convergent in } \mathrm{Hol}(V, X)\}.$$

Then Ω is open and non-empty. Let $\lambda_1 \in \partial\Omega$. The above reasoning implies that there exist open neighborhoods $U' \ni g(\lambda_1)$ and $V' \ni \lambda_1$ such that

$$f_j(V') \subset U', \quad \forall j \geq j_0,$$

and $\theta^{-1}(U')$ has SVP. Hence $\{f_j|_{V'}\}$ converges in $\mathrm{Hol}(V', X)$. This implies $\lambda_1 \in \Omega$. Consequently, $\Omega = A$. This finishes our proof. ■

REMARK. (i) By the same proof as in [K, Theorem 3.2.15], we can prove an analogous result to Proposition 3.10 where Vitali properties are replaced by complete hyperbolicity.

(ii) When X, Y are complex manifolds, an analogous result to Proposition 3.11, where Vitali properties are replaced by tautness, is obtained in [K, Proposition 5.1.8]. Notice that it was assumed there that the holomorphic map θ is *proper*. Since Vitali properties and tautness coincide in the case of complex manifolds [KDK, Theorem 3.14], Proposition 3.11 is somewhat stronger than the above mentioned result in [K].

(iii) Let Ω be a domain having SVP in a Banach space, and φ be a continuous plurisubharmonic function on Ω. Consider the Hartogs domain $\Omega' \subset \Omega \times \mathbb{C}$ defined by

$$\Omega' := \{(z, w) : z \in \Omega, \log|w| + \varphi(z) < 0\}.$$

Then we have the projection map $\theta : \Omega' \to \Omega, (z, w) \mapsto z$. Fix $z_0 \in \Omega$, choose $r > 0$ so small such that $U := \mathbb{B}(z_0, r) \subset \Omega$. It follows that

$$U' := \theta^{-1}(U) = \{(z, w) \in U \times \mathbb{C} : \psi(z, w) := \log|w| + \varphi(z) < 0\}.$$

Then each boundary point $\partial U'$ admits the barrier $\max\{\psi(z, w), \|z - z_0\| - r\}$. Then by Corollary 3.10(ii) we conclude that U' has SVP. Thus, Ω' has SVP by Proposition 3.11.

Acknowledgements. We are greatly indebted to Professor Nguyen Van Khue for sharing with us his deep insight and useful suggestions during the preparation of this manuscript. This work would not have been written without his generous help. We are also grateful the the referee for his/her constructive comments that helped to improve our exposition. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2016.07.

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