Hence $(a+bi)\sqrt{f} = A^2$, $A \in Q^{\mathrm{mc}}$ and $(a-bi)\sqrt{f} = \bar{A}^2$. Hence $A\bar{A} = f$ since f is positive. Hence

$$\sqrt{f}(a+\sqrt{f}) = \frac{(a+bi)\sqrt{f}+(a-bi)\sqrt{f}+2f}{2} = \frac{A^2+\bar{A}^2+2A\bar{A}}{2} = \left(\frac{A+\bar{A}}{\sqrt{2}}\right)^2,$$

where $(A+\bar{A})/\sqrt{2} \in Q^{\mathrm{mc}}$. The proof is complete.

References

[1] H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, Teil I: *Klassenkörpertheorie*, Teil Ia: *Beweise zu Teil I*, Würzburg–Wien 1970.

[2] — *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, Teil II: *Reziprozitätsgesetz*, Würzburg–Wien 1970.

[3] — *Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörper*, Abh. Deutsche Akad. Wiss. Berlin, Jahrgang 1948, No 2, Berlin 1950.

[4] Henry B. Mann, *Introduction to algebraic number theory*, Columbus 1955.

[5] Warren May, *Unit groups of infinite abelian extensions*, Proc. Amer. Math. Soc. 25 (1970), pp. 680–683.

[6] A. Rotkiewicz, *On the prime factor of the number* $2^{p-1}-1$, Glasgow Mathematical Journal 9 (1968), pp. 83–86.

[7] A. Schinzel, *Abelian polynomials, power residues and exponential congruences*, Acta Arith. 32 (1977), pp. 245–274.

[8] W. Y. Vélez, *On normal binomials*, Acta Arith. 36 (1980), pp. 113–124.

[9] J. Wójcik, *On the composite Lehmer numbers with prime indices II*, Prace Mat. 9 (1965), pp. 105–113.

[10] — *On the composite Lehmer numbers with prime indices III*, Colloq. Math. (in press).

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Kummer congruences for the coefficients of Hurwitz series

by

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1. Introduction. In L. Carlitz [3], it is shown that Hurwitz series $f(x)$ satisfying the differential equation

$$(f')^2 = 1+\sum_{i=1}^{4} a_i f^i \quad (a_i \in \boldsymbol{Z})$$

possess Kummer congruences. (These concepts are defined below.) However once the polynomial function on the right-hand side of the above equation has degree greater than four, Carlitz's methods fail to yield information about Kummer congruences. Nevertheless, he believed that when $f(x)$ satisfies

$$(f')^2 = 1+f^6$$

then f has Kummer congruences.

In this article we refine the machinery developed by Carlitz and solve the above problem in the affirmative. Moreover we show that of all Hurwitz series $f(x)$ satisfying in particular

$$(f')^2 = 1+f^m$$

for m an integer greater than 4, only for $m = 6$ does f have Kummer congruences.

Although this is the only application of the machinery developed that is given, the methods may be applied to other Hurwitz series satisfying more general differential equations.

2. An analysis of the Ω_p operator. Let R be an integral domain containing $\boldsymbol{Z}$, the rational integers.

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DEFINITION 1. A *Hurwitz series over* R (or *H-series*, for short) $H(x)$ is a formal power series of the form

$$H(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \quad \text{with} \quad a_n \in R.$$

The element a_n is called the *n-th coefficient of* $H(x)$.

With respect to the power series operations of addition and multiplication, the set of all H-series forms an integral domain containing R.

Many of the results here may be obtained almost directly from the papers of Carlitz. These results will therefore be stated without proof.

PROPOSITION 1. *If $H(x)$ is an H-series defined as above and $a_0 = 0$, then for any positive integer k*

$$\big(H(x)\big)^k \equiv 0 \bmod (k!).$$

(The congruence is to be considered ideal theoretically.)

Proof. Cf. L. Carlitz [2].

PROPOSITION 2. *If $H(x)$ is as defined above, $a_0 = 0$, and a_1 is a unit in R, then there exists a unique H-series $L(x)$ such that*

$$H\big(L(x)\big) = x = L\big(H(x)\big).$$

($L(x)$ is called the *composition inverse of* $H(x)$.)

Proof. Cf. L. Carlitz [2].

HYPOTHESIS. Throughout the rest of this paper, with the exception of the next proposition, we will assume the following:

(1) $f(x) = \sum_{n=1}^{\infty} c_n \frac{x^n}{n!}$ is an H-series over R with $c_1 = 1$.

(2) The composition inverse $\lambda(x)$ of $f(x)$ has the form

$$\lambda(x) = \sum_{n=1}^{\infty} (n-1)!\,\varepsilon_n \frac{x^n}{n!} = \sum_{n=1}^{\infty} \varepsilon_n \frac{x^n}{n} \quad \text{with} \quad \varepsilon_n \in R.$$

PROPOSITION 3. *Suppose $f(x)$ only satisfies assumption* (1) *of the above Hypothesis. Then assumption* (2) *is valid if and only if*

$$f'(x) = \sum_{\nu=0}^{\infty} d_\nu f^\nu \quad \textit{where} \quad d_\nu \in R \textit{ and } d_0 = 1.$$

($f'(x)$ is the formal power series derivative of $f(x)$ with respect to x.)

Proof. Cf. L. Carlitz [2].

PROPOSITION 4. *For any rational prime p and $m \geqslant 1$*

$$c_{m+(p-1)} \equiv c_p c_m \bmod (p).$$

Proof. Cf. L. Carlitz [4].

DEFINITION 2. The H-series $f(x)$ is said to *possess Kummer congruences at a rational prime p* (or "f has Kc(p)" for short) if and only if for every positive integer $r \geqslant 1$ and every integer $m \geqslant r$

$$d_{r,m} := \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} c_p^{r-i} c_{m+i(p-1)} \equiv 0 \bmod (p^r).$$

If $f(x)$ has Kc(p) for all primes p, we say $f(x)$ has Kummer congruences.

In order to test $f(x)$ for Kummer congruences we introduce Carlitz's Ω_p operator for each prime p.

DEFINITION 3. Let p be a rational prime. Then we define

$$\Omega_p f := (D_x^p - c_p D_x) f,$$

where D_x is the formal differentiation operator with respect to x.

PROPOSITION 5. *For each prime p and positive integer r,*

$$\Omega_p^r f = \sum_{m=r}^{\infty} d_{r,m} \frac{x^{m-r}}{(m-r)!}.$$

Proof. Cf. L. Carlitz [3].

Thus we may check whether or not f has Kc(p) by considering the coefficients of $\Omega_p^r f \bmod (p^r)$ for all $r \geqslant 1$. We now reduce this problem to a one-step procedure.

PROPOSITION 6. *If p is a prime, then*

$$D_x^{p-1} f - c_p f = b_0 + p \sum_{i=1}^{p-1} b_i f^i + \sum_{\nu=p}^{\infty} b_\nu f^\nu$$

where $b_\mu \in R$ for all $\mu \geqslant 0$.

Proof. Cf. L. Carlitz [3].

COROLLARY. $\Omega_p f = \sum_{\nu=0}^{\infty} \eta_\nu f^\nu$ *where $\eta_\nu \in R$ for all $\nu \geqslant 0$ and $\eta_\nu \equiv 0\ (p)$ for $\nu < p$.*

Proof. Cf. L. Carlitz [3].

THEOREM 1. *Let p be a prime. Then f has Kc(p) if and only if*

$$\Omega_p f = \sum_{\nu=0}^{\infty} \eta_\nu f^\nu \quad \textit{where} \quad \eta_\nu \equiv 0 \bmod (p) \textit{ for all } \nu < p^2.$$

The proof will follow by establishing the following lemmas, propositions, and corollaries.

PROPOSITION 7. *Let p be a prime and r a positive integer. Let*

$$\Omega_p^r f = \sum_{\nu=0}^{\infty} \eta_\nu^{(r)} f^\nu.$$

(It follows easily from Proposition 3 that $\eta_\nu^{(r)} \in R$ for all $\nu \geqslant 0$.) *Then*

$$\Omega_p^r f \equiv 0 \bmod (p^r) \qquad (\textit{as an } H\textit{-series})$$

if and only if

$$\eta_\nu^{(r)} \equiv 0 \bmod (p^{X(r-\operatorname{ord}_p\nu!)})$$

for all $\nu \in N$ where for any integer z, we define $X(z) = \max(0, z)$ and $\operatorname{ord}_p z$ as the exact exponent of p in the prime decomposition of z.

Proof. By representing $\Omega_p^r f$ as a power series in x we obtain

$$\Omega_p^r f = \sum_{m=0}^{\infty} \Big(\sum_{\nu=0}^{m} \eta_\nu^{(r)} c_m^{(\nu)} \Big) \frac{x^m}{m!},$$

where $c_m^{(\nu)}$ is the mth coefficient of the H-series $(f(x))^\nu$.

If $\eta_\nu^{(r)} \equiv 0 \bmod (p^{X(r-\operatorname{ord}_p\nu!)})$ for all $\nu \in N$, then Proposition 1 applied to f implies $\Omega_p^r f \equiv 0 \bmod (p^r)$.

Conversely, suppose $\Omega_p^r f \equiv 0 \bmod (p^r)$, i.e. $\sum_{\nu=0}^{m} \eta_\nu^{(r)} c_m^{(\nu)} \equiv 0 \bmod (p^r)$ for all m. Then a straightforward induction argument on m establishes the result.

LEMMA 1. *Let $\Omega_p^r f = \sum_{\nu=0}^{\infty} \eta_\nu^{(r)} f^\nu$, as above. Then*

$$\Omega_p^{r+1} f = \Omega_p f \sum_{\nu=0}^{\infty} (\nu+1) \eta_\nu^{(r)} f^\nu + \sum_{i=1}^{p-1} \binom{p}{i} \sum_{\mu,\nu=1}^{\infty} \eta_{\nu+\mu}^{(r)} f^{\mu-1} D_x^i f^\nu D_x^{p-i} f.$$

Proof. We first establish that for all $m \in N$

(1) $$\Omega_p^{r+1} f = \Omega_p f \Big(\sum_{\nu=0}^{m-1} (\nu+1) \eta_{\nu+1}^{(r)} f^\nu + m \sum_{\nu=m}^{\infty} \eta_\nu^{(r)} f^\nu \Big) + \\ + \sum_{\mu=1}^{m} f^{\mu-1} \sum_{\nu=\mu+1}^{\infty} \eta_\nu^{(r)} \theta(\nu-\mu) + f^m \sum_{\nu=1}^{\infty} \eta_{\nu+m}^{(r)} \Omega_p(f^\nu),$$

where $\theta(k) = \sum_{i=1}^{p-1} \binom{p}{i} D_x^i f^k D_x^{p-i} f$ for any $k \in N$. This is proved by induction on m. For $m = 0$, the result follows by the linearity of Ω_p.

Now suppose it is true for m. We now show it then true for $m+1$. Since $\Omega_p f^\nu = f \Omega_p f^{\nu-1} + f^{\nu-1} \Omega_p f + \theta(\nu-1)$ for $\nu \geqslant 1$ as is easily verified, we have

$$f^m \sum_{\nu=1}^{\infty} \eta_{\nu+m}^{(r)} \Omega_p(f^\nu) = f^m \sum_{\nu=1}^{\infty} \eta_{\nu+m}^{(r)} \big(f \Omega_p f^{\nu-1} + f^{\nu-1} \Omega_p f + \theta(\nu-1) \big).$$

The right-hand side is equal to

$$\Omega_p f \sum_{\nu=0}^{\infty} \eta_{\nu+(m+1)}^{(r)} f^{\nu+m} + f^m \sum_{\nu=1}^{\infty} \eta_{\nu+m}^{(r)} \theta(\nu-1) + f^{m+1} \sum_{\nu=1}^{\infty} \eta_{\nu+(m+1)}^{(r)} \Omega_p f^\nu$$

and this in turn equals

$$\Omega_p f \sum_{\nu=m}^{\infty} \eta_{\nu+1}^{(r)} f^\nu + f^m \sum_{\nu=m+2}^{\infty} \eta_\nu^{(r)} \theta(\nu-m-1) + f^{m+1} \sum_{\nu=1}^{\infty} \eta_{\nu+(m+1)}^{(r)} \Omega_p f^\nu.$$

By replacing $f^m \sum_{\nu=1}^{\infty} \eta_{\nu+m}^{(r)} \Omega_p(f^\nu)$ in (1) by the above expression and by combining the appropriate terms, we obtain (1) for $m+1$.

Finally letting m approach infinity establishes the lemma.

PROPOSITION 8. *Let $\Omega_p^r f = \sum_{\nu=0}^{\infty} \eta_\nu^{(r)} f^\nu$ for all $r \geqslant 1$ and define*

$$\nu_0^{(r)} = \begin{cases} \min\{\nu\colon \eta_\nu^{(r)} \not\equiv 0 \bmod (p^r)\} & \textit{if } \{\nu\colon \eta_\nu^{(r)} \not\equiv 0\ (p^r)\} \neq \emptyset, \\ \infty & \textit{otherwise}. \end{cases}$$

Suppose $\nu_0^{(1)} < p^2$. Then

$$\nu_0^{(r)} = \nu_0^{(1)} - (r-1)p$$

for $r \leqslant \nu_0^{(1)}/p + 1$.

Proof. We establish the proposition by induction on r. It is clear for $r = 1$. Now assume the proposition for r, i.e. if $r \leqslant \nu_0^{(1)}/p + 1$ then $\nu_0^{(r)} = \nu_0^{(1)} - (r-1)p$. From this we shall show that $r+1 \leqslant \nu_0^{(1)}/p$ implies $\nu_0^{(r+1)} = \nu_0^{(1)} - rp$.

By Lemma 1,

(2) $$\Omega_p^{r+1} f = \Omega_p f \sum_{\nu=0}^{\infty} (\nu+1) \eta_{\nu+1}^{(r)} f^\nu + \sum_{i=1}^{p-1} \binom{p}{i} \sum_{\mu,\nu=1}^{\infty} \eta_{\nu+\mu}^{(r)} f^{\mu-1} D_x^i f^\nu D_x^{p-i} f^\nu \\ = \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} (j+1) \eta_{j+1}^{(r)} \eta_{k-j}^{(1)} \Big) f^k + \sum_{i=1}^{p-1} \binom{p}{i} \sum_{\nu,\mu=1}^{\infty} \eta_{\nu+\mu}^{(r)} f^{\mu-1} D_x^i f^\nu D_x^{p-i} f^\nu.$$

First, notice that for $k \leqslant \nu_0^{(r)} - p$,

$$\sum_{j=0}^{k} (j+1) \eta_{j+1}^{(r)} \eta_{k-j}^{(1)} \equiv 0 \bmod (p^{r+1})$$

since for each $j = 0, \ldots, k$, $j+1$ is then less than $\nu_0^{(r)}$ implying $\eta_{j+1}^{(r)} \equiv 0 \bmod (p^r)$ and $k-j$ is less than $\nu_0^{(1)}$ so that $\eta_{k-j}^{(1)} \equiv 0 \bmod (p)$.

Therefore to establish that

$$\nu_0^{(r+1)} = \nu_0^{(r)} - p,$$

which is equivalent to our goal, we must only consider the second sum on the right-hand side of equation (2).

To this end, we have

$$\sum_{i=1}^{p-1}\binom{p}{i}D_x^i f^\nu D_x^{p-i}f = \sum_{k=X(\nu-p+1)}^{\infty}\delta_k^{(\nu)}f^k$$

where $\delta_k^{(\nu)}\in R$ and X is as defined in Proposition 7. Notice that $\delta_k^{(\nu)}\equiv 0 \bmod (p)$ for all ν and k. A brute force calculation shows that as a power series in f,

$$\sum_{i=1}^{p-1}\binom{p}{i}\sum_{\mu,\nu=1}^{\infty}\eta_{\nu+\mu}^{(r)}f^{\mu-1}D_x^i f^\nu D_x^{p-i}f = \sum_{k=0}^{\infty}a_k f^k$$

where

$$a_k = \sum_{\nu=1}^{k+p-1}\sum_{\mu=1}^{k+1-X(\nu-p+1)}\eta_{\nu+\mu}^{(r)}\delta_{k+1-\mu}^{(\nu)}. \tag{3}$$

Now notice that for all k, $\nu+\mu\leqslant k+p$ for all ν and μ such that $1\leqslant\nu\leqslant k+p-1$ and $1\leqslant\mu\leqslant k+1-X(\nu-p+1)$. Moreover $\nu+\mu=k+p$ precisely when $p-1\leqslant\nu\leqslant k+p-1$ and $\mu=k-\nu+p$. This implies that $a_k\equiv 0 \bmod (p^{r+1})$ for all $k<\nu_0^{(r)}-p$, since $\nu+\mu<\nu_0^{(r)}$ so $\eta_{\nu+\mu}^{(r)}\equiv 0 \bmod (p^r)$, and since $\delta_{k+1-\mu}^{(\nu)}\equiv 0 \bmod (p)$.

Now consider the crucial value $k=\nu_0^{(r)}-p$. Then

$$a_k = \eta_{\nu_0^{(r)}}^{(r)}\sum_{\nu=p-1}^{\nu_0^{(r)}-1}\delta_{\nu-p+1}^{(\nu)} + {\sum_{\mu,\nu\geqslant 1}}' \eta_{\nu+\mu}^{(r)}\delta_{\nu_0^{(r)}-p+1-\mu}^{(\nu)}$$

where $\sum'$ is the restriction of the summation to those μ and ν with $\nu+\mu<\nu_0^{(r)}$. But this implies that this second summation is congruent to $0 \bmod (p^{r+1})$. On the other hand,

$$\eta_{\nu_0^{(r)}}^{(r)}\not\equiv 0 \bmod (p^r).$$

We now have only to determine $\sum_{\nu=p-1}^{\nu_0^{(r)}-1}\delta_{\nu-(p-1)}^{(\nu)}$. Since $\nu\geqslant p-1$,

$$\sum_{i=1}^{p-1}\binom{p}{i}D_x^i f^\nu D_x^{p-i}f = \sum_{k=\nu-(p-1)}^{\infty}\delta_k^{(\nu)}f^k. \tag{4}$$

Notice from the left-hand side of (4), the term $\delta_{\nu-(p-1)}^{(\nu)}$ only occurs if $i=p-1$ and therefore the only contribution to $\delta_{\nu-(p-1)}^{(\nu)}$ is in the first term of

$$\binom{p}{p-1}D_x^{p-1}f^\nu D_x f.$$

Thus $\delta_{\nu-(p-1)}^{(\nu)} = p\nu(\nu-1)\ldots(\nu-p+2)$. (Remember that $D_x f = 1+\sum_{\nu=1}^{\infty}d_\nu f^\nu$.)

But then

$$\sum_{\nu=p-1}^{\nu_0^{(r)}-1}\delta_{\nu-(p-1)}^{(\nu)} = p\sum_{\nu=p-1}^{\nu_0^{(r)}-1}\nu(\nu-1)\ldots(\nu-p+2) \equiv p\sum_{\substack{\nu=p-1\\ \nu\equiv -1(p)}}^{\nu_0^{(r)}-1}(p-1)!$$

$$\equiv -p\left[\frac{\nu_0^{(r)}}{p}\right] \bmod (p^2).$$

Since $0<\nu_0^{(r)}\leqslant\nu_0^{(1)}<p^2$, $\left[\frac{\nu_0^{(r)}}{p}\right]\not\equiv \bmod (p)$ and thus

$$\sum_{\nu=p-1}^{\nu_0^{(r)}-1}\delta_{\nu-(p-1)}^{(\nu)}\not\equiv 0 \bmod (p^2).$$

It then follows easily that

$$\eta_{\nu_0^{(r)}}^{(r)}\sum_{\nu=p-1}^{\nu_0^{(r)}-1}\delta_{\nu-(p-1)}^{(\nu)}\not\equiv 0 \bmod (p^{r+1}).$$

Therefore $\nu_0^{(r+1)} = \nu_0^{(r)}-p = \nu_0^{(1)}-pr$ as desired.

Corollary. *Let $\Omega_p f = \sum_{\nu=0}^{\infty}\eta_\nu f^\nu$. Suppose further that there exists $\nu<p^2$ such that $\eta_\nu\not\equiv 0 \bmod (p)$. Then f does not possess Kummer congruences at p.*

Proof. This is a direct consequence of Propositions 5, 7, and 8 with the appropriate choice of r.

Lemma 2. *Let $\sum_{i=1}^{p-1}\binom{p}{i}D_x^i f^\nu D_x^{p-i}f = \sum_{n=0}^{\infty}\delta_n^{(\nu)}f^n$ for $\nu\geqslant 1$. Then there exist polynomials $p_m(X_1,\ldots,X_{p-1})\in p\mathbf{Z}[X_1,\ldots,X_{p-1}]$ for $m=1,\ldots,p-1$ independent of ν such that*

$$\sum_{i=1}^{p-1}\binom{p}{i}D_x^i f^\nu D_x^{p-i}f = \sum_{m=1}^{p-1}\nu(\nu-1)\ldots(\nu-m+1)f^{\nu-m}p_m(D_x f,\ldots,D_x^{p-1}f).$$

Proof. First, it is easy to establish that for each $i=1,\ldots,p-1$

$$D_x^i f^\nu = \sum_{m=1}^{i}\nu(\nu-1)\ldots(\nu-m+1)f^{\nu-m}p_{im}(Df,\ldots,D^i f)$$

where $p_{im}(X_1,\ldots,X_i)\in\mathbf{Z}[X_1,\ldots,X_i]$ and is independent of ν. This is done by induction on i and we shall not carry out the details here.

Now

$$\sum_{i=1}^{p-1}\binom{p}{i}D_x^i f^{\nu} D_x^{p-i} f$$

$$= \sum_{m=1}^{p-1} \nu(\nu-1)\ldots(\nu-m+1) f^{\nu-m} \sum_{i=1}^{p-1}\binom{p}{i} p_{im}(D_x f, \ldots, D_x^i f) D_x^{p-i} f.$$

Taking $p_m(X_1, \ldots, X_{p-1}) := \sum_{i=m}^{p-1}\binom{p}{i} p_{im}(X_1, \ldots, X_i) X_{p-i}$ establishes the lemma.

Corollary. *Let $\delta_n^{(\nu)}$ be defined as above. Then*

$$\delta_n^{(\nu)} \equiv \delta_{n+p}^{(\nu+p)} \bmod (p^2).$$

Proof. Let $p_m(D_x f, \ldots, D_x^{p-1} f) = \sum_{k=0}^{\infty} a_{mk} f^k$. (By the lemma $a_{mk} \equiv 0 \bmod (p)$ for all k, m.) Then we have

$$\sum_{m=1}^{p-1} \nu(\nu-1)\ldots(\nu-m+1) f^{\nu-m} p_m(D_x f, \ldots, D_x^{p-1} f)$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{m=1}^{p-1} \nu(\nu-1)\ldots(\nu-m+1) a_{m,m+n-\nu}\Big) f^n$$

where a_{mk} is interpreted as 0 if $k < 0$. Thus

$$\delta_n^{(\nu)} = \sum_{m=1}^{p-1} \nu(\nu-1)\ldots(\nu-m+1) a_{m,m+n-\nu}$$

whereas

$$\delta_{n+p}^{(\nu+p)} = \sum_{m=1}^{p-1} (\nu+p)(\nu+p-1)\ldots(\nu+p-m+1) a_{m,m+n-\nu}.$$

But this implies that $\delta_n^{(\nu)} \equiv \delta_{n+p}^{(\nu+p)} \bmod (p^2)$ as desired.

Proposition 9. *Suppose $\Omega_p f = \sum_{\nu=0}^{\infty} \eta_\nu f^\nu$ where $\eta_\nu \equiv 0 \bmod (p)$ for all $\nu < p^e$ where e is a fixed integer greater than 1. Then $\Omega_p^r f = \sum_{k=0}^{\infty} \eta_k^{(r)} f^k$ where*

$$\eta_k^{(r)} \equiv 0 \bmod \big(p^{X(r-[\frac{k}{p^e}])}\big).$$

Proof. The proposition is established by induction on r. The result is true by assumption if $r = 1$. Now assume it true for r. We shall then show it true for $r+1$. To this end, let the index $k = qp^e + i$ with $0 \leqslant i < p^e$.

We use induction on q to establish the result for $r+1$. Using equations (2) and (3) along with the identity

$$\sum_{\nu=1}^{k+p-1} \sum_{\mu=1}^{k+1-X(\nu-p+1)} \eta_{\nu+\mu}^{(r)} \delta_{k+1-\mu}^{(\nu)} = \sum_{n=2}^{k+p} \eta_n^{(r)} \sum_{\nu=\max(1,n-k-1)}^{n-1} \delta_{k+1-n+\nu}^{(\nu)} \quad \text{for } k \geqslant 0$$

we have

$$\eta_k^{(r+1)} = \sum_{j=0}^{k} (j+1) \eta_{j+1}^{(r)} \eta_{k-j} + \sum_{n=2}^{k+p} \eta_n^{(r)} \sum_{\nu=\max(1,n-k-1)}^{n-1} \delta_{\nu-(n-k-1)}^{(\nu)}. \tag{5}$$

Now suppose $q = 0$ so that $k = i < p^e$. We show that $\eta_i^{(r+1)} \equiv 0 \bmod (p^{r+1})$. We consider the two summands in (5) separately. From the hypothesis of the proposition and the induction hypothesis on r, it is clear that

$$\sum_{j=0}^{i} (j+1) \eta_{j+1}^{(r)} \eta_{i-j} \equiv 0 \bmod (p^{r+1}).$$

Moreover, if $i+p < p^e$, then

$$\sum_{n=2}^{i+p} \eta_n^{(r)} \sum_{\nu=\max(1,n-i-1)}^{n-1} \delta_{\nu-(n-i-1)}^{(\nu)} \equiv 0 \bmod (p^{r+1}).$$

Now suppose $p^e - p \leqslant i < p^e$. We then have two cases to consider.

Case 1. Suppose $n-i-1 \geqslant 1$. Let $i \equiv s \bmod (p)$ for $0 \leqslant s \leqslant p-1$. Then

$$\sum_{n=2}^{i+p} \eta_n^{(r)} \sum_{\nu=n-i-1}^{n-1} \delta_{\nu-(n-i-1)}^{(\nu)} \equiv \sum_{n=p^e}^{i+p} \eta_n^{(r)} \sum_{\nu=n-i-1}^{n-1} \delta_{\nu-(n-i-1)}^{(\nu)}$$

$$= \sum_{n=p^e}^{i+p} \eta_n^{(r)} \sum_{\mu=0}^{i} \delta_\mu^{(n-i-1+\mu)} \bmod (p^{r+1}).$$

But

$$\sum_{\mu=0}^{i} \delta_\mu^{(n-i-1+\mu)} = \sum_{j=0}^{s} \sum_{\substack{\mu=0 \\ \mu\equiv j(p)}}^{i} \delta_\mu^{(n-i-1+\mu)} + \sum_{j=s+1}^{p-1} \sum_{\substack{\mu=0 \\ \mu\equiv j(p)}}^{i} \delta_\mu^{(n-i-1+\mu)}$$

$$\equiv \sum_{j=0}^{s} p^{e-1} \delta_j^{(n-i-1+j)} + \sum_{j=s+1}^{p-1} (p^{e-1}-1) \delta_j^{(n-i-1+j)} \bmod (p^2).$$

The above congruence follows by the Corollary to Lemma 2. The first summand in the right-hand side of the above expression is congruent to $0 \bmod (p^2)$. Moreover the second summand is also congruent to $0 \bmod (p^2)$. This we show by establishing that for each $j \geqslant s+1$,

$$\delta_j^{(n-i-1+j)} \equiv 0 \bmod (p^2). \tag{6}$$

This is accomplished by noticing that by Lemma 2 $\delta_j^{(n-i-1+j)}$ is the coefficient of f^j in the expansion with respect to f of

$$\sum_{m=1}^{p-1}(n-i-1+j)(n-i-1+j-1)\dots(n-i-1+j-m+1)\times$$
$$\times f^{n-i-1+j-m}p_m(D_xf,\dots,D_x^{p-1}f).$$

Now $n-i-1+j\geqslant p$ since $j\geqslant s+1$. Moreover the only possible nonzero contribution to $\delta_j^{(n-i-1+j)}$ occurs when $n-i-1+j-m\leqslant j$ and since $j\leqslant p-1$, we obtain

$$n-i-1+j-m+1\leqslant p.$$

Thus $(n-i-1+j)(n-i-1+j-1)\dots(n-i-1+j-m+1)\equiv 0 \bmod (p)$. Since $p_m(X_1,\dots,X_{p-1})\in p\mathbf{Z}[X_1,\dots,X_{p-1}]$, we have the congruence (6).

These results establish Case 1 since ther estriction on i implies that $\eta_n^{(r)}\equiv 0 \bmod (p^{r-1})$ for $p^e\leqslant n\leqslant i+p$ by the induction assumption. Thus

$$\sum_{n=2}^{i+p}\eta_n^{(r)}\sum_{\nu=n-i-1}^{n-1}\delta^{(\nu)}_{\nu-(n-i-1)}\equiv 0 \bmod (p^{r+1}).$$

Case 2. Suppose $n-i-1<1$. Then as above

$$\sum_{n=2}^{i+p}\eta_n^{(r)}\sum_{\nu=1}^{n-1}\delta^{(\nu)}_{\nu-(n-i-1)}\equiv\sum_{n=p^e}^{i+p}\eta_n^{(r)}\sum_{\nu=1}^{n-1}\delta^{(\nu)}_{\nu-(n-i-1)} \bmod (p^{r+1}).$$

Hence we need only consider $p^e\leqslant n\leqslant i+p$. Since $n-i-1<1$ and $p^e-p\leqslant i<p^e$, we have only the possibility of $n=p^e$ and $i=p^e-1$. As in Case 1 we are reduced to showing that

$$\sum_{\nu=1}^{p^e-1}\delta_\nu^{(\nu)}\equiv 0 \bmod (p^2).$$

So

$$\sum_{\nu=1}^{p^e-1}\delta_\nu^{(\nu)}=\sum_{j=1}^{p}\sum_{\substack{\nu=1\\ \nu\equiv j(p)}}^{p^e-1}\delta_\nu^{(\nu)}\equiv\sum_{j=1}^{p-1}p^{e-1}\delta_j^{(j)}+(p^e-1)\,\delta_p^{(p)} \bmod (p^2).$$

Both terms are easily seen to be congruent to 0 mod (p^2).

These two cases establish that if $q=0$, then

$$\eta_k^{(r+1)}\equiv 0 \bmod (p^{r+1}).$$

We now assume the proposition is valid for $k=qp^e+i$, $0\leqslant i<p^e$. That is,

$$\eta_k^{(r+1)}\equiv 0 \bmod (p^{X(r+1-q)}).$$

We now show the result holds for $k=(q+1)p^e+i$, i.e.

$$\eta_k^{(r+1)}\equiv 0 \bmod (p^{X(r-q)}).$$

Without loss of generality we may assume $r>q$ (otherwise the congruence is already true). Fix $k=(q+1)p^e+i$ with $0\leqslant i<p^e$.

We shall show that $\eta_k^{(r+1)}\equiv 0 \bmod (p^{r-q})$ by showing that each of the summands on the right-hand side of equation (5) is congruent to zero mod (p^{r-q}).

We have

$$\sum_{j=0}^{k}(j+1)\eta_{j+1}^{(r)}\eta_{k-j}\equiv 0 \bmod (p^{r-q}),$$

for if $j<(q+1)p^e$, then $(j+1)\eta_{j+1}^{(r)}\equiv 0 \bmod (p^{r-q})$; and if $j\geqslant(q+1)p^e$, then $(j+1)\eta_{j+1}^{(r)}\equiv 0 \bmod (p^{r-q-1})$ and since $k-j\leqslant i$, $\eta_{k-j}\equiv 0 \bmod (p)$.

Now consider the second summand,

$$\sum_{n=2}^{k+p}\eta_n^{(r)}\sum_{\nu=\max(1,n-k-1)}^{n-1}\delta^{(\nu)}_{\nu-(n-k-1)}.$$

If $i+p<p^e$, then $\eta_n^{(r)}\equiv 0 \bmod (p^{r-q-1})$. Since $\delta^{(\nu)}_{\nu-(n-k-1)}\equiv 0 \bmod (p)$ the above expression is congruent to 0 mod (p^{r-q}).

Now suppose that $i\geqslant p^e-p$. Then

$$\sum_{n=2}^{k+p}\eta_n^{(r)}\sum_{\nu=\max(1,n-k-1)}^{n-1}\delta^{(\nu)}_{\nu-(n-k-1)}\equiv\sum_{n=(q+2)p^e}^{k+p}\eta_n^{(r)}\sum_{\nu=\max(1,n-k-1)}^{n-1}\delta^{(\nu)}_{\nu-(n-k-1)} \bmod (p^{r-q}). \tag{7}$$

As before it suffices to show that

$$\sum_{\nu=\max(1,n-k-1)}^{n-1}\delta^{(\nu)}_{\nu-(n-k-1)}\equiv 0 \bmod (p^2).$$

The proof is divided into two cases.

Case 1. Suppose $n-k-1\geqslant 1$. Let $i\equiv s \bmod p$ with $0\leqslant s\leqslant p-1$. Then

$$\sum_{\nu=n-k-1}^{n-1}\delta^{(\nu)}_{\nu-(n-k-1)}=\sum_{\mu=0}^{k}\delta_\mu^{(n-k-1+\mu)}$$
$$=\sum_{j=0}^{s}\sum_{\substack{\mu=0\\ \mu\equiv j(p)}}^{k}\delta_\mu^{(n-k-1+\mu)}+\sum_{j=s+1}^{p-1}\sum_{\substack{\mu=0\\ \mu\equiv j(p)}}^{k}\delta_\mu^{(n-k-1+\mu)}$$
$$=\sum_{j=0}^{k}(q+2)p^{e-1}\delta_j^{(n-k-1+j)}+\sum_{j=k+1}^{p-1}\big((q+2)p^{e-1}-1\big)\,\delta_\mu^{(n-k-1+\mu)}.$$

It is immediate that the first sum is congruent to 0 mod (p^2). For the second summand an argument completely analogous to the one for $q = 0$ shows that each $\delta_\mu^{(n-k-1+\mu)} \equiv 0 \bmod (p^2)$.

Case 2. Suppose $n-k-1 < 1$. Since we need only consider $n \geqslant (q+2)p^e$ (by (7)), the only possibility occurs when $n = (q+2)p^e$ and $i = p^e-1$. Hence as in Case 2 for $q = 0$ we need to show that

$$\sum_{\nu=1}^{k-1} \delta_\nu^{(\nu)} \equiv 0 \bmod (p^2).$$

But

$$\sum_{\nu=1}^{k-1} \delta_\nu^{(\nu)} = \sum_{j=1}^{p} \sum_{\substack{\nu=1 \\ \nu\equiv j(p)}}^{k-1} \delta_\nu^{(\nu)} \equiv \sum_{j=1}^{p-1} (q+2)p^{e-1}\delta_j^{(j)} + \big((q+2)p^{e-1}-1\big)\,\delta_p^{(p)} \equiv 0 \bmod (p^2).$$

These two cases and the previous arguments establish that $\eta_k^{(r+1)} \equiv 0 \bmod (p^{r-q})$ for $k = (q+1)p^e+i$.

This establishes the induction step and thus the proposition.

Corollary. *Let* $\Omega_p f = \sum_{\nu=0}^{\infty} \eta_\nu f^\nu$. *Suppose that for each* $\nu < p^2$ $\eta_\nu \equiv 0 \bmod (p)$. *Then f has Kummer congruences at p.*

Proof. The hypothesis of Proposition 9 is fulfilled for $e = 2$. Thus for each $r \geqslant 1$, we have

$$\eta_\nu^{(r)} \equiv 0 \bmod \big(p^{\chi\left(r-\left[\frac{\nu}{p^2}\right]\right)}\big)$$

or each $\nu \geqslant 0$. Thus since $[\nu/p^2] \leqslant \operatorname{ord}_p \nu!$,

$$\eta_\nu^{(r)} \equiv 0 \bmod (p^{\chi(r-\operatorname{ord}_p \nu!)})$$

for all $\nu \geqslant 0$. The corollary then follows from Propositions 7 and 5.

The above results prove Theorem 1.

3. A further analysis of the Ω_p operator and applications. By Theorem 1, we need only determine $\eta_\nu \bmod (p)$ for $\nu < p^2$ in order to check whether or not f has $\mathrm{Kc}(p)$. We now simplify this procedure further.

Theorem 2. *For any ring R of characteristic p with unity, for all derivations D on R, and for any $u \in R$,*

$$(uD)^{p-2}(u) = -D^{p-2}u^{p-1}.$$

Proof. We first claim that if R is any commutative ring with unity and $u_1, \ldots, u_n \in R$ for some fixed positive integer n, then

$$D\Big(u_n D\big(u_{n-1} D(\ldots D(u_2 D u_1)\ldots)\big)\Big) = \sum_{\mu_1,\ldots,\mu_n \geqslant 0} d_{\mu_1\ldots\mu_n} D^{\mu_1}u_1 \ldots D^{\mu_n}u_n$$

where the d_μ are determined by the polynomial equation:

$$X_1(X_1+X_2)\ldots(X_1+\ldots+X_n) = \sum_{\mu_1,\ldots,\mu_n\geqslant 0} d_{\mu_1\ldots\mu_n} X_1^{\mu_1}\ldots X_n^{\mu_n}.$$

This follows easily by an induction argument on n.

Next, we have for any positive integer k

$$D^k(u_1\ldots u_n) = \sum_{\mu_1,\ldots,\mu_n\geqslant 0} c_{\mu_1\ldots\mu_n} D^{\mu_1}u_1\ldots D^{\mu_n}u_n$$

where the c_μ are determined by the polynomial equation:

$$(X_1+\ldots+X_n)^k = \sum_{\mu_1,\ldots,\mu_n\geqslant 0} c_{\mu_1\ldots\mu_n} X_1^{\mu_1}\ldots X_n^{\mu_n}.$$

The proof follows by an induction on k.

Now by the result of G. Baron and A. Schinzel [1]

$$\sum_{\sigma\in S_{p-1}} X_{\sigma(1)}(X_{\sigma(1)}+X_{\sigma(2)})\ldots(X_{\sigma(1)}+\ldots+X_{\sigma(p-2)}) = (X_1+\ldots+X_{p-1})^{p-2}.$$

Then the above two results show that if R is a commutative ring of characteristic p with unity, then

$$\sum_{\sigma\in S_{p-1}} u_{\sigma(p-1)} D\Big(u_{\sigma(p-2)} D\big(\ldots(u_{\sigma(2)} D u_{\sigma(1)})\ldots\big)\Big) = D^{p-2}(u_1\ldots u_{p-1}).$$

If $u_i = u$ for all i, we have $(p-1)!\,(uD)^{p-2}(u) = D^{p-2}u^{p-1}$. Since $(p-1)! = -1$, we obtain $(uD)^{p-2}(u) = -D^{p-2}(u^{p-1})$.

This theorem will greatly simplify the task of determining whether or not f has $\mathrm{Kc}(p)$.

Let $f(x)$ have all the assumptions previously specified. Let

$$D_x f = \sum_{\nu=0}^{\infty} d_\nu f^\nu, \quad D_x^{p-1} f = \sum_{\nu=0}^{\infty} a_\nu f^\nu, \quad \text{and} \quad (D_x f)^{p-1} = \sum_{\nu=0}^{\infty} d_\nu^{(p-1)} f^\nu.$$

Notice that $D_x = f' D_f$ where $f' = D_x f$ and therefore

$$D_x^{p-1} f = (f'D_f)^{p-1} f = (f'D_f)^{p-2}(f').$$

Thus by the above theorem

$$(f'D_f)^{p-2}(f') \equiv -D_f^{p-2}\big((f')^{p-1}\big) \bmod (pR[[f]]).$$

Since

$$D_f^{p-2}\big((f')^{p-1}\big) = \sum_{\mu=0}^{\infty} (\mu+1)\ldots(\mu+p-2)\, d_{\mu+p-2}^{(p-1)} f^\mu,$$

we obtain

$$a_\mu \equiv (\mu+1)\ldots(\mu+p-2)\, d_{\mu+p-2}^{(p-1)} \bmod (pR). \tag{8}$$

DEFINITION 4.
$$\nu_0 = \begin{cases}\min\{\nu:\ \eta_\nu \not\equiv 0 \bmod p\} & \text{if it exists,}\\ \infty & \text{otherwise;}\end{cases}$$
$$\mu_0 = \begin{cases}\min\{\mu:\ \mu \equiv -1(p),\ \mu \geqslant p,\ d_\mu^{(p-1)} \not\equiv 0(p)\} & \text{if it exists,}\\ \infty & \text{otherwise.}\end{cases}$$

PROPOSITION 10. *Let μ_0 and ν_0 be as in Definition 4. Then*
$$\mu_0 = \nu_0 + p - 1.$$

Proof. We have
$$\sum_{\nu=0}^{\infty} \eta_\nu f^\nu = \Omega_p f = D_x(D_x^{p-1} - c_p) f = f' D_f(D_x^{p-1} f - c_p f)$$
$$= \Big((a_1 - c_p) + \sum_{\mu=1}^{\infty} (\mu+1) a_{\mu+1} f^\mu\Big)\Big(\sum_{\nu=0}^{\infty} d_\nu f^\nu\Big)$$
$$\equiv -\sum_{\mu=1}^{\infty} (\mu+1)\ldots(\mu+p-1)\, d_{\mu+p-1}^{(p-1)} f^\mu \sum_{\nu=0}^{\infty} d_\nu f^\nu \bmod (p)$$
since by Proposition 6 $a_1 - c_p \equiv 0 \bmod (p)$ (and by use of (8)). Multiplying out the right-hand side of the above congruence, we obtain
$$\eta_\nu \equiv \sum_{\substack{p \leqslant \mu \leqslant \nu+p-1 \\ \mu \equiv -1(p)}} d_\mu^{(p-1)} d_{\nu+p-1-\mu} \bmod (p).$$

Now assume μ_0 is finite. Then if $\nu < \mu_0 - p + 1$, we see that $\eta_\nu \equiv 0 \bmod (p)$. Moreover if $\nu = \mu_0 - p + 1$, then clearly $\eta_\nu \not\equiv 0 \bmod (p)$. Thus $\nu_0 = \mu_0 - p + 1$ and is finite.

Next suppose ν_0 is finite. Let $\mu < \nu_0 + p - 1$ such that $\mu \equiv -1(p)$. Then $d_\mu^{(p-1)} \equiv 0 \bmod (p)$ by the preceding argument. Whereas, if $\mu = \nu_0 + p - 1$, then
$$0 \not\equiv \eta_{\nu_0} \equiv \sum_{\substack{p \leqslant \mu \leqslant \nu_0+p-1 \\ \mu \equiv -1(p)}} d_\mu^{(p-1)} d_{\nu_0+p-1-\mu} \equiv d_{\nu_0+p-1}^{(p-1)} \bmod (p).$$
Thus $\mu_0 = \nu_0 + p - 1$ and hence μ_0 is finite.

We are now in a position to give an application of the above results.

PROPOSITION 11. *Assume*
$$f(x) = \sum_{n=1}^{\infty} c_n \frac{x^n}{n!} \quad \textit{with} \quad c_1 = 1$$
and that
$$(f')^2 = 1 + d f^m$$

where m is a positive integer and $d \neq 0$ is contained in some field of characteristic zero. (Notice then that $f(x)$ is an H-series over $R := \mathbf{Z}[\frac{1}{2}, d]$ satisfying the Hypothesis stated earlier.)

Then f has Kummer congruences if and only if $m = 1, 2, 3, 4$, or 6. For all other m, there exist infinitely many rational primes at which f does not possess Kummer congruences.

Proof. Since 2 is a unit in R, f clearly satisfies $\mathrm{Kc}(2)$. Next notice that for all odd primes p,
$$\sum_{\mu=0}^{mr} d_\mu^{(p-1)} f^\mu = (f')^{p-1} = \sum_{k=0}^{r} \binom{r}{k} d^k f^{km}$$
where $r = (p-1)/2$. Thus $d_{km}^{(p-1)} = \binom{r}{k} d^k$ for $0 \leqslant k \leqslant r$, and $d_\mu^{(p-1)} = 0$ if $\mu \not\equiv 0\ (m)$ or $\mu > mr$.

Now let μ_0 and ν_0 be as in Definition 4. Then either $\mu_0 \leqslant m\dfrac{p-1}{2}$ or $\mu_0 = \infty$. By Proposition 10 this is equivalent to $\nu_0 \leqslant (m-2)\dfrac{p-1}{2}$ or $\nu_0 = \infty$. If $p > (m-2)/2$, notice that the above is equivalent to
$$\nu_0 < p^2 \quad \text{or} \quad \nu_0 = \infty.$$
Theorem 1 then implies that if $p > (m-2)/2$, then f has $\mathrm{Kc}(p)$ if and only if $\nu_0 = \infty$ (or equivalently if and only if $\mu_0 = \infty$).

We now investigate when $\mu_0 < \infty$. By the corollary to Proposition 6, $\nu_0 \geqslant p$. Thus $\mu_0 \geqslant 2p - 1$ by Proposition 10. We thus obtain $\mu_0 < \infty$ if and only if there exists an integer k such that
$$2p - 1 \leqslant kp - 1 \leqslant m\frac{p-1}{2}, \qquad kp - 1 \equiv 0\ (m)$$
and $d^{(kp-1)/m} \not\equiv 0 \bmod (p)$.

The above inequality can be rewritten as
$$2 \leqslant k \leqslant \frac{m}{2} - \frac{m-2}{2p}.$$

Notice that if $m \leqslant 4$, no such k can exist. Moreover the restriction $p > (m-2)/2$ is no restriction at all. Thus f has Kummer congruences in this case. (Actually this is a special case of L. Carlitz [2].)

If $m = 6$, then $\mu_0 < \infty$ would imply $k = 2$ in which case $2p - 1 \not\equiv 0\ (6)$. Hence $\mu_0 = \infty$. Moreover, since $(m-2)/2 = 2$, the inequality $p > (m-2)/2$ restricts the prime to all the odd ones. Therefore, f possesses Kummer congruences when $m = 6$.

For all other cases of m, assume f has $\mathrm{Kc}(p)$ for all but finitely many primes p. In particular, we must have for all $p > N$ for some $N > m-2$ that $\mu_0 = \infty$.

Notice next that there exists $k_0 \in N$ such that for all $p > N$ we have $2 \leqslant k_0 \leqslant \frac{m}{2} - \frac{m-2}{2p}$ and $(k_0, m) = 1$. For, if m is odd, then take $k_0 = 2$; if m is even, say $m = 2n$, then take

$$k_0 = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n-2 & \text{if } n \text{ is odd.} \end{cases}$$

Let $\tilde{k}_0$ be some integer with $\tilde{k}_0 k_0 \equiv 1\ (m)$. Also let

$$P = \{p\colon p \text{ is a prime, } p > N, \text{ and } p \equiv \tilde{k}_0\ (m)\}.$$

We then have

$$2p-1 \leqslant k_0 p - 1 \leqslant m\,\frac{p-1}{2} \quad \text{and} \quad k_0 p - 1 \equiv 0\ (m)$$

for all $p \in P$. Thus since $\mu_0 = \infty$, we must have

$$d^{(k_0 p-1)/m} \equiv 0 \bmod (p).$$

This implies that $d \in \bigcap_{p\in P} \mathrm{Rad}(pR)$.

But since P is an infinite set and $R = Z[\frac{1}{2}, d]$ we have

$$\bigcap_{p\in P} \mathrm{Rad}(pR) = (0).$$

Thus $d = 0$, a contradiction. Hence for $m = 5$, and $m \geqslant 7$, there exist infinitely many primes at which f has no Kummer congruences.

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References

[1] G. Baron and A. Schinzel, *An extension of Wilson's theorem*, C. R. Math. Rep. Acad. Sci. Canada 1 (1979), pp. 115–118.
[2] L. Carlitz, *The coefficients of the reciprocal of a series*, Duke Math. Journ. 8 (1941), pp. 689–700.

[3] L. Carlitz, *Congruences for the coefficients of the Jacobi elliptic functions*, ibid. 16 (1949), pp. 297–302.
[4] — *Some properties of Hurwitz series*, ibid. 16 (1949), pp. 285–295.
[5] — *Criteria for Kummer's congruences*, Acta Arith. 6 (1961), pp. 375–391.
[6] N. Nielson, *Traité élémentaires des nombres de Bernoulli*, Gauthier-Villars, Paris 1973.

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