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On bases and unconditional bases in the spaces $L^p(d\mu)$, $1 \leqslant p < \infty$

by

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Abstract. Necessary and sufficient conditions are found for a Borel measure μ in order that the system of functions $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ resulting from the Haar system by removing finitely many members be a basis in the space $L^p(d\mu)$, $1 \leqslant p < \infty$. It is also shown that if such a cofinite subsystem of the Haar system constitutes a basis in $L^p(d\mu)$, $1 < p < \infty$, then it actually constitutes an unconditional basis in that space.

1. Introduction. Let E be a Borel set on the real line and μ be a finite positive Borel measure on E. The symbol $L_E^p(d\mu)$, $1 \leqslant p < \infty$, will denote the Banach space of all functions f such that

$$\|f\|_{L_E^p(d\mu)} = \Big(\int_E |f(x)|^p\, d\mu\Big)^{1/p} < \infty. \tag{1}$$

Further, by definition,

$$\|f\|_{L_E^\infty(d\mu)} = \operatorname*{sup\,ess}_{x\in E} |f(x)| \qquad (\text{relative to } \mu). \tag{2}$$

In case where μ is the Lebesgue measure, we write just L_E^p instead of $L_E^p(d\mu)$.

In most problems concerning the Haar system and the Walsh system, the functions which enter those systems can be given quite arbitrary values at the discontinuity points. In this paper, however, we are dealing with general Borel measures, and so these values gain significance. Thus, we define the exact values of Haar and Walsh functions in such a way as to obtain closed systems in $C[0,1]$. Namely, by convention, the value at an interior discontinuity point is to be equal to the arithmetic mean of the one-sided limits at this point and the value at an endpoint is to be just the appropriate one-sided limit value.

A system of functions $\{f_n(x)\}$ is said to be *closed* in the space $L^p(d\mu)$, $1 \leqslant p < \infty$, iff every function in $L^p(d\mu)$ can be norm-approximated by finite linear combinations of the f_n's. A system $\{f_n(x)\}$ in $L^q(d\mu)$, q denoting the conjugate exponent to p, $1 \leqslant p < \infty$, is said to be *total* with respect to $L^p(d\mu)$ iff the only function in $L^p(d\mu)$ orthogonal to all the f_n's is the zero function. Following A.I. Markushevitch [13], we shall call a system of functions $\{f_n(x)\}$ a *basis in broad sense* for $L^p(d\mu)$, $1\leqslant p<\infty$, iff that system is minimal closed in $L^p(d\mu)$ and the system conjugate to $\{f_n(x)\}$ is total with respect to $L^p(d\mu)$.

The problem of the existence of conditional bases in a Hilbert space has remained open for a longish time. This problem was solved by K.I. Babenko [1] who showed that the systems resulting from the trigonometric system $\{e^{inx}\}_{n=-\infty}^{\infty}$ by multiplication by the functions $M_\alpha(x)=|x|^\alpha$, $0<\alpha<1/2$, constitute conditional bases in the space $L^p_{[-\pi,\pi]}$.

Observe that if a system $\{f_n(x)\}$ is such that multiplication of the f_n's by some function $M(x)$ produces a basis (or a basis in broad sense) in some L^p, $1\leqslant p<\infty$, then the system itself is a basis (a basis in broad sense) in the space $L^p[\psi(x)dx]$ where $\psi(x)=|M(x)|^p$, and conversely. In the sequel we shall often use this fact without further comment.

In 1972 R. Hunt, B. Muckenhoupt and R. Wheeden [6] found a characterization of the class of all functions which—multiplied by the trigonometric system—produce a basis in $L^p_{[0,2\pi]}$.

THEOREM (Hunt, Muckenhoupt, Wheeden). *Let $W(x)$ be a non-negative 2π-periodic function. The trigonometric system $\{e^{inx}\}_{n=-\infty}^{\infty}$ is a basis in the space $L^p_{[0,2\pi]}[W(x)dx]$, $1<p<\infty$, if and only if there exists an absolute constant K_p such that the following estimate is satisfied for any interval I:*

$$(A_p)\qquad \left(\frac{1}{|I|}\int_I W(x)\,dx\right)\left(\frac{1}{|I|}\int_I (W(x))^{-1/(p-1)}\,dx\right)^{p-1}\leqslant K_p,$$

$|I|$ denoting the length of I.

In 1971 A.S. Krantzberg [12] described all positive Borel measures μ for which the Haar system is a basis in $L^p(d\mu)$, $1\leqslant p<\infty$.

THEOREM (Krantzberg). *Let μ be a positive Borel measure on $[0,1]$. The Haar system $\{\chi_n(x)\}_{n=1}^{\infty}$ is a basis in the space $L^p(d\mu)$, $1\leqslant p<\infty$, if and only if μ is of the form*

$$(2)\qquad d\mu(x)=\psi(x)\,dx,$$

where $\psi(x)$ is a non-negative Lebesgue integrable function satisfying for any dyadic interval $\Delta=((m-1)/2^n, m/2^n)$ $(n=1,2,\ldots;\ m=1,\ldots,2^n)$ the estimate

$$(3)\qquad \left(\frac{1}{|\Delta|}\int_\Delta \psi(x)\,dx\right)\left(\frac{1}{|\Delta|}\int_\Delta (\psi(x))^{-1/(p-1)}\,dx\right)^{p-1}\leqslant K_p$$

with an absolute constant K_p depending on p only.

It also turns out that, under conditions (2) and (3), the Haar system constitutes an unconditional basis and the Walsh system a basis in the space $L^p(d\mu)$, $1<p<\infty$.

In the author's paper [9] it has been shown that, for any natural N, the Haar system with the initial N terms removed can be made, by multiplication by a bounded function, into a basis for all the spaces $L^p_{[0,1]}$, $1\leqslant p<\infty$. It has been also remarked that the above remains valid for the Haar system with finitely many of its members removed. In the present paper we find necessary and sufficient conditions for such a co-finite subsystem of the Haar system and for a Borel measure μ in order that the subsystem in question be a basis for $L^p(d\mu)$, $1\leqslant p<\infty$. It is also shown that if such a subsystem is a basis for $L^p(d\mu)$, $1<p<\infty$, it is also an unconditional basis for that space.

In what follows, dyadic intervals of the form $((m-1)/2^n, m/2^n)$ $(n=1,2,\ldots;\ m=1,\ldots,2^n)$ will be called *Haar intervals*. By the *support of the k-th Haar function* $\chi_k(x)$ we shall mean the Haar interval Δ_k on which $\chi_k(x)$ is non-zero (except of at a single point).

The symbol $L^q(d\mu)$ will always denote the dual space of $L^p(d\mu)$, $1\leqslant p<\infty$: $1/p+1/q=1$ (as usual, we assume $1/\infty=0$ and $1/0=\infty$ so that $p=1$ forces $q=\infty$). δ_{nm} is the Kronecker delta (0 for $n\neq m$, 1 for $n=m$).

The results of this paper for the case of weighted spaces have been published in the author's earlier paper [10].

It is a pleasant debt of mine to express my gratitude to Professor A.A. Talalian for his attention to my work and helpful criticism.

2. On the absolute continuity of μ. We are going to show that in all cases which are of interest for us the measure μ is absolutely continuous.

LEMMA 1. *Suppose that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a minimal closed system in $C[a,b]$ and $\{\psi_n(x)\}_{n=1}^{\infty}$ is a system biorthonormal to $\{\varphi_n(x)\}$. Further, suppose that Ω is a finite (or empty) set of natural numbers and Ω^c denotes the complement of Ω to the set of all naturals. If the system $\{\varphi_n(x)\}_{n\in\Omega^c}$ constitutes a basis in broad sense in $L^p_{[a,b]}(d\mu)$, $1\leqslant p<\infty$, for some finite positive Borel measure μ, then μ is absolutely continuous.*

Proof. Since the system $\{\varphi_n(x)\}_{n\in\Omega^c}$ is minimal closed in $L^p_{[a,b]}(d\mu)$, there exists a system $\{f_n(x)\}_{n\in\Omega^c}$ in $L^q_{[a,b]}(d\mu)$ biorthogonal to $\{\varphi_n(x)\}_{n\in\Omega^c}$ $(1/p+1/q=1)$,

$$(4)\qquad \int_a^b \varphi_n(x)f_m(x)\,d\mu(x)=\delta_{nm}\qquad (n,m\in\Omega^c).$$

Write

$$(5)\qquad a_i^{(m)}=\int_a^b \varphi_i(x)f_m(x)\,d\mu(x)\qquad (i\in\Omega,\ m\in\Omega^c)$$

and

$$(6)\qquad d\nu_m(x)=f_m(x)\,d\mu(x)\qquad (m\in\Omega^c).$$

Define measures ν'_m ($m \in \Omega^c$) as follows:

(7) $$d\nu'_m(x) = \Big(\varphi_m(x) + \sum_{i\in\Omega} a_i^{(m)}\psi_i(x)\Big)\,dx.$$

Conditions (4)–(6) yield

$$\int_a^b \varphi_n(x)\,d[\nu_m(x) - \nu'_m(x)] = 0 \qquad (n = 1, 2, \ldots;\ m \in \Omega^c).$$

Since the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is closed in $C[a, b]$, we obtain

(8) $$d[\nu_m(x) - \nu'_m(x)] = 0 \qquad (m \in \Omega^c).$$

The formulas

$$c_m = \int_a^b f(x) f_m(x)\,d\mu(x) = \int_a^b f(x)\,d\nu_m(x) \qquad (m \in \Omega^c)$$

define the coefficients of the development of $f \in L^p_{[a,b]}(d\mu)$. Hence in view of (7), (8) it follows that

$$c_m = \int_a^b f(x)\Big(\varphi_m(x) + \sum_{i\in\Omega} a_i^{(m)}\psi_i(x)\Big)\,dx \qquad (m \in \Omega^c).$$

Thus, assuming that μ is not absolutely continuous, we arrive at the conclusion that there is a non-zero function $f \in L^p_{[a,b]}(d\mu)$ whose all coefficients are zero. This contradicts the assumption that the system $\{\varphi_n(x)\}_{n\in\Omega^c}$ is a basis in broad sense. The lemma is proved.

We prepare one more useful lemma, which may be of interest in itself.

LEMMA 2. *Suppose that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a system of bounded measurable functions, total with respect to $L^1_{[a,b]}$ and constituting a basis in broad sense for $L^p_{[a,b]}$, $1 \leqslant p < \infty$. Let Ω and Ω^c have the same meaning as in Lemma 1.*

If the system $\{\varphi_n(x)\}_{n\in\Omega^c}$ is minimal in $L^p_{[a,b]}(\psi(x)\,dx)$ for some Lebesgue integrable non-negative function $\psi(x)$, then $\psi(x)$ is positive almost everywhere on $[a, b]$.

Proof. Since $\{\varphi_n(x)\}_{n\in\Omega^c}$ is minimal in $L^p_{[a,b]}(\psi(x)\,dx)$, there exists a conjugate system $\{f_n(x)\}_{n\in\Omega^c}$ in $L^q_{[a,b]}(\psi(x)\,dx)$,

(9) $$\int_a^b \varphi_m(x) f_n(x)\psi(x)\,dx = \delta_{nm} \qquad (n, m \in \Omega^c).$$

Let $\{\psi_n(x)\}_{n=1}^{\infty}$ denote the system in $L^q_{[a,b]}$ conjugate to $\{\varphi_n(x)\}_{n=1}^{\infty}$. Writing

(10) $$a_\nu^{(n)} = \int_a^b \varphi_\nu(x) f_n(x)\psi(x)\,dx \qquad (\nu \in \Omega,\ n \in \Omega^c)$$

we get, in view of (9),

$$\int_a^b \varphi_m(x)\Big[f_n(x)\psi(x) - \psi_n(x) - \sum_{\nu\in\Omega} a_\nu^{(n)}\psi_\nu(x)\Big]\,dx = 0$$
$$(m = 1, 2, \ldots,\ n \in \Omega^c).$$

The system $\{\varphi_m(x)\}_{m=1}^{\infty}$ being total over $L^1_{[a,b]}$, it hence follows that

(11) $$f_n(x)\psi(x) = \psi_n(x) + \sum_{\nu\in\Omega} a_\nu^{(n)}\psi_\nu(x) \qquad (n \in \Omega^c).$$

Supposing that $\psi(x)$ is zero on a set $F \subset [a, b]$ of positive measure we now easily arrive at a contradiction: taking a bounded function $\Phi(x)$ vanishing on $[a, b]\setminus F$ and such that

$$\int_a^b \Phi(x)\psi_\nu(x)\,dx = 0 \quad \text{for all } \nu \in \Omega,$$

we obtain by (11)

$$\int_a^b \Phi(x)\psi_n(x)\,dx = 0 \qquad (n \in \Omega^c),$$

contrary to the fact that the system $\{\psi_n(x)\}_{n=1}^{\infty}$ is total with respect to $L^p_{[a,b]}$. This proves the lemma.

Remark. The following statement is also true; the proof is analogous to the proof of Lemma 2.

Suppose that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a system constituting a basis in broad sense for $L^p_{[a,b]}$, $1 \leqslant p < \infty$. Let Ω and Ω^c be as above. If the system $\{\varphi_n(x)\}_{n\in\Omega^c}$ is minimal in $L^p_{[a,b]}(\psi(x)\,dx)$ for some non-negative bounded function $\psi(x)$, then $\psi(x)$ is positive almost everywhere on $[a, b]$.

Lemmas 1 and 2 jointly with Theorem 3 of the paper [9] result in the following

THEOREM 1. *Let $\{f_n(x)\}_{n=1}^{\infty}$ denote either the Walsh system or the trigonometric system, in an arbitrary numbering. Then for any positive integer N and for any positive Borel measure μ the system $\{f_n(x)\}_{n=N+1}^{\infty}$ is a basis in neither of the spaces $L^p(d\mu)$, $1 \leqslant p < \infty$.*

Evidently, the measure μ has to be finite.

Assuming that the assertion does not hold and applying Lemmas 1 and 2 we infer that $d\mu(x) = \psi(x)\,dx$ for some function $\psi(x)$ which is positive-valued on the entire interval. Hence and from Theorem 3 of [9] follows our theorem.

3. $\{\chi_n(x)\}_{n=N+1}^{\infty}$ being a basis. To begin with we state and prove the result for the case of the Haar system with the very first member removed; the proof for the remaining cases is a consequence of the above.

THEOREM 2. *The following conditions* (a)–(d) *are necessary and sufficient for the system* $\{\chi_n(x)\}_{n=2}^{\infty}$ *to be a basis in the space* $L^p(d\mu)$, $1 \leqslant p < \infty$:

(a) *there exists a Lebesgue integrable function* $\psi(x)$ *such that* $d\mu(x) = \psi(x)\,dx$;

(b) $\psi(x) > 0$ *almost everywhere on* $[0, 1]$;

(c) *there exists a sequence of Haar intervals*(*)

$$\Delta_0^{(i_0)} \supset \Delta_1^{(i_1)} \supset \ldots \supset \Delta_k^{(i_k)} \supset \ldots$$

with $|\Delta_k^{(i_k)}| = 2^{-k}$ $(k = 0, 1, \ldots)$, *such that*

$$[\psi(x)]^{-1} \notin L^{1/(p-1)}_{\Delta_k^{(i_k)}} \quad \text{and} \quad [\psi(x)]^{-1} \in L^{1/(p-1)}_{C\Delta_k^{(i_k)}};$$

(d) *there exists a number* $B_p > 0$ *such that for any* $k = 0, 1, \ldots$ *we have*

$$\left(\frac{1}{|\Delta_k^{(i_k)}|}\int\limits_{\Delta_k^{(i_k)}} \psi(x)\,dx\right)\left(\frac{1}{|\Delta_k^{(i_k)}|}\int\limits_{C\Delta_k^{(i_k)}} [\psi(x)]^{1/(p-1)}\,dx\right)^{p-1} \leqslant B_p$$

and such that for every Haar interval Δ *which is not identical to any of the* $\Delta_k^{(i_k)}$ $(k = 0, 1, \ldots)$ *we have*

$$\left(\frac{1}{|\Delta|}\int\limits_{\Delta} \psi(x)\,dx\right)\left(\frac{1}{|\Delta|}\int\limits_{\Delta} [\psi(x)]^{-1/(p-1)}\,dx\right)^{p-1} \leqslant B_p;$$

where $C\Delta_k^{(i_k)} = [0, 1] \setminus \Delta_k^{(i_k)}$.

In the proof of Theorem 2 we shall use the following fact, which is a direct consequence of Lemmas 1 and 2, of Lemma 1 from [9] and of Theorem 6 from [11].

THEOREM 3. *In order that* $\{\chi_n(x)\}_{n=2}^{\infty}$ *be a basis in broad sense for* $L^p(d\mu)$, $1 \leqslant p < \infty$, *conditions* (a), (b) *and* (c) *of Theorem* 2 *are necessary and sufficient.*

Now we prove a lemma, which constitutes the final step in the proof of Theorem 2. Assume that the measure μ fulfils conditions (a), (b), (c), that means, the system $\{\chi_n(x)\}_{n=2}^{\infty}$ is a basis in broad sense for $L^p(d\mu)$, $1 \leqslant p < \infty$. Denote by $S_n(f, x)$ the partial sums of the development of $f \in L^p(d\mu)$ in terms of that system. Consider the Haar intervals of maximal length on which all the functions $\chi_n(x)$, $n = 2, \ldots, N$, are constant; clearly, given an integer $N \geqslant 2$, there are N such intervals. Exactly one of them coincides with one of the intervals occurring in condition (c); denote this unique interval by Γ_N and the remaining ones by $\Gamma_1, \ldots, \Gamma_{N-1}$. Using this notation we state

* The upper index is introduced for convenience; the sense of this will be made clear during the proof.

LEMMA 3. *Let* μ *be a measure satisfying conditions* (a), (b), (c) *of Theorem* 2. *Then for every* $N \geqslant 2$ *we have*

$$S_N(f, x) = \begin{cases} \dfrac{1}{|\Gamma_n|}\displaystyle\int\limits_{\Gamma_n} f(t)\,dt & \text{if} \quad x \in \Gamma_n \ (1 \leqslant n \leqslant N-1), \\ -\dfrac{1}{|\Gamma_N|}\displaystyle\int\limits_{C\Gamma_N} f(t)\,dt & \text{if} \quad x \in \Gamma_N. \end{cases} \tag{12}$$

Proof. By assumption, $\{\chi_i(x)\}_{i=2}^{\infty}$ is a basis in broad sense for $L^p(d\mu)$. We will construct a conjugate system $\{\psi_i(x)\}_{i=2}^{\infty}$. First of all, we give a precise meaning to the upper indices i_k (see the footnote to condition (c)). For any $k = 0, 1, \ldots$ we define i_k to be either 1 or 2, according as the subsequent interval is either the left or the right half of $\Delta_k^{(i_k)}$.

For a given $n = 0, 1, \ldots$ and k, $1 \leqslant k \leqslant 2^n$, we put

(13)

$$\psi_{2^n+k}(x) = \begin{cases} [\psi(x)]^{-1}[\chi_{2^n+k}(x) + (-1)^{i_n}2^{n/2}] & \text{if} \quad \Delta_n^{(i_n)} \subset \Delta_{2^n+k}, \\ [\psi(x)]^{-1}\chi_{2^n+k}(x) & \text{if} \quad \Delta_n^{(i_n)} \cap \Delta_{2^n+k} = \emptyset. \end{cases}$$

According to condition (c), the system $\{\psi_i\}_{i=2}^{\infty}$ is contained in $L^q(d\mu)$, $1/p + 1/q = 1$. A straightforward verification shows that $\{\psi_i(x)\}_{i=2}^{\infty}$ is the system conjugate to $\{\chi_i(x)\}_{i=2}^{\infty}$. And thus, for any $N \geqslant 2$ and $f \in L^p(d\mu)$ we have

$$S_N(f, x) = \sum_{i=2}^{N} c_i\chi_i(x), \tag{14}$$

where

$$c_i = \int\limits_0^1 f(x)\psi_i(x)\,d\mu(x). \tag{15}$$

The proof of the lemma is carried by induction. For $N = 2$ we have

$$\begin{aligned} c_2 &= \int\limits_0^1 f(x)[\psi(x)]^{-1}[\chi_2(x) + (-1)^{i_1}]\psi(x)\,dx \\ &= 2\int\limits_{C\Delta_1^{(i_1)}} [\operatorname{sign}\chi_2(x)]f(x)\,dx. \end{aligned}$$

Hence

$$S_2(f, x) = \begin{cases} 2\displaystyle\int\limits_{C\Delta_1^{(i_1)}} f(x)\,dx & \text{for} \quad x \in C\Delta_1^{(i_1)}, \\ -2\displaystyle\int\limits_{C\Delta_1^{(i_1)}} f(x)\,dx & \text{for} \quad x \in \Delta_1^{(i_1)}. \end{cases}$$

Now assume that (12) is true for some N and calculate $S_{N+1}(f,x)$:

(16) $$S_{N+1}(f,x) = S_N(f,x)+c_{N+1}\chi_{N+1}(x).$$

From the definitions of the Haar system and of the sets Γ_i, $1\leqslant i\leqslant N$, it follows that Δ_{N+1}, the support of χ_{N+1}, coincides with one of the intervals Γ_i, $1\leqslant i\leqslant N$.

First consider the case where Δ_{N+1} is one of the Γ_i, $1\leqslant i\leqslant N-1$. Let Δ'_{N+1} and Δ''_{N+1} denote the left and the right half of Δ_{N+1}, respectively. Equalities (13) and (15) yield

$$c_{N+1} = \int_0^1 f(x)[\psi(x)]^{-1}\chi_{N+1}(x)\psi(x)\,dx = \int_0^1 f(x)\chi_{N+1}(x)\,dx.$$

Hence, by (12) and (16) and in view of the fact that $\chi_{N+1}(x) = \pm|\Delta_{N+1}|^{-1/2}$ (plus in Δ'_{N+1} and minus in Δ''_{N+1}), we obtain

(17) $$S_{N+1}(f,x) = \begin{cases} \dfrac{2}{|\Delta_{N+1}|}\displaystyle\int_{\Delta'_{N+1}} f(x)\,dx & \text{for} \quad x\in\Delta'_{N+1}, \\ \dfrac{2}{|\Delta_{N+1}|}\displaystyle\int_{\Delta''_{N+1}} f(x)\,dx & \text{for} \quad x\in\Delta''_{N+1}. \end{cases}$$

Since $\chi_{N+1}(x)$ is zero outside Δ_{N+1} and since the functions $\chi_i(x)$, $1\leqslant i\leqslant N+1$, are constant on the intervals Δ'_{N+1}, Δ''_{N+1}, the inductive assertion follows in view of (12) and (17).

It remains to consider the case of $\Delta_{N+1} = \Gamma_N = \Delta_k^{(i_k)}$, where $k = \log_2(1/|\Delta_{N+1}|)$. Conditions (15), (13) and (a) then force

(18) $$c_{N+1} = \int_0^1 f(x)\left[\chi_{N+1}(x)+(-1)^{i_k}\left(\frac{1}{|\Delta_{N+1}|}\right)^{1/2}\right]dx$$
$$= 2\frac{1}{|\Delta_{N+1}|}\int_{\Delta_k^{(i_k)}\setminus\Delta_{k+1}^{(i_k+1)}} f(x)\chi_{N+1}(x)\,dx+(-1)^{i_k}\left(\frac{1}{|\Delta_{N+1}|}\right)^{1/2}\int_{C\Delta_k^{(i_k)}} f(x)\,dx.$$

Now, we have

$$(-1)^{i_k}\operatorname{sign}\chi_{N+1}(x) = 1 \quad \text{for} \quad x\in\Delta_k^{(i_k)}\setminus\Delta_{k+1}^{(i_k+1)}.$$

Hence, inserting the values of c_{N+1} and $S_N(f,x)$ into (16), we get

$$S_{N+1}(f,x) = \begin{cases} \dfrac{2}{|\Delta_{N+1}|}\displaystyle\int_{\Delta_k^{(i_k)}\setminus\Delta_{k+1}^{(i_k+1)}} f(x)\,dx & \text{for} \quad x\in\Delta_k^{(i_k)}\setminus\Delta_{k+1}^{(i_k+1)}, \\ -\dfrac{2}{|\Delta_{N+1}|}\displaystyle\int_{C\Delta_{k+1}^{(i_k+1)}} f(x)\,dx & \text{for} \quad x\in\Delta_{k+1}^{(i_k+1)}. \end{cases}$$

This concludes the induction and the proof of Lemma 3.

Proof of Theorem 2. It is well known that a neccessary and sufficient condition for a basis in broad sense in $L^p(d\mu)$ to be actually a basis is that the norms of the operators of taking partial sums of the developments be commonly bounded. Thus, in view of Theorem 3, in order to establish the sufficiency of conditions (a)–(d) it just remains to estimate the norms of $S_N(f,x)$. According to Lemma 3 we have

$$\int_0^1 |S_N(f,x)|^p\psi(x)\,dx = \sum_{i=1}^N\int_{\Gamma_i}|S_N(f,x)|^p\psi(x)\,dx$$
$$= \sum_{i=1}^{N-1}\left|\frac{1}{|\Gamma_i|}\int_{\Gamma_i} f(x)\,dx\right|^p\int_{\Gamma_i}\psi(x)\,dx+\left|\frac{1}{|\Gamma_N|}\int_{C\Gamma_N} f(x)\,dx\right|^p\int_{\Gamma_N}\psi(x)\,dx$$
$$\leqslant \sum_{i=1}^{N-1}\frac{1}{|\Gamma_i|^p}\int_{\Gamma_i}|f(x)|^p\psi(x)\,dx\left[\int_{\Gamma_i}[\psi(x)]^{-q/p}\,dx\right]^{p/q}\int_{\Gamma_i}\psi(x)\,dx+$$
$$+\frac{1}{|\Gamma_N|^p}\int_{C\Gamma_N}|f(x)|^p\psi(x)\,dx\left[\int_{C\Gamma_N}[\psi(x)]^{-q/p}\,dx\right]^{p/q}\int_{\Gamma_N}\psi(x)\,dx$$
$$\leqslant 2B_p\int_0^1|f(x)|^p\psi(x)\,dx.$$

The sufficiency in Theorem 2 is thus proved. To show the necessity, we use the fact that if the system $\{\chi_i(x)\}_{i=2}^\infty$ is a basis in $L^p(d\mu)$, then the norms of the operators S_N are commonly bounded:

(19) $$\|S_N\|\leqslant M<+\infty.$$

On the other hand, by virtue of Theorem 3, Lemma 3 and by the boundedness of the operators S_N, we have

(20) $$\|S_N\| = \sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\|S_N(f)\|_{L^p(d\mu)}\geqslant \max_{1\leqslant i\leqslant N-1}\sup\|S_N(f,x)\|_{L^p(d\mu)},$$

the last supremum being taken over all f with

$$\|f\|_{L^p(d\mu)}\leqslant 1 \quad \text{and} \quad f(x) = 0 \text{ for } x\in C\Gamma_i.$$

Equality (12) shows that for $i = 1,\ldots,N-1$ this last supremum is equal to

(21) $$\sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\|S_N(f,x)\|_{L^p_{\Gamma_i}(d\mu)}$$
$$= \left[\int_{\Gamma_i}\psi(x)\,dx\right]^{1/p}\sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\left|\frac{1}{|\Gamma_i|}\int_{\Gamma_i}f(x)\,dx\right|.$$

Writing

$$f(x) = f(x)[\psi(x)]^{-1}\psi(x)$$

we obtain immediately

$$\sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\Big|\int_{\Gamma_i} f(x)\,dx\Big| = \|[\psi(x)]^{-1}\|_{L^q_{\Gamma_i}(d\mu)} = \Big(\int_{\Gamma_i}[\psi(x)]^{-1/(p-1)}\,dx\Big)^{(p-1)/p}.$$

Hence and by relations (19)–(21) follows the necessity of condition (d) for those Haar intervals Δ which do not coincide with any of the $\Delta_k^{(i_k)}$ $(k=0,1,2,\ldots)$.

The necessity of (d) for the latters is proved quite analogously, as shows the following calculation:

$$\begin{aligned}
\|S_N\| &= \sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\|S_N(f,x)\|_{L^p(d\mu)}\\
&\geqslant \sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\|S_N(f,x)\|_{L^p_{\Gamma_N}(d\mu)}\\
&= \Big[\int_{\Gamma_N}\psi(x)\,dx\Big]^{1/p}\sup_{\|f\|_{L^p(d\mu)}\leqslant 1}\Big|\frac{1}{|\Gamma_N|}\int_{C\Gamma_N} f(x)\,dx\Big|\\
&= \frac{1}{|\Gamma_N|}\Big[\int_{\Gamma_N}\psi(x)\,dx\Big]^{1/p}\|[\psi(x)]^{-1}\|_{L^q_{C\Gamma_N}(d\mu)}\\
&= \Big[\frac{1}{|\Gamma_N|}\int_{\Gamma_N}\psi(x)\,dx\Big]^{1/p}\Big[\int_{C\Gamma_N}[\psi(x)]^{-1/(p-1)}\,dx\Big]^{(p-1)/p}.
\end{aligned}$$

The proof of Theorem 2 is complete.

We now give concrete examples of positive measures for which conditions (a)–(d) of Theorem 2 are satisfied.

COROLLARY 1. *Let $\{\chi_n(x)\}_{n=1}^{\infty}$ be the Haar system and let p_1 be any number, $1<p_1<\infty$. Define:*

$$\psi(x) = x^{p_1-1}\quad \textit{for}\quad x\in[0,1]. \tag{22}$$

Then the system $\{\chi_n(x)\}_{n=2}^{\infty}$ is a basis in $L^p(\psi(x)\,dx)$ for all $p\in[1,p_1)$, is a basis in broad sense in $L^{p_1}(\psi(x)\,dx)$, and for every $r>p_1$ this system is not closed in $L^r(\psi(x)\,dx)$.

Proof. Clearly, for $p\in[1,p_1]$ we have

$$[\psi(x)]^{-1}\notin L^{1/(p-1)}_{[0,2^{-n}]}$$

and

$$[\psi(x)]^{-1}\in L^{1/(p-1)}_{[2^{-n},1]}$$

$(n=0,1,2,\ldots)$. Hence by Theorem 3 it follows that the system $\{\chi_n(x)\}_{n=2}^{\infty}$ is a basis in broad sense in $L^p(\psi(x)\,dx)$, for any $p\in[1,p_1]$. Further, for $p>p_1$ we have

$$[\psi(x)]^{-1}\in L^{1/(p-1)}_{[0,1]},$$

whence

$$[\psi(x)]^{-1}\in L^q(\psi(x)\,dx),\qquad\text{where}\qquad p^{-1}+q^{-1}=1,$$

and

$$\int_0^1[\psi(x)]^{-1}\chi_n(x)\psi(x)\,dx = 0\qquad (n=2,3,\ldots).$$

Thus the system $\{\chi_n(x)\}_{n=2}^{\infty}$ is not closed in any $L^p(\psi(x)\,dx)$, $p>p_1$. Finally, for $p\in[1,p_1)$ condition (d) is easily verified. The corollary is thus proved.

In the sequel we shall deal with a more general case; therefore, at present, we just describe necessary and sufficient conditions for the system $\{\chi_n(x)\}_{n=N+1}^{\infty}$ to be a basis in $L^p(d\mu)$, without giving them a precise form.

In view of Lemmas 1 and 2 it is clear that conditions (a) and (b) of Theorem 2 are necessary. Consider the intervals of the greatest length on which all the functions $\chi_n(x)$, $1\leqslant n\leqslant N$, are constant; on each of those intervals take into account all the functions $\chi_n(x)$, $n\geqslant N+1$, which do not vanish identically there. By a linear transformation which carries the interval considered onto $(0,1)$ the resulting system can be identified with the system $\{\chi_n(x)\}_{n=2}^{\infty}$, up to a constant. Consequently, in order that the system $\{\chi_n(x)\}_{n=N+1}^{\infty}$ be a basis in $L^p(d\mu)$, $1\leqslant p<\infty$, it is necessary and sufficient that, apart of conditions (a) and (b) of Theorem 2, conditions analogous to (c) and (d) be satisfied on each of those maximal intervals of constancy of all the $\chi_n(x)$, $1\leqslant n\leqslant N$.

4. $\{\chi_n(x)\}_{n=1}^{\infty}$ being an unconditional basis; $\{w_n(x)\}_{n=1}^{\infty}$ being a basis. In 1935 R. Paley [16] showed that the Walsh system constitutes a basis in each $L^p_{[0,1]}$, $1<p<\infty$. The basic tool in the proof was furnished by two inequalities, which now have become classical. Owing to their origin, they are known under the name of Paley's inequalities. Using those inequalities J. Marcinkiewicz [14] showed in 1937 that the Walsh system is actually an unconditional basis in $L^p_{[0,1]}$, $1<p<\infty$.

In 1974 R. Gundy and R. Wheeden [5] obtained a remarkable result stating that if a weight function $\psi(x)$ fulfils inequality (3), then Paley's weighted inequalities (with weight $\psi(x)$) are also satisfied in that case. Their result is basic for the following theorem.

THEOREM 4. *Let μ be a positive Borel measure on $[0,1]$ and let $p>1$. Then each one of the following five conditions implies the remaining ones:*

(α) *μ is absolutely continuous and there exists an absolute constant K_p such that for any Haar interval $\Delta=(k/2^n,(k+1)/2^n)$, $0\leqslant k\leqslant 2^n$, $n=0,1,2,\ldots$, the weight function given by (2) satisfies inequality (3);*

(β) *the Haar system is a basis for $L^p(d\mu)$;*

(γ) *the Haar system is an unconditional basis for $L^p(d\mu)$;*

(δ) *the Walsh system is a basis for* $L^p(d\mu)$;

(ε) *the Haar system is a basis in broad sense for* $L^p(d\mu)$ *and, moreover, Paley's generalized inequalities are satisfied*:

$$c_p\Big(\int_0^1 |f(x)|^p\,d\mu\Big)^{1/p} \leqslant \Big(\int_0^1\Big(\sum_{k=1}^{\infty} a_k^2\chi_k^2(x)\Big)^{p/2} d\mu\Big)^{1/p} \leqslant C_p\Big(\int_0^1 |f(x)|^p\,d\mu\Big)^{1/p},$$

where a_k *denote the coefficients of the development of* $f \in L^p(d\mu)$ *in terms of* $\{\chi_k(x)\}_{k=1}^{\infty}$ *and* c_p, C_p *are positive constants depending only on* p.

Proof. The equivalence between (α) and (β) is just the content of Krantzberg's theorem [12]. Hence, in virtue of Gundy–Wheeden's theorem [5] follows the implication (α) ⇒ (ε). The implication (ε) ⇒ (γ) is a consequence of Lemma 1 and Gaposhkin's criterion [3] for deciding whether or not a given system is an unconditional basis in L^p, $p > 1$. The proof of the implication (ε) ⇒ (δ) is analogous to Paley's proof of the fact that the Walsh system is a basis in $L^p_{[0,1]}$. Nevertheless, to be precise, we now carry in detail some of Paley's arguments.

Write

$$D_n(x, t) = \sum_{k=1}^{n} w_k(x)w_k(t).$$

Our further considerations are based on the equality

$$D_n(x, t)w_{n+1}(x)w_{n+1}(t) = \sum_{i=1}^{m}\big(D_{2^{n_i+1}}(x, t) - D_{2^{n_i}}(x, t)\big), \tag{23}$$

where

$$n = \sum_{i=1}^{m} 2^{n_i}, \quad n_m > n_{m-1} > \ldots > n_1, \quad n_m = [\log_2 n];$$

equality (23) is due to Paley.

In Paley's paper the Walsh system has been examined exclusively in Paley's numbering. However, as it has been observed in [11], equality (23) is valid for Walsh'es numbering, just as well.

Condition (ε) implies, according to Lemma 1, that μ is absolutely continuous:

$$d\mu(x) = \psi(x)\,dx,$$

where $\psi(x)$ is a Lebesgue integrable function. It is hence readily seen that the system $\{[\psi(x)]^{-1}\chi_k(x)\}_{k=1}^{\infty}$ is conjugate to $\{\chi_k(x)\}_{k=1}^{\infty}$ in $L^p(d\mu)$, and so the coefficients a_k are given by the formulas

$$a_k = \int_0^1 f(t)[\psi(t)]^{-1}\chi_k(t)\psi(t)\,dt = \int_0^1 f(t)\chi_k(t)\,dt.$$

In view of the fact that the system $\{w_k(x)\}_{k=2^i+1}^{2^{i+1}}$ results from $\{\chi_k(x)\}_{k=2^i+1}^{2^{i+1}}$ by a certain orthogonal transformation ($i = 0, 1, 2, \ldots$), it is clear that the system $\{w_k(x)\}_{k=1}^{\infty}$ is also a basis in broad sense for $L^p(d\mu)$.

Observe that

$$\sum_{k=1}^{\infty} a_k^2\chi_k^2(x) = b_1^2 + \sum_{i=0}^{\infty}\Big(\sum_{k=2^i+1}^{2^{i+1}} b_k w_k(x)\Big)^2, \tag{24}$$

where the numbers

$$b_k = \int_0^1 f(t)w_k(t)\,dt \quad (k = 0, 1, 2, \ldots) \tag{25}$$

are the coeffcients of the development of f in terms of $\{w_k(x)\}_{k=1}^{\infty}$. Denote by $S_n(f, x)$ the nth partial sum of that development.

The following inequality can be derived from condition (ε) without much difficulty

$$\Big\|\sum_{i=1}^{m}\big(S_{2^{n_i+1}}(f, x) - S_{2^{n_i}}(f, x)\big)\Big\|_{L^p(d\mu)} \leqslant \frac{C_p}{c_p}\|f\|_{L^p(d\mu)},$$

holding for arbitrary naturals $n_1 < n_2 < \ldots < n_i < \ldots$ and for $m = 1, 2, \ldots$ Applying identity (23) we hence infer that for any function $f \in L^p(d\mu)$ and any positive integer n the partial sum $S_n(f, x)$ satisfies in norm the estimate (the meaning of n_i's being as in (23)):

$$\|S_n(f, x)\|_{L^p(d\mu)} = \Big\|\sum_{i=1}^{m}\big(S_{2^{n_i+1}}(fw_{n+1}, x) - S_{2^{n_i}}(fw_{n+1}, x)\big)w_{n+1}(x)\Big\|_{L^p(d\mu)}$$
$$\leqslant 2\frac{C_p}{c_p}\|fw_{n+1}\|_{L^p(d\mu)} = 2\frac{C_p}{c_p}\|f\|_{L^p(d\mu)}.$$

This concludes the proof of the implication (ε) ⇒ (δ).

To obtain the implication (δ) ⇒ (β), let us notice that if the Walsh system is a basis for $L^p(d\mu)$, then there exists a constant B'_p such that

$$\Big\|\sum_{k=1}^{n} b_k w_k(x)\Big\|_{L^p(d\mu)} \leqslant B'_p\|f\|_{L^p(d\mu)} \quad (n = 1, 2, \ldots), \tag{26}$$

where b_k's are given by (25). Using inequality (26) and writing $n' = 2^{[\log_2 n]}$ we obtain

$$\Big\|\sum_{k=1}^{n} a_k\chi_k\Big\|_{L^p(d\mu)} \leqslant \Big\|\sum_{k=1}^{n'} a_k\chi_k\Big\|_{L^p(d\mu)} + \Big\|\sum_{k=n'+1}^{n} a_k\chi_k\Big\|_{L^p(d\mu)}$$
$$\leqslant \Big\|\sum_{k=1}^{n'} b_k w_k\Big\|_{L^p(d\mu)} + \Big\|\sum_{k=n'+1}^{2n'} b_k w_k\Big\|_{L^p(d\mu)} \leqslant 3B'_p\|f\|_{L^p(d\mu)},$$

and this implies condition (β).

The proof of Theorem 4 is complete.

The Walsh system does not constitute an unconditional basis in $L^p(d\mu)$, for any measure μ and any exponent $p \neq 2$ $(p > 1)$. This is an easy consequence of a theorem of V.F. Gaposhkin [3].

COROLLARY TO GAPOSHKIN'S THEOREM. Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ *be a system of measurable functions on* $[a, b]$. *Suppose that these functions are commonly bounded and normalized in* $L^2_{[a,b]}$. *Let further* $\psi(x)$ *be a positive Lebesgue integrable function. Then the system* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *is not an unconditional basis in* $L^p_{[a,b]}(\psi(x)\,dx)$, *for any* $p \neq 2$ $(p > 1)$.

Proof. We first consider the case of $p > 2$.

Assume that $\{(\psi(x))^{1/p}\varphi_n(x)\}_{n=1}^{\infty}$ is an unconditional basis for $L^p_{[a,b]}$. Then, according to Gaposhkin's theorem, the following inequalities hold:

$$(27)\qquad m_p\Big(\int_a^b\Big(\sum_{n=1}^{\infty} a_n^2(\psi(x))^{2/p}\varphi_n^2(x)\Big)^{p/2}dx\Big)^{1/p} \leqslant \|f\|_{L^p_{[a,b]}} \leqslant M_p\Big(\int_a^b\Big(\sum_{n=1}^{\infty} a_n^2(\psi(x))^{2/p}\varphi_n^2(x)\Big)^{p/2}dx\Big)^{1/2},$$

here m_p and M_p denote certain absolute constants and a_n's are the coefficients of the development of $f(x)$. Now, on the one hand, the common boundedness of the functions of the system implies that

$$(28)\qquad \Big(\int_a^b\Big(\sum_{n=1}^{\infty} a_n^2\varphi_n^2(x)\Big)^{p/2}\psi(x)\,dx\Big)^{1/p} \leqslant M_p'\Big(\sum_{n=1}^{\infty} a_n^2\Big)^{1/2},$$

M_p' denoting a certain constant. On the other hand, we have

$$\Big(\int_a^b\Big(\sum_{n=1}^{\infty} a_n^2\varphi_n^2(x)\Big)^{p/2}\psi(x)\,dx\Big)^{1/p} \geqslant \Big(\int_a^b\Big(\sum_{n=1}^{\infty} a_n^2\varphi_n^2(x)\Big)\psi(x)\,dx\Big)^{1/2}\Big(\int_a^b\psi(x)\,dx\Big)^{-(p-2)/2p},$$

and since the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is commonly bounded and square-normalized, we hence obtain

$$(29)\qquad \Big(\int_a^b\Big(\sum_{n=1}^{\infty} a_n^2\varphi_n^2(x)\Big)^{p/2}\psi(x)\,dx\Big)^{1/p} \geqslant m_p'\Big(\sum_{n=1}^{\infty} a_n^2\Big)^{1/2}$$

with a suitable constant m_p'. Inequalities (27), (28), (29) show that the correspondence between functions $f \in L^p_{[a,b]}$ and the sequences of their coefficients establishes an isomorphism between the spaces $L^p_{[a,b]}$ and l^2; and this is impossible, in virtue of a theorem of Banach (see [2], p. 75).

As regards the case of $1 < p < 2$, observe that if the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is an unconditional basis in $L^p_{[a,b]}(\psi(x)\,dx)$, then the system $\{[\psi(x)]^{-1}\varphi_n(x)\}_{n=1}^{\infty}$ is an unconditional basis in $L^q_{[a,b]}(\psi(x)\,dx)$, where $1/p+1/q=1$ (and so $q > 2$). Thus the assertion in this case follows immediately from the preceding case, and this ends the proof of the corollary.

It is well known that, given a uniformly bounded orthonormal system of functions on $[a, b]$ and a measurable function $M(x)$, in order that multiplication by $M(x)$ should make the system into an unconditional basis for $L^2_{[a,b]}$, it is necessary and sufficient that $M(x)$ and $[M(x)]^{-1}$ be almost everywhere bounded (see the papers by Babenko [1] and Olevskiĭ [15]). Accordingly, our Theorem 4 shows that a necessary and sufficient condition for the system $\{M(x)w_n(x)\}_{n=1}^{\infty}$ to constitute a conditional basis in $L^2_{[0,1]}$ is that the function $\psi(x) = (M(x))^p$ should satisfy condition (α) and one of the functions $M(x)$, $[M(x)]^{-1}$ be unbounded.

Applying Theorem 4 we can easily construct an example of a system of functions, which is a basis in $L^p_{[0,1]}$ for all $p \notin (c, d)$, where (c, d) is any prescribed interval, $1 \leqslant c < d \leqslant \infty$; we can also construct examples of conditional bases in $L^2_{[0,1]}$.

The following statement is easily demonstrated by a straightforward verification of condition (3):

PROPOSITION. *Consider the Haar system* $\{\chi_n(x)\}_{n=1}^{\infty}$ *and let* $p_1 > 1$ *be any number. Write*

$$(30)\qquad M_{\varepsilon,p_1}(x) = x^{\varepsilon/p_1}, \quad x \in [0, 1], \quad \varepsilon = \pm 1,$$

and consider the system $\{M_{\varepsilon,p_1}(x)\chi_n(x)\}_{n=1}^{\infty}$. *For* $\varepsilon = -1$ *this system is a basis in* $L^p_{[0,1]}$ *if and only if* $p \in [1, p_1)$; *for* $\varepsilon = 1$ *this system is a basis in* $L^p_{[0,1]}$ *if and only if* $p \in (p_1/(p_1-1), \infty)$.

This proposition together with Theorem 4 lead directly to the following two theorems:

THEOREM 5. *Consider the Haar system* $\{\chi_n(x)\}_{n=1}^{\infty}$ *and let* (c, d) *be any interval*, $1 \leqslant c < d \leqslant \infty$. *Write*

$$(31)\qquad M(x) = \begin{cases} M_{-1,d}(2x) & \text{for} \quad x \in [0, 1/2], \\ M_{1,c/(c-1)}(2(x-1/2)) & \text{for} \quad x \in (1/2, 1] \end{cases}$$

(*where* $M_{1,1/0}(x) \equiv 1$). *The system* $\{M(x)\chi_n(x)\}_{n=1}^{\infty}$ *is an unconditional basis in* $L^p_{[0,1]}$ *if and only if* $p \in (c, d)$.

THEOREM 6. *Consider the Walsh system* $\{w_n(x)\}_{n=1}^{\infty}$. *Then the system* $\{M(x)w_n(x)\}_{n=1}^{\infty}$, *where*

$$M(x) = x^r, \quad x \in [0, 1],$$

and r *is any number in the interval* $-1/2 < r < 1/2$, *constitutes a conditional basis in* $L^2_{[0,1]}$.

5. $\{\chi_n(x)\}_{n=N+1}^{\infty}$ being an unconditional basis. In this section we prove the following theorem:

THEOREM 7. *Consider the Haar system $\{\chi_n(x)\}_{n=1}^{\infty}$. Let p be an exponent, $1<p<\infty$, and N be a positive integer. The system $\{\chi_n(x)\}_{n=N+1}^{\infty}$ is an unconditional basis in $L^p(d\mu)$ if and only if it is a basis in $L^p(d\mu)$.*

Proof. The "only if" part is obvious. We shall prove the "if" part for the case of the system $\{\chi_n(x)\}_{n=2}^{\infty}$. The assertion in the general case then follows by virtue of the considerations at the end of Section 3.

We shall need several lemmas.

LEMMA 4. *Let*

$$\Delta_{i(0)}\supset\Delta_{i(1)}\supset\ldots\supset\Delta_{i(k)}\supset\ldots,\qquad |\Delta_{i(k)}|=1/2^k\qquad (k=0,1,2,\ldots) \tag{32}$$

be a sequence of Haar intervals, $i(0)=2$, and let $\chi_{i(k)}(x)$ denote the Haar function supported by $\Delta_{i(k)}$. Given a series

$$\sum_{k=0}^{\infty} a_k\chi_{i(k)}(x)=F(x), \tag{33}$$

let us write

$$F(x)=c_l\quad \text{for}\quad x\in\Delta_{i(l)}\setminus\Delta_{i(l+1)}\qquad (l=0,1,2,\ldots). \tag{34}$$

Then for any sequence $\{\varepsilon_k\}_{k=0}^{\infty}$, where $\varepsilon_k=\pm1$, we have the inequality

$$\Big|\sum_{k=0}^{\infty}\varepsilon_k a_k\chi_{i(k)}(x)\Big|\leqslant\sum_{l=0}^{j}2^{j-l}|c_l|\qquad \text{for}\quad x\in\Delta_{i(j)}\setminus\Delta_{i(j+1)}\qquad (j=0,1,2,\ldots). \tag{35}$$

Proof. We shall show by induction that

$$a_j\chi_{i(j)}(x)=c_j+\sum_{l=0}^{j-1}2^{j-l-1}c_l\qquad \text{for}\quad x\in\Delta_{i(j)}\setminus\Delta_{i(j+1)}\qquad (j=0,1,2,\ldots); \tag{36}$$

inequality (35) is a direct consequence of this.

For $j=0$ we have

$$\sum_{k=1}^{\infty}a_k\chi_{i(k)}(x)=F(x)-a_0\chi_{i(0)}(x). \tag{37}$$

Since $i(0)=2$ and since the function $\chi_{i(0)}(x)$ is the unique term of the series (33) that does not vanish on the interval $[0,1]\setminus\Delta_{i(1)}$, we obtain in view of (34)

$$a_0\chi_2(x)=c_0\quad \text{for}\quad x\in[0,1]\setminus\Delta_{i(1)}. \tag{38}$$

Now suppose that (36) holds true for $j=0,\ldots,j_0$. Then

$$a_j\chi_{i(j)}(x)=-\Big(c_j+\sum_{l=0}^{j-1}2^{j-l-1}c_l\Big)\quad \text{for}\quad x\in\Delta_{i(j_0+1)}\qquad (j\leqslant j_0). \tag{39}$$

We have, by (33) and (34),

$$\sum_{j=0}^{j_0+1}a_j\chi_{i(j)}(x)=c_{j_0+1}\quad \text{for}\quad x\in\Delta_{i(j_0+1)}\setminus\Delta_{i(j_0+2)}. \tag{40}$$

Hence by (39) follows equality (36) for $j=j_0+1$. This concludes the induction and ends the proof.

LEMMA 5. *Let $\psi(x)$ be a positive-valued Lebesgue integrable function and let $p>1$. Let further $\{\Delta_{i(k)}\}_{k=0}^{\infty}$ be a sequence of Haar intervals satisfying conditions (32) and let $B_p>0$ be a constant such that*

$$\Big(\frac{1}{|\Delta_{i(k)}|}\int_{\Delta_{i(k)}}\psi(x)\,dx\Big)\Big(\frac{1}{|\Delta_{i(k)}|}\int_{C\Delta_{i(k)}}[\psi(x)]^{1/(p-1)}\,dx\Big)^{p-1}\leqslant B_p\qquad (k=0,1,2,\ldots). \tag{41}$$

Then there exist constants B and $S>1$, depending only on B_p and p, and such that the following estimate is valid for any positive integers j and l $(1\leqslant l\leqslant j)$:

$$\frac{1}{|\Delta_{i(j)}|}\Big(\int_{\Delta_{i(j)}}\psi(x)\,dx\Big)^{1/p}\Big(\int_{C\Delta_{i(l)}}[\psi(x)]^{1/(p-1)}\,dx\Big)^{(p-1)/p}\leqslant B(1/S)^{j-l}.$$

Proof. We have, for any positive integer k,

$$\begin{aligned}
&\Big(\int_{C\Delta_{i(k+1)}}[\psi(x)]^{-1/(p-1)}\,dx\Big)\Big(\int_{C\Delta_{i(k)}}[\psi(x)]^{-1/(p-1)}\,dx\Big)^{-1}\\
&=1+\Big(\int_{\Delta_{i(k)}\setminus\Delta_{i(k+1)}}[\psi(x)]^{-1/(p-1)}\,dx\Big)\Big(\int_{C\Delta_{i(k)}}[\psi(x)]^{-1/(p-1)}\,dx\Big)^{-1}\\
&\geqslant 1+\Big(\int_{\Delta_{i(k)}\setminus\Delta_{i(k+1)}}[\psi(x)]^{-1/(p-1)}\,dx\Big)\cdot B_p^{-1/(p-1)}\Big(\frac{1}{|\Delta_{i(k)}|}\Big)^{p/(p-1)}\Big(\int_{\Delta_{i(k)}}\psi(x)\,dx\Big)^{1/(p-1)}\\
&\geqslant 1+B_p^{-1/(p-1)}\Big(\frac{1}{|\Delta_{i(k)}|}\Big)^{p/(p-1)}\Big(\int_{\Delta_{i(k)}\setminus\Delta_{i(k+1)}}\psi(x)\,dx\Big)^{1/(p-1)}\times\\
&\qquad\times\Big(\int_{\Delta_{i(k)}\setminus\Delta_{i(k+1)}}[\psi(x)]^{-1/(p-1)}\,dx\Big).
\end{aligned} \tag{42}$$

Since the inequality

$$1 \leqslant \frac{1}{|I|}\Big(\int_I \psi(x)\,dx\Big)^{1/p}\Big(\int_I [\psi(x)]^{-1/(p-1)}\,dx\Big)^{(p-1)/p}$$

is true for an arbitrary interval I, we obtain from (41) and (42):

$$(43)\qquad \Big(\int_{C\Delta_{i(k+1)}} [\psi(x)]^{-1/(p-1)}\,dx\Big)\Big(\int_{C\Delta_{i(k)}} [\psi(x)]^{-1/(p-1)}\,dx\Big)^{-1} \geqslant 1+B_p^{-1/(p-1)}(1/2)^{p/(p-1)} = S^{p/(p-1)}.$$

Using condition (41) once more, we hence get the estimate

$$\frac{1}{|\Delta_{i(j)}|}\Big(\int_{\Delta_{i(j)}} \psi(x)\,dx\Big)^{1/p}\Big(\int_{C\Delta_{i(l)}} [\psi(x)]^{-1/(p-1)}\,dx\Big)^{(p-1)/p}$$
$$\leqslant B_p^{1/p}\Big(\int_{C\Delta_{i(l)}} [\psi(x)]^{-1/(p-1)}\,dx\Big)^{(p-1)/p}\Big(\int_{C\Delta_{i(j)}} [\psi(x)]^{-1/(p-1)}\,dx\Big)^{-(p-1)/p}$$
$$\leqslant B(1/S)^{j-l},$$

holding for all j and l $(1\leqslant l\leqslant j)$, as asserted.

We now formulate one more lemma, which is just a specific case of a theorem of Hardy, Littlewood and Polya (see [16], p. 198).

LEMMA 6. *Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be numerical sequences, $\{u_n\}\in l^1$, $\{v_n\}\in l^p$, $1\leqslant p\leqslant\infty$, and let $\{w_n\}_{n=0}^{\infty}$ denote their Cauchy product: $w_n = \sum_{k=0}^{n} u_{n-k}v_k$. Then $\{w_n\}\in l^p$ and we have*

$$\|\{w_n\}\|_{l^p}\leqslant \|\{u_n\}\|_{l^1}\|\{v_n\}\|_{l^p}.$$

Proof of Theorem 7 for the case for $N=1$. Assume that the system $\{\chi_n(x)\}_{n=2}^{\infty}$ is a basis in $L^p(d\mu)$, which means that conditions (a)–(d) of Theorem 2 are satisfied. For any function $f\in L^p(d\mu)$ there is a unique series

$$(44)\qquad \sum_{n=2}^{\infty} a_n\chi_n(x) = f(x)\qquad (\text{convergence in } L^p(d\mu)).$$

By virtue of a theorem of W. Orlicz (see [8], p. 30) this series converges unconditionally in $L^p(d\mu)$ if and only if the series

$$\sum_{n=2}^{\infty} \varepsilon_n a_n\chi_n(x)$$

converges in $L^p(d\mu)$ for any sequence $\{\varepsilon_n\}_{n=2}^{\infty}$, $\varepsilon_n=\pm 1$.

Let $i(k)$ denote the index of the Haar function supported by the interval $\Delta_k^{(i_k)}$ occurring in condition (c) of Theorem 2 $(k=0,1,2,\ldots;$ $i(0)=2)$. Let us split the series (44) into the sum

$$(45)\qquad \sum_{n=2}^{\infty} a_n\chi_n(x) = \sum_{k=0}^{\infty} a_{i(k)}\chi_{i(k)}(x) + {\sum}' a_n\chi_n(x),$$

where Σ' denotes summation with ommision of the terms distinguished in the first sum.

Theorem 2 and Lemma 3 imply

$$(46)\qquad \sum_{k=0}^{\infty} a_{i(k)}\chi_{i(k)}(x) = \frac{1}{2|\Delta_{i(j)}|}\int_{\Delta_{i(j)}\setminus\Delta_{i(j+1)}} f(t)\,dt = c_j$$
$$\text{for}\quad x\in\Delta_{i(j)}\setminus\Delta_{i(j+1)}\qquad (j=0,1,2,\ldots).$$

Since $\{c_j\}\in l^p$ and since the measure μ satisfies condition (a) of Theorem 2, the convergence in $L^p(d\mu)$ of the series (44) forces the convergence in $L^p(d\mu)$ of the series (46).

According to Lemma 4 we have

$$\Big|\sum_{k=0}^{\infty}\varepsilon_k a_k\chi_k(x)\Big| \leqslant \sum_{l=0}^{j} 2^{j-l}|c_l|\qquad \text{for}\quad x\in\Delta_{i(j)}\setminus\Delta_{i(j+1)}\qquad (j=0,1,2,\ldots).$$

Writing

$$(47)\qquad \sum_{k=0}^{\infty}\varepsilon_k a_k\chi_{i(k)}(x) = G(x),$$

we obtain

$$\|G(x)\|_{L^p(d\mu)} \leqslant \Big(\sum_{j=0}^{\infty}\Big(\sum_{l=0}^{j} 2^{j-l}|c_l|\Big)^p \int_{\Delta_{i(j)}\setminus\Delta_{i(j+1)}} \psi(x)\,dx\Big)^{1/p}$$
$$\leqslant\Big(\sum_{j=0}^{\infty}\Big(2^j\sum_{l=0}^{j}\Big(\int_{\Delta_{i(l)}\setminus\Delta_{i(l+1)}} |f(x)|^p\psi(x)\,dx\Big)^{1/p}\Big(\int_{\Delta_{i(l)}\setminus\Delta_{i(l+1)}} [\psi(x)]^{-1/(p-1)}\,dx\Big)^{(p-1)/p}\times$$
$$\times\Big(\int_{\Delta_{i(j)}\setminus\Delta_{i(j+1)}}\psi(x)\,dx\Big)^{1/p}\Big)^p\Big)^{1/p}$$
$$\leqslant\Big(\sum_{j=0}^{\infty}\Big(\sum_{l=0}^{j-1}\Big(\int_{\Delta_{i(l)}\setminus\Delta_{i(l+1)}} |f(x)|^p\psi(x)\,dx\Big)^{1/p}\frac{1}{|\Delta_{i(j)}|}\Big(\int_{C\Delta_{i(l+1)}}[\psi(x)]^{-1/(p-1)}\,dx\Big)^{(p-1)/p}\times$$
$$\times\Big(\int_{\Delta_{i(j)}}\psi(x)\,dx\Big)^{1/p}\Big)^p\Big)^{1/p}+\Big(\sum_{j=0}^{\infty}B_p\int_{\Delta_{i(j)}\setminus\Delta_{i(j+1)}}|f(x)|^p\psi(x)\,dx\Big)^{1/p}.$$

Hence by Lemma 5 we get

$$\|G(x)\|_{L^p(d\mu)}\leqslant B_p^{1/p}\|f\|_{L^p(d\mu)}+$$
$$+B\Big(\sum_{j=0}^{\infty}\Big(\sum_{l=0}^{j}\Big(\int_{\Delta_{i(l)}\setminus\Delta_{i(l+1)}}|f(x)|^p\psi(x)\,dx\Big)^{1/p}(1/S)^{j-l-1}\Big)^p\Big)^{1/p},$$

where $S>1$.

Finally, applying Lemma 6, we conclude that there exists a constant $B_p' > 0$ such that

$$\Big\| \sum_{k=0}^{\infty} \varepsilon_k a_k \chi_{i(k)}(x) \Big\|_{L^p(d\mu)} \leqslant B_p' \|f\|_{L^p(d\mu)}. \tag{48}$$

Now, the function $G(x)$ belongs to $L^p(d\mu)$ and its development in the basis $\{\chi_n(x)\}_{n=2}^{\infty}$ reduces to terms with indices $i(k)$ $(k = 0, 1, 2, \ldots)$; other coefficients are equal to zero. The coefficients of expansion in terms of the system $\{\chi_{i(k)}(x)\}_{k=0}^{\infty}$ are uniquely determined by the values of the sum on the intervals of constancy. These observations serve as a motivation for the convergence in $L^p(d\mu)$ of the series (47).

Applying Theorem 4 to each of the intervals $\Delta_{i(j)} \setminus \Delta_{i(j+1)}$ and to the series $\sum' \varepsilon_n a_n \chi_n(x)$ we see that this series converges in $L^p(d\mu)$, independently of the choice of the ε_n's $(\varepsilon_n = \pm 1)$.

This ends the proof of Theorem 7 in the case of the system $\{\chi_n(x)\}_{n=2}^{\infty}$. As it has been already remarked, the proof in the general case is a simple consequence of the above.

6. On the Haar system with finitely many members removed. In dealing with such a system it is advisable to use notation and definitions introduced in the paper [9]. Let us recall them briefly.

The symbols Δ_n' and Δ_n'' denote, respectively, the left and the right half-interval of Δ_n, the support of the nth Haar function $\chi_n(x)$, $n \geqslant 2$. For $n = 1$ we set $\Delta_1' = \Delta_1'' = [0, 1]$. The field of sets generated by the intervals Δ_{k_i}', Δ_{k_i}'', $1 \leqslant i \leqslant N$, and the whole segment $[0, 1]$ is denoted by $S(\Delta_{k_1}', \Delta_{k_1}'', \ldots, \Delta_{k_N}', \Delta_{k_N}'')$. It is considered as a measure field, with the Lebesgue measure.

Definition. Let $\{L_i\}_{i=1}^N$ be a finite collection of atoms of the field $S(\Delta_{k_1}', \Delta_{k_1}'', \ldots, \Delta_{k_N}', \Delta_{k_N}'')$, $1 \leqslant k_1 < k_2 < \ldots < k_N$. We call $\{L^i\}_{i=1}^N$ an *admissible collection* iff $L_i \cap \Delta_{k_i} \neq \emptyset$, $1 \leqslant i \leqslant N$, and for any integer j $(1 \leqslant j \leqslant N)$ the field $S(\Delta_{k_1}', \Delta_{k_1}'', \ldots, \Delta_{k_j}', \Delta_{k_j}'')$ has exactly j distinct atoms containing (each of them) at least one of the sets L_i, $1 \leqslant i \leqslant j$.

It is easy to show by induction (see [11]) that for any N integers k_i, $1 \leqslant k_1 < k_2 < \ldots < k_N$, one can find an admissible collection of N atoms in the field $S(\Delta_{k_1}', \Delta_{k_1}'', \ldots, \Delta_{k_N}', \Delta_{k_N}'')$. Theorem 8 of the paper [11] and Lemmas 1 and 2 of the present article jointly result in the following:

Theorem 8. *Consider the Haar system $\{\chi_n(x)\}_{n=1}^{\infty}$ and the system $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ obtained from the Haar system by removing from it a finite collection of functions $\{\chi_{k_j}(x)\}_{j=1}^N$, $1 \leqslant k_1 < k_2 \ldots < k_N$, N denoting any positive integer. Let μ be a positive Borel measure on $[0, 1]$ and let $1 \leqslant p < \infty$. The following conditions are necessary and sufficient for the system $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ to be a basis in broad sense for the space $L^p(d\mu)$:*

(a_1) $d\mu(x) = \psi(x)\,dx$;

(b_1) $\psi(x) > 0$ a.e. *on* $[0, 1]$;

(c_1) *there exist an admissible collection of atoms $\{L_j\}_{j=1}^N$ in the field $S(\Delta_{k_1}', \Delta_{k_1}'', \ldots, \Delta_{k_N}', \Delta_{k_N}'')$ and sequences of Haar intervals $\{\Delta_k^{(j)}\}_{k=0}^{\infty}$, $1 \leqslant j \leqslant N$, with*

$$L_j \supset \Delta_0^{(j)} \supset \Delta_1^{(j)} \supset \ldots \supset \Delta_k^{(j)} \supset \ldots,$$

$$|\Delta_0^{(j)}| = \tfrac{1}{2}|\Delta_{k_N}|,$$

$$|\Delta_k^{(j)}| = 2\,|\Delta_{k+1}^{(j)}| \quad (1 \leqslant j \leqslant N;\ k = 0, 1, 2, \ldots),$$

such that

$$[\psi(x)]^{-1} \notin L_{L_j}^{1/(p-1)}, \quad [\psi(x)]^{-1} \in L_{L_j \setminus \Delta_k^{(j)}}^{1/(p-1)} \quad (1 \leqslant j \leqslant N;\ k = 0, 1, 2, \ldots)$$

and

$$[\psi(x)]^{-1} \in L_{C \bigcup_{j=1}^N L_j}^{1/(p-1)}.$$

From Theorems 2 and 8 we easily derive:

Theorem 9. *Let $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$, $\{\chi_{k_j}(x)\}_{j=1}^N$, $1 \leqslant k_1 < k_2 \ldots < k_N$, μ and p have the same meaning as in the preceding theorem. Then conditions* (a_1), (b_1), (c_1) *of Theorem 8 together with condition* (d_1) *formulated below are necessary and sufficient for the system $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ to be a basis in the space $L^p(d\mu)$:*

(d_1) *there exists a number $B_p > 0$ such that, for any j and k, $1 \leqslant j \leqslant N$, $k = 0, 1, 2, \ldots$, we have*

$$\Big(\frac{1}{|\Delta_k^{(j)}|} \int_{\Delta_k^{(j)}} \psi(x)\,dx\Big)\Big(\frac{1}{|\Delta_k^{(j)}|} \int_{\Delta_0^{(j)} \setminus \Delta_k^{(j)}} [\psi(x)]^{-1/(p-1)}\,dx\Big) \leqslant B_p$$

and such that for every Haar interval Δ with $|\Delta| \leqslant \frac{1}{2}|\Delta_{k_N}|$ which is not identical to any of the $\Delta_k^{(j)}$ $(1 \leqslant j \leqslant N;\ k = 0, 1, 2, \ldots)$ we have

$$\Big(\frac{1}{|\Delta|} \int_{\Delta} \psi(x)\,dx\Big)\Big(\frac{1}{|\Delta|} \int_{\Delta} [\psi(x)]^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant B_p.$$

Proof. *Necessity.* Conditions (a_1), (b_1), (c_1) are satisfied in view of Theorem 8. It remains to verify condition (d_1). To this end, consider the system $\{\Phi_{n_i}(x)\}_{i=1}^{\infty}$ conjugate to $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$. Denote by $\{\chi_{j_k}(x)\}_{k=1}^{\infty}$ the set of all Haar functions whose supports lie in $\Delta_0^{(j)}$, $1 \leqslant j \leqslant N$. Put

$$\varphi_{j_k}(x) = \begin{cases} \Phi_{j_k}(x) & \text{for } x \in \Delta_0^{(j)}, \\ 0 & \text{for } x \in C\Delta_0^{(j)} \end{cases}$$

$$(1 \leqslant j \leqslant N;\ k = 1, 2, \ldots).$$

Obviously, the systems $\{\chi_{j_k}(x)\}_{k=1}^{\infty}$ and $\{\varphi_{j_k}(x)\}_{k=1}^{\infty}$ are biorthogonal with respect to μ.

Fix j, $1 \leqslant j \leqslant N$, and apply a linear transposition of the interval $\Delta_0^{(j)}$ onto $(0,1)$. The function $\psi(x)$, restricted to $\Delta_0^{(j)}$, is carried into a function $\psi_j(x)$ defined on $(0,1)$; the systems $\{\chi_{j_k}(x)\}_{k=1}^{\infty}$ and $\{\varphi_{j_k}(x)\}_{k=1}^{\infty}$ transform into systems on $(0,1)$ biorthogonal with respect to the measure $d\mu_j(x) = \psi_j(x)dx$. Now, the system resulting by transposition from $\{\chi_{j_k}(x)\}_{k=1}^{\infty}$ coincides, up to a constant, with the system $\{\chi_n(x)\}_{n=2}^{\infty}$, and the latter is a basis in broad sense for $L^p(d\mu_j)$, on account of condition (c_1).

We have assumed that $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ is a basis for $L^p(d\mu)$. Hence, in view of the above considerations, we infer that the system $\{\chi_n(x)\}_{n=2}^{\infty}$ constitutes a basis in $L^p(d\mu_j)$. An appeal to Theorem 2 concludes the proof of necessity.

Sufficiency. Conditions (a_1), (b_1), (c_1) imply, in virtue of Theorem 8, that the system $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ is a basis in broad sense for the space $L^p(d\mu)$. Denote by $\{\Phi_{n_i}(x)\}_{i=1}^{\infty}$ the system conjugate to $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$. Using an argument similar to that applied in the proof of necessity (now in the opposite direction) and resorting to Theorem 2 we end the proof of Theorem 9.

In just the same way as we proved Theorem 9, from Theorems 7 and 9 we derive the following:

THEOREM 10. *Let* $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$, $\{\chi_{k_j}(x)\}_{j=1}^{N}$, $1 \leqslant k_1 < k_2 < \ldots < k_N$, *$\mu$ and p have the same meaning as in Theorems* 8 *and* 9, *and assume* $p > 1$. *Then each of the following statements implies the remaining two ones*:

(α_1) *The system* $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ *constitutes a basis in* $L^p(d\mu)$.

(α_2) *The system* $\{\chi_{n_i}(x)\}_{i=1}^{\infty}$ *constitutes an unconditional basis in* $L^p(d\mu)$.

(α_3) *Conditions* (a_1), (b_1), (c_1), (d_1) *are satisfied.*

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