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A negative result in differentiation theory

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Abstract. Given a differentiation basis in $\boldsymbol{R}^m$, we construct counterexamples to a.e. differentiability which remain counterexamples after any rotation of the basis, and we prove that they are typical elements (in the sense of Baire's category) of the Orlicz spaces to which they belong.

Notation. A *Busemann–Feller differentiation basis* $\mathscr{B}$ is a family of open subsets of $\boldsymbol{R}^m$ with every $x \in \boldsymbol{R}^m$ belonging to sets of arbitrarily small diameter which are in $\mathscr{B}$. $|A|$ will denote the Lebesgue measure, and we refer to

$$D\int f(x) = \lim\Big\{(1/|R|)\int_R f,\ x \in R \in \mathscr{B}\Big\}$$

and to

$$\bar{D}\int f(x) = \limsup\Big\{(1/|R|)\int_R f,\ x \in R \in \mathscr{B}\Big\}$$

as the derivative and the upper derivative of $\int f$ relative to $\mathscr{B}$ at the point x, the lim's being taken as the diameter of R decreases to 0. M will be the maximal operator associated with $\mathscr{B}$:

$$Mf(x) = \sup\Big\{(1/|R|)\int_R |f|,\ x \in R \in \mathscr{B}\Big\}.$$

C will be some positive constant, not always the same.

Introduction. It is a well known fact that if we substitute n-dimensional intervals for the balls or cubes in Lebesgue's differentiation theorem, it turns out to be false for some $f \in L^1$, and even in one sense (Baire's category) for almost every $f \in L^1$ at every point of $\boldsymbol{R}^m$ (Saks' rarity theorem, see [3]). To see the relevance of this fact, let us suppose for instance that we are dealing with the rectangular Cesáro means of a function $f \in L^1(\boldsymbol{R}^2)$ at the point (x_0, y_0):

$$f_{R,S}\ (x_0, y_0) = \int f(x_0 - x, y_0 - y) K_R(x) K_S(y)\,dx\,dy,$$

where $K_R(x) = (1/2\pi^2 R)\big((1-\cos(2\pi R x))/x^2\big)$ is the Fejér kernel.

We have $\lim_{R\to\infty} f_{R,R}(x_0, y_0) = f(x_0, y_0)$ for a.e. (x_0, y_0), but we cannot expect to have the same for the unrestricted $R, S\to\infty$ limit because $K_R(x) > CR\chi_{(-1/R,1/R)}$, so that $f_{R,S}$ controls the mean of f over the interval $(-1/R, 1/R)\times(-1/S, 1/S)$ and the existence of the latter limit would imply Lebesgue's differentiation theorem for the interval basis.

Saks' rarity theorem is only an example of the following general situation: let $\mathscr{B}$ be a Busemann–Feller differentiation basis in $\boldsymbol{R}^m$ that is homothecy invariant and let Φ be its halo function, i.e.,

$$\Phi(u) = \sup\{|\{M\chi_A > 1/u\}|,\ A\subset \boldsymbol{R}^m,\ |A| = 1\}.$$

Then in every Orlicz class $\Psi(L)$ with Ψ smaller than Φ at infinity, we can find some f which is not differentiated by the basis (see [3]), and this implies that the same is true for every $f\in\Psi(L)$ except those in a set of the first category, as Moriyón [6] has proved.

If $\mathscr{B}$ is again the interval basis in $\boldsymbol{R}^2$, we know from the theorem of Jessen, Marcinkiewicz and Zygmund [4] that it differentiates every $f\in L(\log^+ L)_{\rm loc}$, so that $\Phi(u)$ cannot be greater at infinity than $u(\log u)$, and direct computation shows that it is exactly of the same order, so that we can substitute anything like $L(\log^+ L)^{1/2}$ or $L(\log^+ L)/\log^+(\log^+ L)$ for L^1 in Saks' rarity theorem.

On the other hand, Zygmund had raised the following question, which may seem natural in view of the previous considerations: given $f\in L^1(\boldsymbol{R}^2)$, will it always be possible to find a pair of rectangular axes such that the interval basis in those axes will differentiate f?

Marstrand [5] answered this in the negative, exhibiting an $f\in L^1$ such that for every rotation of the axes the upper derivative of f relative to the rotated interval basis is $+\infty$ almost everywhere, and this was improved upon by El Helou [2], who found such an f in $\bigcap_{\alpha<1} L(\log^+ L)^\alpha$.

Our purpose here is to show:

(a) that the idea in Marstrand's construction works with any translation invariant Busemann–Feller differentiation basis, and

(b) that things are, here also, as they are in Saks' rarity theorem.

These two ideas were suggested to me by Antonio Córdoba and Miguel de Guzmán, whom I am glad to thank here.

Result. We suppose first that $\mathscr{B}$ is also homothecy invariant and define, for $0 < r < 1$,

$$H(r) = \bigcup\{R\in\mathscr{B},\ R\subset B_1,\ |R\cap B_r|/|R| > r\},\qquad \tilde{\Phi}(u) = u^m|H(1/u)|,$$

where B_r is the ball with radius r around 0. (For the definition in the general case, see Remark 1.)

THEOREM. *Let Ψ: $(1,\infty)\to\boldsymbol{R}$ be an increasing, convex function such that*

$$\lim_{u\to\infty}(\Psi(u)/\tilde{\Phi}(u)) = 0.$$

Then every $f\in\Psi(L)$ except those in a set of the first category in $\Psi(L)$ verifies that: for every rotation γ of $\boldsymbol{R}^m$ the upper derivative of $\int f$ relative to the basis $\mathscr{B}_\gamma$ obtained by rotating $\mathscr{B}$ through γ, is $+\infty$ almost everywhere.

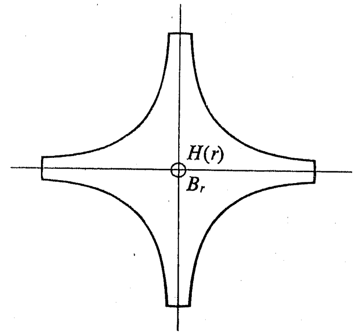

Fig. 1

We note that for the interval basis in $\boldsymbol{R}^2$ (see Fig. 1) an easy computation shows $\tilde{\Phi}(u)\simeq Cu(\log u)$, which gives the result of [5] in the stronger version of [2] if we take for instance $\Psi(u) = u(\log u)(\log\log u)^{-1}$.

A small-scale model of the proof. We shall now use some of the ideas of the proof below to give a demonstration of this known fact: there is an $f\in L^1(\boldsymbol{R}^2)$ which is not differentiated by the interval basis. In fact, we shall construct $f\in L^1$ such that $\bar{D}\int f(x) = \infty$ on a set of positive measure.

For each n, we divide the unit square into n^2 equal squares and consider the n^2 balls with radius $n^{-1}r_n$ concentric with them. We denote the union of these balls by C_n. Next we consider the images of $H(r_n)$ by the homothecies that apply the ball B_{r_n} onto each of the balls in C_n, and we call K_n their union.

Observe that $|C_n| = |B_{r_n}|$ and $|K_n| = |H(r_n)|\simeq r_n\log(1/r_n)$.

Now we take $f_n = \lambda_n\chi_{C_n}$ and $f = \sup_n f_n$ (the numbers $r_n, \lambda_n > 0$ will be determined later).

To get $f \in L^1(\mathbf{R}^2)$ it is enough to have

$$(1) \qquad \sum_n r_n^2 \lambda_n < \infty.$$

By the definition of $H(r_n)$ and the homothecy invariance of the interval basis, K_n is a union of intervals R with diameter less than n^{-1} and such that

$$\lambda_n r_n < (1/|R|) \int_R f_n \leqslant (1/|R|) \int_R f,$$

so that if $x \in \limsup_n K_n$ and

$$(2) \qquad \lim_n \lambda_n r_n = \infty,$$

x belongs to a contracting sequence of these intervals, and we have $\bar{D} \int f(x) = \infty$. So, let us look at the measure $|\limsup_n K_n|$. As it stands, it need not be greater than $\lim_n |K_n| \simeq \lim_n r_n \log(1/r_n)$, which must be 0 to fulfil (1), but if we "move" the K_n to prevent their intersecting too much, that measure is likely to increase. In fact, an easy application of the Borel–Cantelli theorem tells us that if we have

$$(3) \qquad \sum_n |K_n| = \infty,$$

then for some $x_n \in \mathbf{R}^2$ we have

$$|\limsup_n (x_n + K_n)| > 0.$$

This means that we shall get our f as the sup of the shifted $f_n(x - x_n)$ if we can find numbers $r_n, \lambda_n > 0$ satisfying (1), (2) and (3). We simply verify that

$$r_n = (n \log^2 n)^{-1}, \qquad \lambda_n = n \log^{5/2} n$$

will do the work.

Proof of the theorem. We construct f_n as in the previous section except for the fact that we divide the unit cube of $\mathbf{R}^m$ not into n^m but into M_n^m equal subcubes, where the integers $M_n \to \infty$ will be determined later. Consequently, the balls in C_n now have radius $M_n^{-1} r_n$, and we have as before $|C_n| = |B_{r_n}|$. Finally, we fix a rotation γ and consider the images of $\gamma(H(r_n))$ to form the set $K_n(\gamma)$, and we still have $|K_n(\gamma)| = |H(r_n)|$.

If we put $f = \sup_n f_n$ and the λ_n are increasing,

$$\int \Psi(f) = \sum_n \int_{C_n - \bigcup_{n<k} C_k} \Psi(\lambda_n) < C \sum_n r_n^m \Psi(\lambda_n),$$

so that if we want $f \in \Psi(L)$ we only need

$$(1) \qquad \sum_n r_n^m \Psi(\lambda_n) < \infty.$$

Next, by the same reasoning as before applied to $K_n(\gamma)$ and the basis $\mathscr{B}_\gamma$, we have $\bar{D}_\gamma \int f(x) = \infty$ for $x \in \limsup_n K_n(\gamma)$ provided we have

$$(2) \qquad \lim_n \lambda_n r_n = \infty.$$

So we again want $|\limsup_n K_n(\gamma)| > 0$. Note that the Borel–Cantelli theorem cannot help us now because the sets $K_n(\gamma)$ are different for each γ and they are very "thin" sets as n increases, so that the required shiftings x_n will be a crazy function of γ. So we abandon random shifting as the mixing method and use the following idea, which is the key to the whole construction: if the M_n increase quickly enough, the sets $K_n(\gamma)$ are probabilistically almost independent, i.e.,

$$|K_n \cap K_{n+1}| \simeq |K_n| \, |K_{n+1}|$$

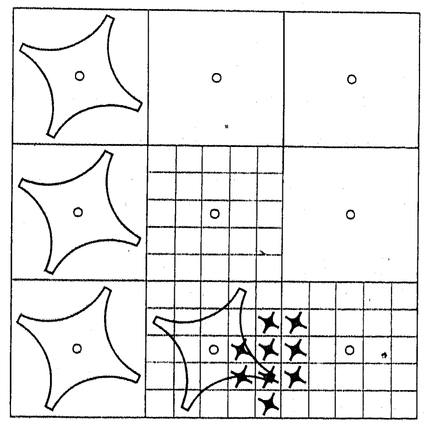

Fig. 2

(see Fig. 2) and the same is true for their complements, so that we obtain the following estimate (see Remark 2):

(E) $$1-\Big|\bigcup_{p\leqslant n\leqslant q} K_n(\gamma)\Big| < \prod_{p\leqslant n\leqslant q}\big(1-(1/2)|H(r_n)|\big).$$

But as

$$\limsup K_n(\gamma) = \bigcap_p \bigcup_{n\geqslant p} K_n(\gamma)$$

and

$$1-|\limsup K_n(\gamma)| \leqslant \sum_p \Big(1-\Big|\bigcup_{p\leqslant n} K_n(\gamma)\Big|\Big),$$

to get $|\limsup K_n(\gamma)| = 1$ we only need, according to (E),

$$\prod_n \big(1-(1/2)|H(r_n)|\big) = 0,$$

or

(3) $$\sum_n |H(r_n)| = \infty.$$

We have arrived at the same three conditions as in the previous section and we have reduced our problem to that of finding numbers r_n, λ_n satisfying (1), (2) and (3), and for this we apply to

$$h(r) = |H(r)| \quad\text{and}\quad g(r) = r^m\Psi(1/r)$$

the following lemma, whose proof is easy:

LEMMA. *Let $h, g\colon (0,1)\to \boldsymbol{R}$ such that*

$$\lim_{r\to 0} g(r) = \lim_{r\to 0}\big(g(r)/h(r)\big) = 0.$$

Then we can find two sequences $r_n\to 0$, $\lambda_n\to\infty$ such that

$$\sum h(r_n) = \infty, \quad \sum (\lambda_n r_n)^m g(1/\lambda_n) < \infty.$$

We have now obtained a function that misbehaves a.e. in the unit cube, and it is a trivial task to extend this to the whole $\boldsymbol{R}^m$ by adding a geometric series of multiples.

Only the "rarity" assertion in the theorem remains to be proved. For a given f and $b, s > 0$ consider the set

$$E(f,\gamma,b,s) = \{x\in\text{unit cube such that } (1/|R|)\int_R f < b \quad \text{if } x\in R\in\mathscr{B}_\gamma \text{ and } \operatorname{diam}(R) < s\}.$$

Given $\varepsilon > 0$, we are going to prove that for the f which we have constructed and a small enough $h\in\Psi(L)$ we have

$$|E(f+h,\gamma,b,s)| < \varepsilon.$$

We take n_0 such that $\forall\, n\geqslant n_0$ $\lambda_n r_n > 2b$ and $M_n^{-1} < s$.

CLAIM. *For more than half the points $y\in H(r)$ (in measure) there is some $\tilde{R}\in\mathscr{B}$ such that*

$$y\in\tilde{R}, \quad |\tilde{R}\cap B_r|/|\tilde{R}| > r$$

and

$$|\tilde{R}\cap B_r| \geqslant a = a(r) > 0$$

where a depends on $\mathscr{B}$ and r but not on y.

Let us then suppose that we took only that half of $H(r)$ as the whole $H(r)$ from the beginning (this does not modify our construction) and we take $h\in\Psi(L)$ such that $\|h\|_1 < \lambda_n a(r_n) M_n^{-m}/2$. Then, for $x\in K_n(\gamma)$ and the homothecy η such that $x\in\eta\circ\gamma\big(H(r_n)\big)$, let $y = (\eta\circ\gamma)^{-1}(x)$, $\tilde{R}$ as in the claim and $R = (\eta\circ\gamma)(\tilde{R})$. We have

$$(1/|R|)\int_R (f+h) \geqslant (1/|R|)\Big(\lambda_n|\eta\circ\gamma(\tilde{R}\cap B_{r_n})| - \int_R |h|\Big)$$

$$> (1/2)(1/|R|)\int_R f_n > \lambda_n r_n/2,$$

so that, if $n\geqslant n_0$, $(1/|R|)\int_R (f+h) > b$ and $\operatorname{diam}(R) < s$, i.e.,

$$x\notin E(f+h,\gamma,b,s).$$

We now take $n_1\geqslant n_0$ such that $\prod_{n_0\leqslant n\leqslant n_1}\big(1-(1/2)|H(r_n)|\big) < \varepsilon$, and this implies, as before:

$$1-\Big|\bigcup_{n_0\leqslant n\leqslant n_1} K_n(\gamma)\Big| < \varepsilon.$$

As none of the preceding constants depended upon γ, we have proved that if $\|h\|_1 < \min\limits_{n_0\leqslant n\leqslant n_1} \lambda_n a(r_n) M_n^{-m}/2$, then

$$\forall\gamma\ \ |E(f+h,\gamma,b,s)| < \varepsilon.$$

But as $\|h\|_1 \leqslant C\|h\|_{\Psi(L)}$ (by the Jensen inequality), we have proved that f is interior in the $\Psi(L)$ topology to each of the sets

$$F_{b,s,\varepsilon} = \{g\in\Psi(L),\ \forall\gamma\, |E(g,\gamma,b,s)| < \varepsilon\}.$$

Given $g\in L^\infty$, we can repeat the same argument beginning with $\lambda_n r_n > 2(b+\|g\|_\infty)$, which gives $(1/|R|)\int_R (f+h) > b+\|g\|_\infty$, i.e., $(1/|R|)\int_R (f+g+h) > b$, so that the interior of $F_{b,s,\varepsilon}$ contains $f+L^\infty$, which is dense in $\Psi(L)$.

It is now an easy exercise to prove that a function $f\in\Psi(L)$ is as bad as our theorem requires if and only if $f\in\bigcap_{b,s,\varepsilon} F_{b,s,\varepsilon} = \bigcap_n F_{n,1/n,1/n}$, which proves the theorem if we justify the claim. But this follows from

$$H(r) = \bigcup_n \bigcup \{R \in \mathscr{B},\ R \subset B_1,\ |R \cap B_r|/|R| > r \text{ and } |R| > 1/n\}.$$

A trivial corollary of the rarity assertion in the theorem is that the typical $f \in \Psi(L)$ is doubly as bad as the one we have constructed: it verifies $\underline{D}_\gamma \int f(x) = -\infty$ as well.

Examples and remarks.

1. If $\mathscr{B}$ is not homothecy invariant, we define

$$H(r, s) = \bigcup \{R \in \mathscr{B},\ R \subset B_s,\ |R \cap B_{rs}|/|R| > r\}, \qquad 0 < r,\ s < 1$$

and we build $K_n(\gamma)$ with translated, rotated copies of $H(r_n, M_n^{-1})$. If we now define $h(r) = \liminf_{s\to 0} |H(r, s)|s^{-m}$, we still have the estimate

$$\text{(E}'\text{)} \qquad 1 - \Big|\bigcup_{p\leqslant q\leqslant n} K_q(\gamma)\Big| < \prod_{p\leqslant q\leqslant n} \big(1-(1/2)h(r_q)\big)$$

(which reduces to (E) if $\mathscr{B}$ is homothecy invariant) and the theorem remains true with $\tilde{\Phi}(u) = u^m h(1/u)$.

2. The validity of our estimate (E), which we entrusted in the proof to the geometric evidence of Fig. 2, can be justified by the following lemma:

Lemma. *Suppose E_k are measurable subsets of the unit cube Q of $\boldsymbol{R}^m$ such that for every subcube Q' in the j-th dyadic subdivision of Q we have, if $k > j$,*

$$|E_k \cap Q'| = |E_k|\,|Q'|.$$

Let $\{A_\alpha\}$ be a family of measurable sets such that for $\varepsilon > 0$ there are a compact K_α and an open V_α verifying $K_\alpha \subset A_\alpha \subset V_\alpha$, $|V_\alpha - K_\alpha| < \varepsilon$ and $\operatorname{dist}(K_\alpha, V_\alpha^c) \geqslant d(\varepsilon) > 0$ *uniformly in α.*

Then we have

$$\lim_{k\to\infty} |A_\alpha \cap E_k| - |A_\alpha|\,|E_k| = 0$$

uniformly in α.

We apply the lemma to get (E′). Suppose we have chosen $M_1, \ldots$ $\ldots, M_{n-1}$. $\{A_\alpha\}$ will be the sets $\bigcap_{p\leqslant q\leqslant n-1} \big(Q - K_q(\gamma)\big)$ for all γ and all $p = 1, \ldots$ $\ldots, n-1$.

E_k will be the complement of the union of 2^{km} rotated, translated copies of $H(r_n, 2^{-k})$, i.e., the candidate to be $Q - K_n(\gamma)$ if we choose $M_n = 2^k$. We have for all p, γ:

$$\Big|\bigcap_{p\leqslant q\leqslant n} \big(Q - K_q(\gamma)\big)\Big| \Big/ \prod_{p\leqslant q\leqslant n-1} \big(1-(1/2)h(r_q)\big)$$
$$< \big(1 - 2^{km}|H(r_n, 2^{-k})|\big) + h(r_n)/4$$
$$< 1 - \big(h(r_n) - h(r_n)/4\big) + h(r_n)/4 = 1 - (1/2)h(r_n).$$

The first inequality is true for all $k >$ some j because of the lemma and the induction hypothesis; the second is true for some $k > j$ because of the definition of $h(r_n)$. Thus we may take $M_n = 2^k$ for this k.

3. We have computed $\tilde{\Phi}(u)$ for some bases that are only translation invariant; they are of the type "intervals with dimensions $s_1 \times \ldots \times s_r \times$ $\times \varphi_1(s_1, \ldots, s_r) \times \ldots \times \varphi_{m-r}(s_1, \ldots, s_r)$", the φ_j's being increasing functions in each s_i; in the case of $r = 2$, $m = 3$, Córdoba [1] has proved $L(\log^+ L)$ to be the border line, and the conjecture for the general case is $L(\log^+ L)^{r-1}$; for our theorem we also obtain $\tilde{\Phi}(u) \simeq Cu(\log u)^{r-1}$ in the computed cases.

For the interval basis in $\boldsymbol{R}^m$ the result is also sharp: $\tilde{\Phi}(u) \simeq Cu(\log u)^{m-1}$, which we know to be as much as is possible from the Jessen–Marcinkiewicz–Zygmund theorem.

4. For the "rectangles in lacunary directions" basis, the computation was already made by Stromberg [8] to get a "lower bound conjecture". In fact, if Φ is the usual halo function and $\Phi_1(L)$ is the largest Orlicz class that $\mathscr{B}$ differentiates, we obviously have at infinity $\tilde{\Phi} < \Phi < \Phi_1$.

For the k-generated set of exponential directions we have $\tilde{\Phi}(u) \simeq Cu(\log u)^{k+1}$ and so far nobody seems to know if $\Phi_1 \simeq \tilde{\Phi}$ (the best result is $\Phi_1(u) < u^p$ $\forall p > 1$, by Nagel, Stein and Wainger [7]).

5. We know that the first inequality in Remark 4 is sometimes a strict one: the "rectangles in all directions" basis gives $\tilde{\Phi}(u) = u^2$, $\Phi(u) = \infty$. It is an open question whether or when the second inequality can be strict (see [3]).

6. By making a wild induction from the preceding examples one could guess our result to be sharp if $\Phi(u) < u^p$ for all $p > 1$, but we can offer no reasons for such a belief but the desire to have an index so easily computable. As the last example of this easiness, we give the result for the "rectangles in a Cantor set of directions" basis: $\Phi(u) \simeq Cu^{1+\varepsilon}$ where $\varepsilon = \log 2/\log 3$ for the usual Cantor set.

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