Since $f(n) \in \mathbf{Z}$ for every n, we have:

$$\sum_{j=1}^{s} \sigma(P_j(n))\,\sigma(r_j)^n = \sum_{j=1}^{s} P_j(n)\, r_j^n.$$

But it is well known that the expression of f in the form (4) is unique, and it follows that the $\sigma(r_j)$ are a permutation of the r_j, and this happens for every σ.

Thus, if in the formula for f some r_j has a polynomial coefficient which is nonzero, then all of its conjugates have the same property. But

$$|f(n)| \geqslant \max_{P_j \neq 0} |r_j|^n \quad \text{for an infinity of } n$$

and, since f is assumed to have polynomial growth, we conclude that $\max_{P_j\neq 0} |r_j| \leqslant 1$, and, by the preceding observation, we have also:

$$\max_{\sigma} \max_{P_j \neq 0} |\sigma(r_j)| \leqslant 1.$$

Since the r_j are algebraic integers, a well known theorem of Kronecker implies that they are roots of unity.

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Selberg's sieve estimate with a one sided hypothesis

by

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1. Introduction. It has been found in many interesting number theory problems that the most successful techniques involve a small sieve. One of the best small sieve techniques known is that of Selberg [8]. This sieve has been investigated by Ankeny–Onishi [1] and Halberstam–Richert [2], among others. The results they obtain using the Selberg sieve rely on assumptions made about the function $\omega(d)$ (defined in Section 2), and the aim of this paper is to obtain similar results with less stringent assumptions.

2. The basis of the sieve and Selberg's λ-method. We follow the notation of Halberstam–Richert ([2] and [4]).

Let $\mathfrak{A}$ be a finite sequence of integers, and let $\mathfrak{A}_d$ denote the subsequence of $\mathfrak{A}$ all of whose elements are divisible by d. We use $|\mathfrak{A}|$ and $|\mathfrak{A}_d|$ to denote the number of elements of $\mathfrak{A}$ and $\mathfrak{A}_d$, respectively.

Let $\mathcal{P}$ be a set of primes and define (the empty product being 1)

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p. \tag{1}$$

Define the sifting function $\mathcal{S}(\mathfrak{A}; \mathcal{P}, z)$ for any z to be

$$\mathcal{S}(\mathfrak{A}; \mathcal{P}, z) = \left|\{a \in \mathfrak{A} \colon (a, P(z)) = 1\}\right|; \tag{2}$$

in other words, $\mathcal{S}(\mathfrak{A}; \mathcal{P}, z)$ is the number of elements of $\mathfrak{A}$ remaining after we have removed all those with prime factors less than z that belong to $\mathcal{P}$.

In order to study the function $\mathcal{S}(\mathfrak{A}; \mathcal{P}, z)$ we need some notation. We choose a convenient approximation to $|\mathfrak{A}|$, call it X, and define

$$R_1 = |\mathfrak{A}| - X.$$

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For each $p \in \mathscr{P}$ we choose a non-negative number $\omega(p)$ and set

$$R_p = |\mathfrak{A}_p| - \frac{\omega(p)}{p} X.$$

We set $\omega(1) = 1$, $\omega(p) = 0$ for $p \notin \mathscr{P}$, and extend the function $\omega(d)$ multiplicatively for square-free d. Thus

$$\omega(d) = \prod_{p|d} \omega(p), \quad \text{for} \quad \mu(d) \neq 0, \tag{3}$$

and we let

$$R_d = |\mathfrak{A}_d| - \frac{\omega(d)}{d} X, \quad \text{for} \quad \mu(d) \neq 0. \tag{4}$$

(In practice $\omega(p)$ is chosen to make the R_d small on average.) We need one more important function, namely

$$W(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right) = \sum_{d|P(z)} \frac{\mu(d)\omega(d)}{d}. \tag{5}$$

Selberg [8] derived his upper bound estimate by defining $\lambda_1 = 1$, λ_d to be arbitrary real numbers, and noting that

$$\mathscr{S}(\mathfrak{A}; \mathscr{P}, z) \leqslant \sum_{a \in \mathfrak{A}} \Big(\sum_{\substack{d|a \\ d|P(z)}} \lambda_d \Big)^2 = \sum_{\substack{d_i|P(z) \\ i=1,2}} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{a \in \mathfrak{A} \\ a \equiv 0 \, (\mathrm{mod}\, [d_1, d_2])}} 1,$$

where $[d_1, d_2]$ denotes the least common multiple of d_1 and d_2. In the notation we have developed, we obtain

$$\begin{aligned} \mathscr{S}(\mathfrak{A}; \mathscr{P}, z) &\leqslant \sum_{\substack{d_i|P(z) \\ i=1,2}} X \lambda_{d_1} \lambda_{d_2} \frac{\omega([d_1, d_2])}{[d_1, d_2]} + \sum_{\substack{d_i|P(z) \\ i=1,2}} |\lambda_{d_1} \lambda_{d_2} R_{[d_1, d_2]}| \\ &= X\Sigma_1 + \Sigma_2, \end{aligned} \tag{6}$$

say.

In order to control the sum Σ_2 Selberg introduced a parameter ξ and restricted the λ_d's so that $\lambda_d = 0$ for $d \geqslant \xi$. He further defined the functions

$$g(d) = \frac{\omega(d)}{d} \prod_{p|d} \left(1 - \frac{\omega(p)}{p}\right)^{-1}, \tag{7}$$

for $\mu(d) \neq 0$, and

$$G(\xi, z) = \sum_{\substack{d<\xi \\ d|P(z)}} g(d). \tag{8}$$

Selberg then minimized the sum Σ_1 by means of Lagrangian multipliers (many other methods can be used) and deduced that the minimal value for Σ_1 is $1/G(\xi, z)$. With the choice of λ_d's that minimize Σ_1 we obtain the upper bound

$$\Sigma_2 \leqslant \sum_{\substack{d<\xi^2 \\ d|P(z)}} 3^{\nu(d)} |R_d|$$

(see [4], (6.1.7)), where $\nu(d)$ is the number of distinct prime divisors of d. By using these upper bounds in (6) we obtain what we refer to as the *Selberg sieve estimate*, namely

$$\mathscr{S}(\mathfrak{A}; \mathscr{P}, z) \leqslant X/G(\xi, z) + \sum_{\substack{d<\xi^2 \\ d|P(z)}} 3^{\nu(d)} |R_d|. \tag{9}$$

3. Analysis of the Selberg sieve estimate. When applying Selberg's sieve estimate one normally begins by choosing ξ in such a way that the sum of remainder terms is smaller than the expected value for the main term. Once this has been done an estimate for $G(\xi, z)$ must be produced. The main problem in using equation (9) is that the function $G(\xi, z)$ is hard to work with in many cases.

Ankeny–Onishi [1] is the first paper to give such estimates applicable to a large number of problems. It shows that by placing certain conditions on a dimension $\varkappa$ sieve problem (defined below) we obtain the asymptotic formula

$$1/G(\xi, z) = \big(W(z)/\sigma(\tau)\big)\{1 + o(1)\}, \quad \text{as} \quad z \to \infty, \tag{10}$$

where $\tau = \log \xi^2/\log z$, and $\sigma(u)$ is the continuous solution of the system

$$\sigma(u) = \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+1)} \left(\frac{u}{2}\right)^{\varkappa}, \quad \text{for} \quad 0 \leqslant u \leqslant 2, \tag{11}$$

$$\big(u^{-\varkappa}\sigma(u)\big)' = -\varkappa u^{-\varkappa-1} \sigma(u-2), \quad \text{for} \quad u > 2. \tag{12}$$

Halberstam–Richert [2] gives explicit error terms in the formula for $1/G(\xi, z)$. The conditions Halberstam–Richert used were

$$0 \leqslant \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A_1}, \tag{Ω_1}$$

for all primes $p \in \mathscr{P}$ and some suitable constant $A_1 > 1$, and

$$-L \leqslant \sum_{w \leqslant p < y} \frac{\omega(p) \log p}{p} - \varkappa \log \frac{y}{w} \leqslant A_2, \tag{$\Omega_2(\varkappa, L)$}$$

if $2 \leqslant w < y$, for some constants L, $A_2 \geqslant 1$, and $\varkappa$. The constant $\varkappa$ is referred to as the dimension of the sieve, and can be thought of as the average number of residue classes sieved out by each prime. With these conditions Halberstam–Richert obtain the following:

THEOREM 1. *Let* (Ω_1), $(\Omega_2(\varkappa, L))$ *hold. For* $\tau = \log\xi^2/\log z$ *we have*

$$1/G(\xi, z) = \frac{W(z)}{\sigma(\tau)}\left\{1+O\left(\frac{L(\log\log\xi)^{2\varkappa+1}}{\log\xi}\right)\right\}, \tag{13}$$

where $\sigma(u)$ *is defined above, and the O-term does not depend on* L.

The presence of the L is somewhat troublesome, since in some applications it can be quite large, and be a function of X. The result we prove in this paper is that after weakening $(\Omega_2(\varkappa, L))$ to

$$(\Omega_2(\varkappa)) \qquad \sum_{w\leqslant p<y} \frac{w(p)\log p}{p} - \varkappa\log\frac{y}{w} \leqslant A_2,$$

if $2 \leqslant w < y$, we can prove:

THEOREM 2. *Let* (Ω_1), $(\Omega_2(\varkappa))$ *hold. For* $\tau = \log\xi^2/\log z$ *we have*

$$1/G(\xi, z) \leqslant \frac{W(z)}{\sigma(\tau)}\left\{1+O\left(\frac{(\log\log\xi)^{2\varkappa+1}}{\log\xi}\right)\right\}. \tag{14}$$

For the purpose of inserting into (9) to obtain a sieve inequality, it is clear that (14) is somewhat better than (13), because of the lack of the constant L.

The proof we give of (14) depends ultimately on (13) and follows from two lemmas. The first lemma shows that we can change the problem to get inequality rather than equality, and the second lemma enables us to use (13) on the changed problem, the new L being a function of $\varkappa$, A_1, and A_2.

4. The reduction lemma. We will eventually use the result of Halberstam and Richert, (13). This result is applicable only if we have an L such that $(\Omega_2(\varkappa, L))$ is satisfied. Since we are only assuming $(\Omega_2(\varkappa))$, we want to change the function $\omega(d)$ to satisfy $(\Omega_2(\varkappa, L))$ with L a suitable function of $\varkappa$, A_1, and A_2. The following lemma shows that increasing $\omega(d)$ preserves the sieve inequality. The result in the case $\xi = z$ is found in the lecture notes of Jurkat [5].

LEMMA 1. *Let* $p > \omega'(p) \geqslant \omega(p)$ *for all primes* p. *If* Q *is a quantity depending on the values* $\{\omega(p)\}$, *let* Q' *be the corresponding quantity depending on the values* $\{\omega'(p)\}$. *Then for all* ξ *and* z *we have*

$$G(\xi, z)W(z) \geqslant G'(\xi, z)W'(z). \tag{15}$$

Proof. Let q be a prime. It suffices to treat the case $\omega'(p) = \omega(p)$ for all $p \neq q$, and $\omega'(q) = \omega(q)+\varepsilon$, for some $\varepsilon \geqslant 0$. We also assume $q < z$, since equality holds in (15) otherwise. In this proof we assume that all sums and products are restricted to divisors of $P(z)$. By using equations (5) and (8) we rewrite (15) as

$$\prod_{p<z}\left(1-\frac{\omega(p)}{p}\right)\sum_{d<\xi} g(d) \geqslant \prod_{p<z}\left(1-\frac{\omega'(p)}{p}\right)\sum_{d<\xi} g'(d). \tag{16}$$

Since $\omega'(p) = \omega(p)$ for all $p \neq q$, we cancel all terms in the products except those involving q. By expanding the sums we see that showing (16) is the same as showing

$$\left(1-\frac{\omega(q)}{q}\right)\Big\{\sum_{\substack{d<\xi\\ q|d}} g(d) + \sum_{\substack{d<\xi\\ q\nmid d}} g(d)\Big\} \geqslant \left(1-\frac{\omega(q)+\varepsilon}{q}\right)\Big\{\sum_{\substack{d<\xi\\ q|d}} g'(d) + \sum_{\substack{d<\xi\\ q\nmid d}} g(d)\Big\},$$

or

$$\left(1-\frac{\omega(q)}{q}\right)\sum_{\substack{d<\xi\\ q|d}} g(d) \geqslant \left(1-\frac{\omega(q)+\varepsilon}{q}\right)\sum_{\substack{d<\xi\\ q|d}} g'(d) - \frac{\varepsilon}{q}\sum_{\substack{d<\xi\\ q\nmid d}} g(d). \tag{17}$$

By using the definition of $g(d)$ (equation (7)) and simplifying again, we see that showing (17) is the same as showing

$$\frac{\omega(q)}{q}\sum_{\substack{d<\xi/q\\ q\nmid d}} g(d) \geqslant \left(\frac{\omega(q)+\varepsilon}{q}\right)\sum_{\substack{d<\xi/q\\ q\nmid d}} g(d) - \frac{\varepsilon}{q}\sum_{\substack{d<\xi\\ q\nmid d}} g(d).$$

This last inequality is clearly true, since we would have equality if the last sum were over $d < \xi/q$. This completes the proof of the lemma. ∎

5. The existence lemma. In this section we prove that under the assumption (Ω_1) and $(\Omega_2(\varkappa))$ the numbers $\omega(p)$ can always be increased to numbers $\omega'(p)$ in such a way that $(\Omega_2(\varkappa, L))$ is satisfied with L a suitable function of $\varkappa$, A_1, and A_2. This enables us to use Lemma 1 and equation (13) to obtain equation (14).

The constants A_1 and A_2 are defined in assumptions (Ω_1) and $(\Omega_2(\varkappa))$, respectively (see Section 3). Let B_1 be such that

$$-B_1 \leqslant \sum_{w\leqslant p<y} \frac{\varkappa\log p}{p} - \varkappa\log\frac{y}{w} \leqslant B_1 \tag{18}$$

holds for all $2 \leqslant w < y$, and let

$$B_2 = \varkappa\max\{1, \log 2\varkappa\}. \tag{19}$$

We note that B_1 and B_2 are functions of $\varkappa$ only.

Define $\omega'(p)$ recursively as follows: for $p \leqslant 2\varkappa$, let

$$\omega'(p) = \omega(p). \tag{20}$$

For $p^* > 2\varkappa$ let $\omega'(p^*) = \omega(p^*)$ if

$$(21)\qquad -A_2 \leqslant \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} + \frac{\omega(p^*)\log p^*}{p^*} - \varkappa\log p^*,$$

and let

$$(22)\qquad \omega'(p^*) = \max\{\omega(p), \varkappa\},$$

otherwise. The idea is that we increase $\omega(p)$ to be at least $\varkappa$ whenever the sum in (21) drops too low. We now prove that these $\omega'(p)$'s are suitable for our purposes.

LEMMA 2. *If we define $A_3 = A_2+B_1+2B_2$, then with the above assumption and definitions, we have modified forms of (Ω_1) and $(\Omega_2(\varkappa, L))$, namely*

$$(\Omega_1)'\qquad \frac{\omega'(p)}{p} \leqslant 1 - \frac{1}{\max(A_1, 2)},$$

and

$$(\Omega_2(\varkappa, 3A_3))'\qquad -3A_3 \leqslant \sum_{w\leqslant p<y} \frac{\omega'(p)\log p}{p} - \varkappa\log\frac{y}{w} \leqslant 3A_3,$$

for all $2\leqslant w<y$ (this last inequality is not sharp).

Proof. The proof of $(\Omega_1)'$ is easy, since either

$$\frac{\omega'(p)}{p} = \frac{\omega(p)}{p} \leqslant 1-\frac{1}{A_1}, \quad\text{or}\quad \frac{\omega(p)}{p} \neq \frac{\omega'(p)}{p} = \frac{\varkappa}{p} < \frac{1}{2},$$

since in the second case $p > 2\varkappa$, by (20).

The proof of $(\Omega_2(\varkappa, 3A_3))'$ is more complicated, and will be done in three steps.

Step 1. We show that

$$(23)\qquad -2A_2-B_1-B_2 \leqslant \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^*$$

holds for all primes p^*.

For $p^* \leqslant 2\varkappa$ we have

$$(24)\qquad \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \geqslant -\varkappa\log 2\varkappa \geqslant -B_2.$$

Define $p_1 < p_2 < \ldots$ by

$$(25)\qquad \omega'(p) = \omega(p) \quad\text{for}\quad p<p_1,$$

$$(26)\qquad \omega'(p) \geqslant \varkappa \quad\text{for}\quad p_{2n+1}\leqslant p<p_{2n+2} \quad (n = 0, 1, 2, \ldots),$$

and

$$(27)\qquad \omega'(p) < \varkappa \quad\text{for}\quad p_{2n}\leqslant p<p_{2n+1} \quad (n = 1, 2, 3, \ldots).$$

For $2\varkappa \leqslant p^* < p_1$, we have

$$(28)\qquad \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \geqslant -A_2 - \frac{\omega(p^*)\log p^*}{p^*} \geqslant -2A_2,$$

by (25), (21), and $(\Omega_2(\varkappa))$ used with $w = p^*$, $y = p^{*+}$.

For $p_{2n} \leqslant p^* < p_{2n+1}$ we have

$$(29)\qquad \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \geqslant -2A_2,$$

by the argument used for (28).

For $p_{2n+1} \leqslant p^* < p_{2n+2}$ we have

$$(30)\qquad \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^*$$
$$= \sum_{2\leqslant p<p_{2n+1}} \frac{\omega'(p)\log p}{p} - \varkappa\log p'_{2n+1} - \varkappa\log\left(\frac{p_{2n+1}}{p'_{2n+1}}\right) + $$
$$+ \sum_{p_{2n+1}\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log\frac{p^*}{p_{2n+1}},$$

where p' is the prime preceding p. By equation (27) we know $\omega'(p_{2n+1}) = \omega(p_{2n+1})$, and so we can bound the first part of (30) from below by $-A_2$, by equation (21). The term involving the ratio of primes can be bounded from below by $-\varkappa\log 2$ by Bertrand's postulate. By (26) $\omega'(p) \geqslant \varkappa$ for all the terms in the last sum, so we may bound the last part from below by $-B_1$, by (18). This gives

$$(31)\qquad \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \geqslant -A_2-B_2-B_1$$

in this case.

We see that (24), (28), (29), and (31) are all included in (23), so Step 1 is finished.

Step 2. We show that

$$(32)\qquad \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \leqslant A_2+B_1+B_2$$

holds for all primes p^*.

Define $p_1 < p_2 < \ldots$ by

(33) $$\omega'(p) = \omega(p) \quad \text{for } p < p_1 \text{ and for } p_{2n} \leqslant p < p_{2n+1} \quad (n = 1, 2, 3, \ldots),$$

and

(34) $$\omega'(p) \neq \omega(p) \quad \text{for} \quad p_{2n+1} \leqslant p < p_{2n+2} \quad (n = 0, 1, 2, \ldots).$$

For $p^* < p_1$ we have

(35) $$\sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa \log p^* \leqslant A_2 - \varkappa\log 2 \leqslant A_2,$$

by (33) and $(\Omega_2(\varkappa))$.

For $p_{2n+1} \leqslant p^* < p_{2n+2}$ we have

(36) $$\sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* = \sum_{2\leqslant p<p_{2n+1}} \frac{\omega'(p)\log p}{p} - \varkappa\log p_{2n+1} + \sum_{p_{2n+1}\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log\left(\frac{p^*}{p_{2n+1}}\right).$$

Since $\omega'(p_{2n+1}) \neq \omega(p_{2n+1})$ by (34) we know that (21) does not hold for $p^* = p_{2n+1}$. This shows that the first part of (36) is negative. In the range of the second sum $\omega'(p) = \varkappa$ by (34), so we can bound the second part from above by B_1, by (18). This gives

(37) $$\sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \leqslant B_1$$

in this case.

For $p_{2n} \leqslant p^* < p_{2n+1}$ we have

(38) $$\sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* = \sum_{2\leqslant p<p_{2n}} \frac{\omega'(p)\log p}{p} - \varkappa\log p'_{2n} - \varkappa\log\left(\frac{p_{2n}}{p'_{2n}}\right) + \sum_{p_{2n}\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log\left(\frac{p^*}{p_{2n}}\right),$$

where p' is the prime preceding p. By (33) we know that (21) is not satisfied for $p^* = p'_{2n}$. This gives

(39) $$\sum_{2\leqslant p<p_{2n}} \frac{\omega'(p)\log p}{p} - \varkappa\log p'_{2n} \leqslant -A_2 + \frac{\varkappa\log p'_{2n}}{p'_{2n}} \leqslant B_2,$$

since $\omega'(p'_{2n}) = \varkappa$, and we know only that $\omega'(p'_{2n}) \geqslant 0$. In the second sum we note $\omega'(p) = \omega(p)$ by (33) and so we use $(\Omega_2(\varkappa))$ to get an upper bound of A_2. We ignore the negative term and use (39) to get

(40) $$\sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \leqslant B_2 + A_2$$

in this case. We note that (35), (37), and (40) are all contained in (32), so Step 2 is finished.

Step 3. We now have the result

(41) $$-2A_2 - B_1 - B_2 \leqslant \sum_{2\leqslant p<p^*} \frac{\omega'(p)\log p}{p} - \varkappa\log p^* \leqslant A_2 + B_1 + B_2$$

for all primes p^*, by (23) and (32). We can replace the p^* in equation (41) with an arbitrary z by noting that

$$\log(p/p') \leqslant \log 2,$$

so we replace the upper bound with $A_2 + B_1 + 2B_2$.

By replacing z with w and y, respectively and subtracting the two, we obtain $(\Omega_2(\varkappa, 3A_3))'$, thereby completing the proof of the lemma. ∎

6. The sieve result. By use of Lemma 1, Lemma 2, and equation (13) used for the $\omega'(p)$'s we now obtain inequality (14). By inserting (14) into (9) we obtain the following sieve inequality.

THEOREM 3. *Let* (Ω_1), $(\Omega_2(\varkappa))$ *hold. If* $\tau = \log\xi^2/\log z > 0$, *then*

$$\mathcal{S}(\mathfrak{A}; \mathcal{P}, z) \leqslant XW(z)\frac{1}{\sigma(\tau)}\left\{1 + O\left(\frac{(\log\log\xi)^{2\varkappa+1}}{\log\xi}\right)\right\} + \sum_{\substack{d<\xi^2 \\ d|P(z)}} 3^{\nu(d)}|R_d|.$$

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