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On multilinear singular integrals on R^n

by

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Abstract. Let C denote a multilinear singular integral on R^n of the form

$$Cf(x) = \text{p.v.} \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n+M}} \prod_{j=1}^{m} P_{m_j}(b^j; x, y) f(y)\, dy$$

where $P_m(b; x, y)$ denotes the mth Taylor series remainder of b at x expanded about y and $M = \sum_{j=1}^{m} m_j$. The main result of this paper is the inequality

$$\|C_* f\|_q < A \prod_{j=1}^{m} \Big(\sum_{|\alpha|=m_j} \|b^j_\alpha\|_{r_j} \Big) \|f\|_p$$

where $1 > \frac{1}{q} = \frac{1}{p} + \sum_{j=1}^{m} \frac{1}{r_j}$, $1 < p < \infty$, Ω satisfies certain symmetry and integrability conditions, and C^* denotes the corresponding maximal operator. Our proof is based upon the method of rotations of Calderón and Zygmund.

Introduction. In this paper we study singular integrals on R^n of the form

$$Cf(x) = \text{p.v.} \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n+M}} \prod_{j=1}^{m} P_{m_j}(b^j; x, y) f(y)\, dy$$

where $\Omega(\cdot)$ is homogeneous of degree zero and integrable over Σ_{n-1}, the unit sphere in R^n, $P_m(b; x, y)$ denotes the mth Taylor series remainder of b at x expanded about y, and $\sum_{j=1}^{m} m_j = M$. The one-dimensional version of our result with $\Omega(\cdot) = \operatorname{sgn}(\cdot)$ was established in [5]. We extend the one-dimensional result to R^n by the "method of rotations" introduced by Calderón and Zygmund in [4]. Special cases of our result in R^1 or R^n include results in [1], [2], [3], and [6]. The special case of our result with the product of two first order remainders was proved by Yee [7].

This paper is divided into seven sections. In the first section we state the main result. In Sections 2 through 5 we deal with the "odd" case of our result. The results in these sections are fairly straightforward extensions of one-dimensional results via the method of rotations. The last two sections deal with the "even" case. The methods needed in these sections are different and more technical than those used in any of the earlier known cases.

1. In this section we state the main result of this paper. We shall refer to result (i) as the "odd" case and to result (ii) as the "even" case.

THEOREM. *Let $\Omega(x)$ be homogeneous of degree zero and integrable on Σ_{n-1} the unit sphere in $\boldsymbol{R}^n$. For $j=1,2,\ldots,m$, let $b^j(x)$ have derivatives of order m_j in L^{r_j}, $1<r_j\leqslant\infty$. Let $1<p<\infty$, $1<q<\infty$, $\frac{1}{q}=\frac{1}{p}+\sum_{j=1}^{m}\frac{1}{r_j}$, and let*

$$C_\varepsilon f(x)=\int\limits_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{j=1}^{m}P_{m_j}(b^j;x,y)f(y)\,dy,$$

$$C_*f(x)=\sup_{\varepsilon>0}|C_\varepsilon f(x)|,$$

where $M=\sum_{j=1}^{m}m_j$. Then

(i) *if $\Omega(-x)=(-1)^{M+1}\Omega(x)$, then the operator C_* is of strong type (p,q) and*

$$\|C_*f\|_q\leqslant C\prod_{j=1}^{m}\Big(\sum_{|\alpha|=m_j}\|b_\alpha^j\|_{r_j}\Big)\|f\|_p\int\limits_{\Sigma}|\Omega(x)|\,d\sigma$$

where $d\sigma$ denotes surface measure on Σ_{n-1} and C depends only on $p, r_1, \ldots, r_m$, M, and n,

(ii) *if $\Omega(-x)=(-1)^M\Omega(x)$, Ω is in $L\log^+L(\Sigma)$ and $\int_\Sigma x^\alpha\Omega(x)\,d\sigma=0$ for all α such that $|\alpha|=M$, then the same conclusions hold as in* (i) *except that $\int_\Sigma|\Omega(x)|\,d\sigma$ has to be replaced by $\int_\Sigma|\Omega(x)|\big(1+\log^+|\Omega(x)|\big)\,d\sigma$.*

The proof of this theorem will be similar to that in [1]. We shall be brief in those areas of the proof which carry over with minimal change. We refer the reader to [1], [4] for additional details. We note that the boundedness of $C_*f(x)$ in the one-dimensional case was established in [5]. Finally, we note that C will denote a constant which may vary from line to line, but which depends only on the parameters mentioned in the theorem.

2. In this section we establish some lemmas which will be needed later. The first lemma is analogous to 3.1 in [1].

LEMMA 1. *Let $N(x)$ be homogeneous of degree 0, and let $\Phi(t)$, $t\geqslant 0$, be such that $t^s\Phi(t)$ is decreasing for some $s\leqslant 0$ and $\int_0^\infty t^{M+n-1}\Phi(t)\,dt<\infty$. Let b^j have m_j-th derivatives in L^{r_j}, $1<r_j<\infty$, and let $1<p\leqslant\infty$, $1<q<\infty$ and $1/q=1/p+\sum_{j=1}^{m}r_j$. Let*

$$D_\varepsilon f(x)=\varepsilon^{-M-n}\int N(x-y)\,\Phi\Big(\frac{|x-y|}{\varepsilon}\Big)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)f(y)\,dy,$$

$$D_*f(x)=\sup_{\varepsilon>0}\Big\{\varepsilon^{-M-n}\int\Big|N(x-y)\,\Phi\Big(\frac{|x-y|}{\varepsilon}\Big)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)f(y)\Big|dy\Big\}.$$

Then

$$\|D_*f\|_q\leqslant C\prod_{j=1}^{m}\Big(\sum_{|\alpha|=m_j}\|b_\alpha\|_{r_j}\Big)\|f\|_p\int\limits_{\Sigma}|N(x)|\,d\sigma.$$

Proof. By standard arguments, it suffices to prove the estimate for D_*f in the one-dimensional case with $f, b^1, \ldots, b^m$ in C_0^∞. Suppose that $\int_0^\infty t^M\Phi(t)\,dt<\infty$ and $t^s\Phi(t)$ is decreasing. Since

$$0\leqslant x^{M-s}(2x)^s\Phi(2x)x\leqslant\int_x^{2x}t^M\Phi(t)\,dt\to 0$$

as $x\to\infty$, it follows that $x^{M+1}\Phi(x)\to 0$ as $x\to\infty$. Similarly, $x^{M+1}\Phi(x)\to 0$ as $x\to 0$. Let $F_x(y)=\int_0^y t^{-s}\prod_{j=1}^{m}|P_{m_j}(b^j;x,t+x)|\,|f(t+x)|\,dt$. Now $|P_{m_j}(b^j;x,t+x)|\leqslant Ct^{m_j}\Lambda(b^j_{m_j})(x)$ where Λ is the Hardy–Littlewood maximal function. We have

$$F_x(y)\leqslant C\prod_{j=1}^{m}\Lambda(b^j_{m_j})(x)\int_0^y t^{M-s}|f(t+x)|\,dt$$
$$\leqslant C|y|^{M-s+1}\Lambda(f)(x)\prod_{j=1}^{m}\Lambda(b^j_{m_j})(x).$$

Integrating by parts and using the above estimate on the boundary terms yields

$$\varepsilon^{-M-1}\int_0^\infty\Big|\Phi\Big(\frac{y}{\varepsilon}\Big)\prod_{j=1}^{m}P_{m_j}(b^j;y+x)f(y+x)\Big|dy$$
$$=\varepsilon^{-M+s-1}\int_0^\infty\Big(\frac{y}{\varepsilon}\Big)^s\Phi\Big(\frac{y}{\varepsilon}\Big)y^{-s}\prod_{j=1}^{m}|P_{m_j}(b^j;x,y+x)|\,|f(y+x)|\,dy$$

$$= -\varepsilon^{-M+s-1}\int_0^\infty F_x(y)\,d\left\{\left(\frac{y}{\varepsilon}\right)^s\Phi\left(\frac{y}{\varepsilon}\right)\right\}$$

$$\leqslant CA(f)(x)\prod_{j=1}^m A(b^j_{m_j})(x)\left(-\int_0^\infty\left(\frac{y}{\varepsilon}\right)^{M-s+1}d\left\{\left(\frac{y}{\varepsilon}\right)^s\Phi\left(\frac{y}{\varepsilon}\right)\right\}\right)$$

$$\leqslant CA(f)(x)\prod_{j=1}^m A(b^j_{m_j})(x)\int_0^\infty y^M\Phi(y)\,dy.$$

The one-dimensional case now follows by an application of Hölder's inequality and the boundedness of the Hardy–Littlewood maximal operator on L^p and L^{r_j}, $j = 1, \ldots, n$. The n-dimensional result follows by a standard rotational argument (see [4]). We note that the restriction $q \geqslant 1$ is not needed in proving the one-dimensional result, but is required to obtain the n-dimensional result by the rotational method.

The second lemma is a multilinear version of Lebesgue's theorem in n variables.

LEMMA 2. *Let $N(x)$ be homogeneous of degree zero with $\int_{\Sigma_{n-1}}|N(t')|\,dt' < \infty$ where dt' represents surface area on Σ_{n-1}. Assume for $j = 1, 2, \ldots, m$, $f_j \in L^{r_j}(\boldsymbol{R}^n)$ with $\sum_{j=1}^m \frac{1}{r_j} = \frac{1}{q} < 1$. For $\varepsilon > 0$, let*

$$T_\varepsilon(f_1, \ldots, f_m)(x) = \int |N(t')|\prod_{j=1}^m\left(\frac{1}{\varepsilon}\int_0^\varepsilon |f_j(x+rt')-f_j(x)|\,dr\right)dt'.$$

Then $\lim_{\varepsilon\to 0} T_\varepsilon(f_1, \ldots, f_m)(x) = 0$ a.e.

Proof. Let $M(f_1, \ldots, f_m)(x) = \int |N(t')|\prod_{j=1}^m f_j^*(x, t')\,dt'$ where $f_j^*(x, t') = \sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_0^\varepsilon |f_j(x+rt')|\,dr\right)$. By Hölder's inequality and the Hardy–Littlewood theorem it follows that

$$\int_{-\infty}^\infty\Big(\prod_{j=1}^m f_j^*(x+rt', t')\Big)^q dr \leqslant C\prod_{j=1}^m\Big(\int_{-\infty}^\infty |f_j(x+rt')|^{r_j}dr\Big)^{q/r_j}.$$

Integrating over the space of lines parallel to t' and using Minkowski's integral inequality, we have

$$(1)\qquad \Big(\iint_{\boldsymbol{R}^n} M(f_1, \ldots, f_m)(x)^q dx\Big)^{1/q} = \Big(\iint_{\boldsymbol{R}^n}\Big(\int |N(t')|\prod_{j=1}^m f_j^*(x, t')\,dt'\Big)^q dx\Big)^{1/q}$$

$$\leqslant \int |N(t')|\Big(\iint_{\boldsymbol{R}^n}\Big(\prod_{j=1}^m f_j^*(x, t')\,dx\Big)^q\Big)^{1/q}dt'$$

$$\leqslant C\Big(\int |N(t')|\,dt'\Big)\prod_{j=1}^m\Big(\iint_{\boldsymbol{R}^n}|f_j(x)|^{r_j}dx\Big)^{1/r_j}.$$

Let $R_\varepsilon(f_1, \ldots, f_m)(x) = \int N(t')\prod_{j=1}^m\left(\frac{1}{\varepsilon}\int_0^\varepsilon f_j(x+rt')-f_j(x)\,dr\right)dt'$. We now show that $R_\varepsilon(f_1, \ldots, f_m)(x) \to 0$ a.e. as $\varepsilon \to 0$. It is clear that $R_\varepsilon(f_1, \ldots, f_m)(x) \to 0$ everywhere if $f_1, \ldots, f_m \in C_0^\infty$. Let $\delta_\nu \downarrow 0$. We keep $f_2, \ldots, f_m$ in C_0^∞ and let $f_1 \in L^{r_1}(\boldsymbol{R}^n)$. We write $f_1 = g_1^\nu + h_1^\nu$ where $g_1^\nu \in C_0^\infty$ and $\|h_1^\nu\|_{L^{r_1}} \leqslant \delta_\nu^2$. Then $R_\varepsilon(f_1, f_2, \ldots, f_m) = R_\varepsilon(g_1^\nu, f_2, \ldots, f_m) + R_\varepsilon(h_1^\nu, f_2, \ldots, f_m)$. Now $R_\varepsilon(g_1^\nu, f_2, \ldots, f_m)(x) \to 0$ everywhere as $\varepsilon \to 0$ since all the functions are in C_0^∞. Also $|R_\varepsilon(h_1^\nu, f_2, \ldots, f_m)(x)| \leqslant 2^m M(h_1^\nu, f_2, \ldots, f_m)(x)$ a.e. Thus by (1), we have

$$\Big|\{x: \overline{\lim_{\varepsilon\to 0}}|R_\varepsilon(f_1, \ldots, f_m)(x)| > \delta_\nu\}\Big| = |\{x: \overline{\lim_{\varepsilon\to 0}}|R_\varepsilon(h_1^\nu, f_2, \ldots, f_m)(x)| > \delta_\nu\}|$$

$$\leqslant |\{x: M(h_1^\nu, f_2, \ldots, f_m)(x) > 2^{-m}\delta\}|$$

$$\leqslant 2^{-mq}\delta_\nu^{-q}\|M(h_1^\nu, f_2, \ldots, f_m)\|_q^q$$

$$\leqslant 2^{-mq}C\delta_\nu^{-q}\Big(\int |N(t')|\,dt'\Big)^q\Big(\prod_{j=2}^m\|f_j\|_{r_j}^q\Big)\|h_1^\nu\|_{r_1}^q$$

$$\leqslant 2^{-mq}C\delta_\nu^q\Big(\int |N(t')|\,dt'\Big)^q\Big(\prod_{j=2}^m\|f_j\|_{r_j}^q\Big).$$

It follows that $R_\varepsilon(f_1, \ldots, f_m)(x) \to 0$ a.e. The result for $f_j \in L^{r_j}(\boldsymbol{R}^n)$, $j = 1, \ldots, m_1$, follows by a simple induction argument on the number of functions not assumed to be in C_0^∞. We note that $R_\varepsilon(f_1, \ldots, f_n) \to 0$ a.e. if $\prod_{j=1}^m f_j$ is locally in L^q for some $q > 1$. Now let $(\varrho_1, \ldots, \varrho_m)$ be an m-tuple of rational numbers. Let J be a subset of $\{1, \ldots, m\}$, and let $|J| = \mathrm{card}(J)$. Now $\prod_{j\in J}|f_j - \varrho_j|$ is locally in L^q. Thus if $J = \{j_1, \ldots, j_s\}$, $R_\varepsilon(|f_{j_1}-\varrho_{j_1}|, \ldots, |f_{j_s}-\varrho_{j_s}|)(x) \to 0$ a.e. (x) as $\varepsilon \to 0$. Let $E(J, \varrho_{j_1}, \ldots, \varrho_{j_s})$ denote the set of measure zero where this fails. Let $E = \bigcup_J \bigcup E(J, \varrho_{j_1}, \ldots, \varrho_{j_s})$ where the second union is taken over all s-tuples of rational numbers, s being the number of elements in J. Then $|E| = 0$. We now show that $T_\varepsilon(f_1, \ldots, f_m)(x) \to 0$ as $\varepsilon \to 0$ if $x \in \boldsymbol{R}^n\setminus E$. For $x \in \boldsymbol{R}^n\setminus E$, we have

$$T_\varepsilon(f_1, \ldots, f_m)(x) = \int |N(t')|\prod_{j=1}^m\left(\frac{1}{\varepsilon}\int_0^\varepsilon |f_j(x+rt')-f_j(x)|\,dr\right)dt'$$

$$\leqslant \int |N(t')| \prod_{j=1}^{m} \left(\frac{1}{\varepsilon} \int_0^{\varepsilon} |f_j(x+rt') - \varrho_j| - |f_j(x) - \varrho_j| + 2|f_j(x) - \varrho_j|\, dr \right) dt'$$

$$= \sum_{k=1}^{m} \sum_{J:|J|=k} 2^{|J'|} \prod_{k\in J'} |f_k(x) - \varrho_k| \int |N(t')| \prod_{j\in J} \left(\frac{1}{\varepsilon} \int_0^{\varepsilon} |f_j(x+rt') - \varrho_j| - \right.$$

$$\left. - |f_j(x) - \varrho_j|\, dr \right) dt' + 2^m \prod_{j=1}^{m} |f_j(x) - \varrho_j| \int |N(t')|\, dt'.$$

Each integral in the first term is $R_\varepsilon(|f_{j_1} - \varrho_{j_1}|, \ldots, |f_{j_s} - \varrho_{j_s}|)(x)$ for a subset J of $\{1, 2, \ldots, n\}$. Since $x \notin E$, it follows that each term in the sum goes to 0 as $\varepsilon \to 0$ for any $(\varrho_1, \ldots, \varrho_n)$. Choosing $(\varrho_1, \ldots, \varrho_n)$ so that $\prod_{j=1}^{m} |f_j(x) - \varrho_j|$ is small, it follows that $T_\varepsilon(f_1, \ldots, f_m)(x) \to 0$ as $\varepsilon \to 0$. This completes the proof of the lemma.

3. In this section we show that the integrals

$$\int_{|x-y|>\varepsilon} \frac{|\Omega(x-y)|}{|x-y|^{n+M}} \prod_{j=1}^{m} |P_{m_j}(b^j; x, y)|\, |f(y)|\, dy$$

are finite almost everywhere. Proceeding as in [1], let S denote a solid sphere in $\boldsymbol{R}^n$. Then

$$\int_S \int_{|x-y|>\varepsilon} \frac{|\Omega(x-y)|}{|x-y|^{M+n}} \prod_{j=1}^{m} |P_{m_j}(b^j; x, y)|\, |f(y)|\, dy\, dx$$

$$= \int_{|y'|=1} |\Omega(y')| \int_S \int_{\varepsilon}^{\infty} r^{-M-1} \prod_{j=1}^{m} |P_{m_j}(b^j; x-ry') f(x-ry')|\, dr\, dx.$$

Now S is contained in $\{x \mid x = z+sy',\ z \perp y',\ |z| < B,\ |s| < B\}$, and hence the inner integral is dominated by

(3.1)
$$\int_{|z|<B} \int_{|s|<B} \int_{\varepsilon}^{\infty} r^{-M-1} \prod_{j=1}^{m} \left|P_{m_j}(b^j; z+sy', z+(s-r)y') f\big(z+(s-r)y'\big)\right| dr\, ds\, dz.$$

For $j = 1, 2, \ldots, m$, let $B^j(s) = b^j(z+sy')$ and observe that for each j, $\left|P_{m_j}(b^j; z+sy', z+(s-r)y')\right| r^{-m_j} \leqslant \Lambda(B^j_{m_j})(s)$ where Λ denotes the one-dimensional Hardy–Littlewood maximal function along the line $\{z+sy';\ s \in \boldsymbol{R}\}$. Thus the integral in (3.1) is dominated by

$$\int_{|z|<B} \int_{|s|<B} \prod_{j=1}^{m} \Lambda(B^j_{m_j})(s) \int_s^{\infty} r^{-1} \left|f\big(z+(s-r)y'\big)\right| dr\, ds\, dz$$

$$\leqslant C_\varepsilon \int_{|z|<B} \int_{|s|<B} \prod_{j=1}^{m} \Lambda(B^j_{m_j})(s)\, ds\, dz \Big(\int_{-\infty}^{\infty} |f(z+\tau y')|^p d\tau \Big)^{1/p}$$

$$\leqslant C_{\varepsilon,B} \int_{|z|<B} \prod_{j=1}^{m} \Big[\int_{-\infty}^{\infty} \Big(\sum_{|\alpha|=m_j} b^j_\alpha(z+ry') \Big)^{r_j} dr \Big]^{1/r_j} \Big(\int_{-\infty}^{\infty} |f(z+\tau y')|^p d\tau \Big)^{1/p} dz.$$

Integrating with respect to z and using Hölder's inequality with exponents $p, r_1, \ldots, r_m, \eta$, where $\dfrac{1}{p} + \sum_{j=1}^{m} \dfrac{1}{r_j} + \dfrac{1}{\eta} = 1$, shows that the integral in (3.1) is bounded by a constant independent of y'. The result stated at the beginning of this section now follows.

4. In this section we establish the existence almost everywhere of limits of the truncated integrals $C_\varepsilon f$ as $\varepsilon \to 0$ under the assumption that Ω and f are smooth and f has compact support. By a standard approximation argument it suffices to prove the theorem with this additional assumption. Let K denote the support of f, let S denote a solid sphere in $\boldsymbol{R}^n$, and let $d = \sup |x-y|$ where $x \in S$, $y \in K$. To prove $C_\varepsilon f$ has limits almost everywhere in S, we use induction on m, the number of factors in the product defining $C_\varepsilon f$. The case $m = 1$ is simply the result in [1]. Assume that $C_\varepsilon g$ has limits almost everywhere in S whenever the product defining $C_\varepsilon g$ contains at most $m-1$ factors and $g \in C_0^\infty$ with supp $g \subset S$. We now consider $C_\varepsilon f$ where the product defining $C_\varepsilon f$ contains m factors. Then for $x \in S$, we have

(4.1)
$$C_\varepsilon f(x) = \int_{d>|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{M+n}} \prod_{j=1}^{m} P_{m_j}(b^j; x, y)[f(y) - f(x)]\, dy +$$
$$+ f(x) \int_{d>|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{M+n}} \prod_{j=1}^{m} P_{m_j}(b^j; x, y)\, dy.$$

Since $|f(y) - f(x)| \leqslant C_f |x-y|$, Lemma 1 with $\Phi(t) = t^{-M-n+1}\chi_{[0,1]}(t)$ shows that

$$\int_{d>|x-y|} \frac{|\Omega(x-y)|}{|x-y|^{M+n-1}} \prod_{j=1}^{m} |P_{m_j}(b^j; x, y)|\, dy$$

is finite almost everywhere and therefore the first integral in (4.1) converges a.e. in S. To prove that the second integral in (4.1) converges almost everywhere in S, we apply Green's theorem. Let $h(x) = \Omega(x)/|x|^{M+n}$. By Euler's formula on homogeneous functions, we have $\sum_{|\alpha|=1} (x^\alpha h(x))_\alpha = -Mh(x)$. We also have for $|\alpha| = 1$,

$$\frac{\partial}{\partial y^\alpha} P_{m_j}(b^j; x, y) = -\sum_{|\gamma|=m_j-1} \frac{b^j_{\gamma+\alpha}(y)}{\gamma!} (x-y)^\gamma. \tag{4.2}$$

By Green's theorem the second integral in (4.1) is equal to a term independent of ε plus

$$-\frac{1}{M} \sum_{|\alpha|=1} \int_{d>|x-y|>\varepsilon} (x-y)^\alpha h(x-y) \frac{\partial}{\partial y^\alpha} \Big\{\prod_{j=1}^m P_{m_j}\{(b^j; x, y)\Big\} dy, \tag{4.3}$$

$$+\int_{|x-y|=\varepsilon} |x-y| h(x-y) \prod_{j=1}^m P_{m_j}(b^j; x, y)\, dy. \tag{4.4}$$

We now show that (4.3) and (4.4) have limits almost everywhere in S as $\varepsilon \to 0$. Using (4.2) we may rewrite (4.3) as

(4.5)

$$\frac{1}{M} \sum_{|\alpha|=1} \prod_{j=1}^m \sum_{|\gamma|=m_j-1} \frac{1}{\gamma!} \int_{d>|x-y|>\varepsilon} (x-y)^{\gamma+\alpha} h(x-y) \prod_{\substack{\nu=1\\ \nu\neq j}}^m P_{m_\nu}(b^\nu; x, y)\, b^j_{\gamma+\alpha}(y)\, dy.$$

Observing that $x^{\gamma+\alpha}h(x)$ is homogeneous of degree $M-m_j$, we obtain by the induction hypothesis that each of the integrals in (4.5) has limits almost everywhere in S. It remains only to prove that the integral in (4.4) has limits almost everywhere in S. The following notation will be convenient. For $1 \leqslant k \leqslant n-1$, let $\mathscr{J}_k$ denote the family of subsets of $\{1, 2, \ldots, n\}$ having k distinct elements. For $J \in \mathscr{J}_k$ let J' denote $\{1, 2, \ldots, n\} \setminus J$. Then if $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ are two collections of n numbers, we have

$$\prod_{j=1}^n (a_j + b_j) = \prod_{j=1}^n a_j + \prod_{j=1}^n b_j + \sum_{k=1}^{n-1} \sum_{J\in\mathscr{J}_k} \prod_{\substack{j\in J\\ k\in J'}} a_j b_k. \tag{4.6}$$

Using the integral form of the remainder, we have

$$\begin{aligned} P_{m_j}(b^j; x, x-u) &= \sum_{|\alpha|=m_j} \frac{m_j}{\alpha!} u'^\alpha \int_0^{|u|} r^{m_j-1} b^j_\alpha(x-ru')\, dr - \\ &\quad - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} b^j_\alpha(x) u^\alpha + \sum_{|\alpha|=m_j} \frac{1}{\alpha!} b^j_\alpha(x) u^\alpha \\ &= \sum_{|\alpha|=m_j} \frac{m_j}{\alpha!} u'^\alpha \int_0^{|u|} r^{m_j-1} [b^j_\alpha(x-ru') - b^j_\alpha(x)]\, dr + \sum_{|\alpha|=m_j} \frac{1}{\alpha!} b^j_\alpha(x) u^\alpha. \end{aligned} \tag{4.7}$$

Using (4.6) and (4.7) we obtain

$$\begin{aligned} &\int_{|x-y|=\varepsilon} |x-y| h(x-y) \prod_{j=1}^m P_{m_j}(b^j; x, y)\, dy \\ &= \varepsilon^{n-1} \int_{|t'|=1} \varepsilon h(\varepsilon t') \prod_{j=1}^m P_{m_j}(b^j; x, x-\varepsilon t')\, dt' \\ &= \varepsilon^{-M} \int_{|t'|=1} h(t') \prod_{j=1}^m \Big[\sum_{|\alpha|=m_j} \frac{m_j}{\alpha!} t'^\alpha \int_0^\varepsilon r^{m_j-1} [b^j_\alpha(x-rt') - b^j_\alpha(x)]\, dr\Big] dt' + \\ &+ \varepsilon^{-M} \int_{|t'|=1} h(t') \prod_{j=1}^m \Big[\sum_{|\alpha|=m_j} \frac{1}{\alpha!} b^j_\alpha(x) (\varepsilon t')^\alpha\Big] dt' + \\ &+ \varepsilon^{-M} \int_{|t'|=1} h(t') \sum_{\nu=1}^{m-1} \sum_{J\in\mathscr{J}_\nu} \prod_{\substack{j\in J\\ k\in J'}} \Big(\sum_{|\alpha|=m_j} \frac{m_j}{\alpha!} t'^\alpha \int_0^\varepsilon r^{m_j-1} [b^j_\alpha(x-rt') - b^j_\alpha(x)]\, dr\Big) \times \\ &\qquad \times \Big(\sum_{|\alpha|=m_k} \frac{1}{\alpha!} b^k_\alpha(x) (\varepsilon t')^\alpha\Big) dt' = I^1_\varepsilon(x) + I^2_\varepsilon(x) + I^3_\varepsilon(x). \end{aligned}$$

Now

$$\begin{aligned} |I^1_\varepsilon(x)| &\leqslant \int_{|t'|=1} |h(t')| \prod_{j=1}^m \Big[\sum_{|\alpha|=m_j} \frac{m_j}{\alpha!} \Big(\frac{1}{\varepsilon} \int_0^\varepsilon |b^j_\alpha(x-rt') - b^j_\alpha(x)|\, dr\Big)\Big] \\ &= \sum_{\substack{\alpha_1,\ldots,\alpha_m\\ |\alpha_j|=m_j}} \int_{|t'|=1} |h(t')| \prod_{j=1}^m \Big(\frac{m_j}{\alpha_j!} \Big(\frac{1}{\varepsilon} \int_0^\varepsilon |b^j_{\alpha_j}(x-rt') - b^j_{\alpha_j}(x)|\, dr\Big)\Big) dt'. \end{aligned}$$

Applying Lemma 2 to each term in the above sum, we conclude that $I^1_\varepsilon(x) \to 0$ a.e. (x) as $\varepsilon \to 0$. For $I^3_\varepsilon(x)$, we have

$$\begin{aligned} I^3_\varepsilon(x) &= \varepsilon^{-M} \sum_{\nu=1}^{m-1} \sum_{J\in\mathscr{J}_\nu} \int_{|t'|=1} h(t') \prod_{\substack{j\in J\\ k\in J'}} \Big(\sum_{|\alpha|=m} \frac{m_j}{\alpha!} t'^\alpha \int_0^\varepsilon r^{m_j-1} [b^j_\alpha(x-rt') - \\ &\qquad - b^j_\alpha(x)]\, dr\Big) \Big(\sum_{|\alpha|=m_k} \frac{1}{\alpha!} b^k_\alpha(x) (\varepsilon t')^\alpha\Big) dt' \\ &= \varepsilon^{-M} \sum_{\nu=1}^{m-1} \sum_{J\in\mathscr{J}_\nu} \sum_{\substack{\alpha_1,\ldots,\alpha_{|J|}\\ \beta_1,\ldots,\beta_{|J'|}}} \int_{|t'|=1} h(t') = \prod_{j\in J} \prod_{k\in J'} \Big(\frac{m_j}{\alpha_j!\beta_k!} t'^{\alpha_j+\beta_k} \varepsilon^{m_k} b^k_{\alpha_k}(x)\Big) \times \\ &\qquad \times \Big(\int_0^\varepsilon r^{m_j-1} [b^j_{\alpha_j}(x-rt') - b^j_{\alpha_j}(x)]\, dr\Big) dt'. \end{aligned}$$

Thus $I_\varepsilon^3(x)$ is a sum of integrals of the form

$$\varepsilon^{-M+\Sigma m_k} C_{j,k} \prod_{p\in J'} b^p_{\alpha_p}(x) \int_{|t'|=1} h(t') \Big\{ \prod_{j\in J} \prod_{k\in J'} t'^{\alpha_j+\beta_k} \int_0^\varepsilon r^{m_j-1} [b^j_{\alpha_j}(x-rt') - b^j_{\alpha_j}(x)] dr \Big\} dt'.$$

Each such integral is dominated by

$$C_{j,k} \prod_{p\in J'} |b^p_{\alpha_p}(x)| \int_{|t'|=1} |h(t')| \prod_{j\in J} \Big(\frac{1}{\varepsilon} \int_0^\varepsilon |b^j_{\alpha_j}(x-rt') - b^j_{\alpha_j}(x)|\, dr \Big) dt'.$$

To each of these integrals, we may apply Lemma 2 and conclude that $I_\varepsilon^3(x) \to 0$ a.e. Finally, we note that $I_\varepsilon^2(x)$ is independent of ε. In fact,

$$I_\varepsilon^2(x) = \int_{|t'|=1} h(t') \sum_{\substack{\alpha_1,\ldots,\alpha_m\\ |\alpha_j|=m_j}} \prod_{j=1}^m \frac{1}{\alpha_j!} b^j_{\alpha_j}(x) t'^{\alpha_j}$$

$$= \sum_{\substack{\alpha_1,\ldots,\alpha_m\\ |\alpha_j|=m_j}} \Big(\int_{|t'|=1} h(t') t'^{\alpha_1+\ldots+\alpha_m} dt' \Big) \prod_{j=1}^m \frac{1}{\alpha_j!} b^j_{\alpha_j}(x)\, dt'.$$

It now follows that the integral in (4.4) has limits almost everywhere, and therefore, the truncated integrals $C_\varepsilon f$ have limits a.e. (x) in S.

5. In this section we prove the theorem under the hypotheses in (i). Since

$$\int_{|x-y|>\varepsilon} \frac{|\Omega(x-y)|}{|x-y|^{M+n}} \Big| \prod_{j=1}^m P_{m_j}(b_j; x, y) \Big| |f(y)|\, dy < \infty \text{ a.e. } (x),$$

we may pass to polar-co-ordinates in expressing $C_\varepsilon f(x)$. Because of the symmetry $\Omega(-x) = (-1)^{M+1}\Omega(x)$ we obtain

$$C_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{M+n}} \prod_{j=1}^m P_{m_j}(b^j; x, y) f(y)\, dy$$

$$= \frac{1}{2} \int_{|t'|=1} \Omega(t') \int_{|r|>\varepsilon} r^{-M-1} \prod_{j=1}^m P_{m_j}(b^j; x, x-rt') f(x-rt')\, dr\, dt'.$$

Thus

$$|C_* f(x)| \leqslant \tfrac{1}{2} \int_{|t'|=1} |\Omega(t')|\, |C_\varepsilon f(x, t')|\, dt' \leqslant \tfrac{1}{2} \int_{|t'|=1} |\Omega(t')| C_* f(x, t')\, dt',$$

where $C_\varepsilon f(x, t')$ denotes the one-dimensional truncated commutator in the direction of t' and $C_* f(x, t')$ denotes the corresponding one-dimensional maximal commutator. By the result in [5], if $x = z + rt'$, $z \perp t'$, then

$$\int_{-\infty}^{\infty} C_* f(z+rt', t')^q dr \leqslant C \prod_{j=1}^m \Big[\int_{-\infty}^{\infty} \Big(\sum_{|\alpha|=m_j} |b^j_\alpha(z+rt')| \Big)^{r_j} dr \Big]^{q/r_j} \Big(\int_{-\infty}^{\infty} |f(z+rt', t')|^p dr \Big)^{q/p}.$$

Integrating over the space of lines parallel to t' and using Hölder's inequality, we obtain

$$\Big(\iint_{\boldsymbol{R}^n} C_* f(x, t')^q dx \Big)^{1/q} \leqslant C \prod_{j=1}^m \Big[\iint_{\boldsymbol{R}^n} \Big(\sum_{|\alpha|=m_j} |b^j_\alpha(x)| \Big)^{r_j} dx \Big]^{1/r_j} \Big(\iint_{\boldsymbol{R}^n} |f(x)^p| dx \Big)^{1/p}.$$

Applying Minkowski's integral inequality, we obtain

$$\Big(\iint_{\boldsymbol{R}^n} |C_* f(x)|^q dx \Big)^{1/q} \leqslant C \Big(\int_{|t'|=1} |\Omega(t')|\, dt' \Big) \Big(\iint_{\boldsymbol{R}^n} C_* f(x, t')^q\, dx \Big)^{1/q}$$

$$\leqslant C \Big(\int_{|t'|=1} |\Omega(t')|\, dt' \Big) \prod_{j=1}^m \Big[\iint_{\boldsymbol{R}^n} \Big(\sum_{|\alpha|=m_j} |b^j_\alpha(x)| \Big)^{r_j} dx \Big]^{1/r_j} \Big(\iint_{\boldsymbol{R}^n} |f(x)|^p dx \Big)^{1/p}$$

$$\leqslant C \Big(\int_{|t'|=1} |\Omega(t')|\, dt' \Big) \prod_{j=1}^m \Big(\sum_{|\alpha|=m_j} \|b^j_\alpha\|_{r_j} \Big) \|f\|_p.$$

This proves the theorem under the hypotheses in (i).

6. We are now ready to prove the main result in the "even" case. Let $h(x) = \Omega(x)/|x|^{M+n}$. Let $Rf(\cdot)$ denote the vector valued function with jth component equal to the jth Riesz transform of f. If $g(\cdot)$ is a vector valued function, let $(R\cdot g)(\cdot)$ denote the scalar function obtained by summing the jth Riesz transform in the jth component. With this notation we may write any $f \in C_0^\infty$ as $cR\cdot(Rf)$ where c is a fixed constant independent of f. Let $N_1(x) = Rh(x)$ where owing to the singularity of h at the origin, we interpret this for $x \neq 0$ as $\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} R(x-y) h(y)\, dy$ (which converges in the mean as $\varepsilon \to 0$ on any compact set not containing 0, see [4]), and the truncated integrals $\int_{|x-y|>\varepsilon} R(x-y) h(y)\, dy$ are interpreted in a manner analogous to that in [1]. Moreover, we note that $N_1(x)$ is homogeneous of degree $-M-n$ in each component and $N_1(-x) = (-1)^{M+1} N_1(x)$. Thus $N_1(x)$ satisfies the conditions of part (i) of the theorem and therefore it suffices to estimate $\sup_{\varepsilon>0} |C_\varepsilon(R\cdot g)(x) - C^1_\varepsilon(Rf)(x)|$ where $g = Rf$ and

$$C^1_\varepsilon(Rf)(x) = c \int_{|x-y|>\varepsilon} N_1(x) \cdot \prod_{j=1}^m P_{m_j}(b^j; x, y) g(y)\, dy.$$

Let $\varphi(t)$, $t>0$, be a C^∞ function which satisfies $\varphi(t)=0$ for $t\leqslant\frac{1}{4}$, $\varphi(t)=1$ for $t\geqslant\frac{3}{4}$ and $0\leqslant\varphi(t)\leqslant 1$ for all t. Now $C_\varepsilon f(x)-C_\varepsilon^1(Rf)(x)=D_\varepsilon f(x)+$ $+E_\varepsilon f(x)+F_\varepsilon f(x)$ where

$$D_\varepsilon f(x)=\int\limits_{|x-y|>\varepsilon}h(x-y)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)(R\cdot g)(y)\,dy-$$

$$-\int h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)(R\cdot g)(y)\,dy,$$

$$E_\varepsilon f(x)=\int\limits_{|x-y|>\varepsilon}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)R\cdot\Big[\prod_{j=1}^{m}P_{m_j}(b^j;x,\cdot)g(\cdot)\Big](y)\,dy-$$

$$-\int\limits_{|x-y|>\varepsilon}N_1(x-y)\cdot\prod_{j=1}^{m}P_{m_j}(b^j;x,y)g(y)\,dy,$$

$$F_\varepsilon f(x)=\int h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\Big\{\prod_{j=1}^{m}P_{m_j}(b^j;x,y)(R\cdot g)(y)-$$

$$-R\cdot\Big[\prod_{j=1}^{m}P_{m_j}(b^j;x,\cdot)g(\cdot)\Big](y)\Big\}\,dy.$$

Let $D_*f(x)$, $E_*f(x)$ and $F_*f(x)$ denote the corresponding maximal operators. We shall estimate each of these operators separately. Now

$$|D_\varepsilon f(x)|=\Big|\int\limits_{\varepsilon>|x-y|>\varepsilon/4}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)(R\cdot g)(y)\,dy\Big|$$

$$\leqslant\varepsilon^{-M-n}\int\limits_{|x-y|<\varepsilon}\left|h\left(\frac{x-y}{|x-y|}\right)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)(R\cdot g)(y)\right|dy.$$

The desired estimate for D_*f now follows by applying Lemma 1 with $\varphi=\chi_{[0,1]}$. The estimate for E_*f is essentially the same as that in [1]. As in [1], [4], we introduce $N_2(x)=\lim\limits_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}R(x-y)h(y)\varphi(|y|)\,dy$ and note that

(i) $|N_2(x)-N_1(x)|\leqslant C|x|^{-M-n-1}$ for $|x|\geqslant 1$,

(ii) $|N_2(x)|\leqslant G(x)$ for $|x|\leqslant 1$ where G is homogeneous of degree zero and satisfies $\int\limits_{|x'|=1}|G(x')|\,dx'\leqslant C\int\limits_{|x'|=1}|h(x')|(1+\log^+|h(x')|)\,dx'$,

(iii) if $g(\cdot)$ is a vector valued function with entries in L^q, $1<q<\infty$, then

$$\int\limits_{R^n}h(y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)(R\cdot g)(y)\,dy=\varepsilon^{-M-n}\int\limits_{R^n}N_2\left(\frac{x-y}{\varepsilon}\right)\cdot g(y)\,dy.$$

One then obtains

$$|E_\varepsilon f(x)|=\Big|\varepsilon^{-M-n}\int\limits_{R^n}N_2\left(\frac{x-y}{\varepsilon}\right)\cdot\prod_{j=1}^{m}P_{m_j}(b^j;x,y)g(y)\,dy-$$

$$-\int\limits_{|x-y|>\varepsilon}N_1(x-y)\cdot\prod_{j=1}^{m}P_{m_j}(b^j;x,y)g(y)\,dy\Big|$$

$$\leqslant C\varepsilon^{-M-n}\int\limits_{|x-y|\geqslant\varepsilon}\left(\frac{|x-y|}{\varepsilon}\right)^{-M-n-1}\Big|\prod_{j=1}^{m}P_{m_j}(b^j;x,y)g(y)\Big|\,dy+$$

$$+\varepsilon^{-M-n}\int\limits_{|x-y|<\varepsilon}\Big|G(x-y)\prod_{j=1}^{m}P_{m_j}(b^j;x,y)g(y)\Big|\,dy.$$

An application of Lemma 1 yields the desired inequality for E_*f.

We now estimate F_*f. This is the most technical part of the proof. Recall that

$$F_\varepsilon f(x)=\int h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\int R(y-t)\cdot\Big\{\prod_{j=1}^{m}P_{m_j}(b^j;x,y)-$$

$$-\prod_{j=1}^{m}P_{m_j}(b^j;x,t)\Big\}g(t)\,dt\,dy$$

where h is homogeneous of degree $-n-M$, $M=\sum\limits_{j=1}^{m}m_j$. Our entire proof is basically an induction argument based on the number of remainders in the commutator. The case of one remainder is simply the result in [1]. We shall refer to the number of remainders as the order of the commutator. We are considering the case of an mth-order commutator and will assume the appropriate boundedness of all commutators of order less than m as our induction hypothesis. The estimate for $F_\varepsilon f$ requires systematically integrating by parts M times. Unlike the case in [1] the inner integral as a function of y does not have derivatives of order M in the appropriate space. After expanding the product in the inner integral as a large sum of terms, it turns out that at each stage of integration those terms which cannot be differentiated again can in fact be written as a composition of lower order commutators and can therefore be removed from the sum. After the final integration the inner integrals are mth order commutators each having an mth order derivative of the Riesz kernel as its kernel. Since the Riesz kernel and all its derivatives fall into case (i) of our result, the "odd" case, we can estimate this term. Finally, at each stage of integration, terms arise in which a derivative has been placed on $\varphi(|x-y|/\varepsilon)$. We shall refer to these as error terms. While these terms can be estimated essentially by Lemma 1, the proofs are technical and somewhat cumbersome.

We now introduce some notation. Let $\mathcal{J}$ denote the collection of all ordered subsets of $\{1, 2, \dots, m\}$. For $J \in \mathcal{J}$ let J' denote the ordered complement of J (e.g. if $m = 6$ and $J = (2, 5)$, then $J' = (1, 3, 4, 6)$.) Given $J' = (p_1, \dots, p_{|J|})$, let $\sigma(p_k)$ denote a multi-index with n entries. For $J \in \mathcal{J}$ and $J' = (p_1, \dots, p_{|J'|})$ we define $Q[J; \sigma(p_1), \dots, \sigma(p_{|J'|})](y, t)$ as the term

$$\prod_{k=1}^{|J'|} \frac{1}{\sigma(p_k)!} P_{m_{p_k}-|\sigma(p_k)|}(b^{p_k}_{\sigma(p_k)}; y, t)(x-y)^{\sigma(p_k)}.$$

We also impose the restriction that whenever we write $Q[J; \sigma(p_1), \dots$ $\dots, \sigma(p_{|J'|})]$, $|\sigma(p_k)| < m_{p_k}$ for $k = 1, 2, \dots, |J'|$. With the aid of the formula

$$P_m(b; x, t) = P_m(b; x, y) + \sum_{|\alpha|<m} \frac{1}{\alpha!} P_{m-|\alpha|}(b; y, t)(x-y)^{\alpha} \tag{6.1}$$

and the above notation, we may now write

$$\prod_{j=1}^{m} P_{m_j}(b^j; x, t) = \prod_{j=1}^{m} \Big\{P_{m_j}(b^j; x, y) + \sum_{|\alpha|<m_j} \frac{1}{\alpha!} P_{m_j-|\alpha|}(b^j_\alpha; y, t)(x-y)^{\alpha}\Big\}$$

$$= \prod_{j=1}^{m} P_{m_j}(b^j; x, y) + \sum_{\substack{J\in\mathcal{J}\\ J'\neq\emptyset}} \prod_{j\in J} P_{m_j}(b_j; x, y) \sum Q[J; \sigma(p_1), \dots, \sigma(p_{|J'|})](y, t)$$

where the inner sum is taken over all possible choices of $\sigma(p_1), \dots, \sigma(p_{|J'|})$. When no confusion can arise, we will abbreviate the inner sum as $\sum Q[J; \sigma(J')]$. Given a term $Q[J; \sigma(J')]$, we define $|Q[J; \sigma(J')]|$, the order of this term, as $\sum_{1\leqslant k\leqslant |J'|} (m_{p_k} - |\sigma(p_k)|)$.

Applying Euler's formula on homogeneous functions and Green's theorem, we have

$$F_\varepsilon f(x) = \int h(x-y)\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\int R(y-t)\cdot$$
$$\cdot\Big\{\prod_{j=1}^{m} P_{m_j}(b^j; x, y) - \prod_{j=1}^{m} P_{m_j}(b^j; x, t)\Big\} g(t)\,dt\,dy$$

$$= -\frac{1}{M}\sum_{|\alpha|=1}\int (x-y)^{\alpha}h(x-y)\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\int R_\alpha(y-t)\cdot$$

$$\cdot\Big\{\prod_{j=1}^{m} P_{m_j}(b^j; x, y) - \prod_{j=1}^{m} P_{m_j}(b^j; x, t)\Big\} g(t)\,dt\,dy -$$

$$-\frac{1}{M}\sum_{|\alpha|=1}\int (x-y)^{\alpha}h(x-y)\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\int R(y-t)\cdot$$
$$\cdot\frac{\partial}{\partial y^{\alpha}}\Big\{\prod_{j=1}^{m} P_{m_j}(b^j; x, y) - \prod_{j=1}^{m} P_{m_j}(b^j; x, t)\Big\} g(t)\,dt\,dy +$$

$$+\frac{1}{M}\int h(x-y)\Big(\frac{|x-y|}{\varepsilon}\Big)\varphi'\Big(\frac{|x-y|}{\varepsilon}\Big)\int R(y-t)\cdot$$
$$\cdot\Big\{\prod_{j=1}^{m} P_{j_m}(b^j; x, y) - \prod_{j=1}^{m} P_{m_j}(b^j; x, t)\Big\} g(t)\,dt\,dy$$

$$\equiv F^1_\varepsilon f(x) + F^2_\varepsilon(x) + F^3_\varepsilon f(x).$$

The third term $F^3_\varepsilon f(x)$ is an error term and will be estimated with all error terms in the final section of this paper. For the second term we have

$$F^2_\varepsilon f(x) = \frac{1}{M}\sum_{|\alpha|=1}\sum_{k=1}^{m}\sum_{|\beta|=m_k-1}\int (x-y)^{\alpha+\beta}h(x-y)\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\times$$
$$\times\prod_{\substack{j=1\\ j\neq k}}^{m} P_{m_j}(b^j; x, y)\, b^k_{\beta+\alpha}(y) f(y)\,dy.$$

Each integral in $F^2_\varepsilon f$ is a commutator of order strictly less then M. By the induction hypothesis and Hölder's inequality it follows that

$$\|\sup_{\varepsilon}|F^2_\varepsilon f(\cdot)|\,\|_q \leqslant C\prod_{j=1}^{m}\Big(\sum_{|\alpha|=m_j}\|b^j_\alpha\|_{r_j}\Big)\|f\|_p\|\Omega\|_{L\log^+L(\Sigma)}.$$

To estimate $F^1_\varepsilon f$ we must integrate by parts again. Before doing so we must expand the expression in brackets in $F^1_\varepsilon f$ and remove some "good" terms. To do this we use the notation introduced earlier. We have

$$F^1_\varepsilon f(x) = -\frac{1}{M}\sum_{|\alpha|=1}\int (x-y)^{\alpha}h(x-y)\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\int R_\alpha(y-t)\cdot$$
$$\cdot\sum_{\substack{J\in\mathcal{J}\\ J'\neq\emptyset}}\Big\{\prod_{j\in J} P_{m_j}(b^j; x, y)\sum Q[J; \sigma(J')](y, t)\Big\} g(t)\,dt\,dy$$

$$= -\frac{1}{M}\sum_{|\alpha|=1}\sum_{\substack{J\in\mathcal{J}\\ J'\neq\emptyset}}\sum_{Q}\int (x-y)^{\alpha}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j; x, y)\cdot$$

$$\cdot\int R_\alpha(y-t)\cdot Q[J; \sigma(J')](y, t)g(t)\,dt$$

$$\equiv F_\varepsilon^4 f(x) + G_\varepsilon^1 f(x)$$

where $G_\varepsilon^1 f$ contains all terms with $|Q[J;\sigma(J')]| = 1$. Such terms occur only when $|J| = m-1$ and if $J' = \{p\}$, $|\sigma(p)| = m_p - 1$. In such a case

$$Q[J;\sigma(J')](y, t) = \frac{1}{\sigma(p)!}P_1(b^p_{\sigma(p)}; y, t)(x-y)^{\sigma(p)}.$$

Thus $G_\varepsilon^1 f$ is a sum of integrals of the type

$$\frac{1}{\sigma(p)!}\int (x-y)^{\alpha+\sigma(p)}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j; x, y)\times$$

$$\times\int R_\alpha(y-t)\cdot P_1(b^p_{\sigma(p)}; y, t)g(t)\,dt\,dy.$$

Noting that $x^{\alpha+\sigma(p)}h(x)$ is homogeneous of degree $-n-M+|\alpha|+|\sigma(p)| = -n-\sum_{j\in J}m_j$, we observe that each integral in $G_\varepsilon^1 f$ is the composition of a commutator of order $m-1$ and a commutator of order 1. By the induction hypothesis and the boundedness of the Riesz transform we obtain

$$\|\sup_\varepsilon |G_\varepsilon^1 f(\cdot)|\|_q \leqslant C\prod_{j=1}^{m}\Big(\sum_{|\alpha|=m_j}\|b^j_\alpha\|_{r_j}\Big)\|f\|_p\|\Omega\|_{L\log^+(\Sigma)}.$$

To summarize, we so far have $F_\varepsilon f = F_\varepsilon^4 f$+error terms+terms with the appropriate boundedness. We now apply integration by parts to $F_\varepsilon^4 f$ and obtain

$$F_\varepsilon^4 f(x) = \frac{(M-2)!}{M!}\sum_{\substack{|\alpha|=1\\|\beta|=1}}\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>1}}\int (x-y)^{\alpha+\beta}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\cdot$$

$$\cdot\int R_{\alpha+\beta}(y-t)\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)g(t)\,dt\,dy+$$

$$+\frac{(M-2)!}{M!}\sum_{\substack{|\alpha|=1\\|\beta|=1}}\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>1}}\int (x-y)^{\alpha+\beta}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\int R_\alpha(y-t)\cdot$$

$$\cdot\frac{\partial}{\partial y^\beta}\Big\{\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)\Big\}g(t)\,dt\,dy$$

$$-\frac{(M-2)!}{M!}\sum_{|\alpha|=1}\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>1}}\int h(x-y)\left(\frac{|x-y|}{\varepsilon}\right)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\int R_\alpha(y-t)\cdot$$

$$\cdot\Big\{\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)\Big\}g(t)\,dt\,dy$$

$$\equiv F_\varepsilon^5 f(x) + F_\varepsilon^6 f(x) + F_\varepsilon^7 f(x).$$

As before, $F_\varepsilon^7 f$ is an error term and will be estimated later. Before estimating $F_\varepsilon^5 f$ and $F_\varepsilon^6 f$, we describe the situation after k integrations by parts. We do this now because the argument required to estimate $F_\varepsilon^6 f$ is essentially the argument which treats the general case. The $(k+1)$st integration by parts is as follows:

$$F_\varepsilon^{4k}f(x) = \frac{(-1)^{k+1}(M-k-1)!}{M!}\sum_{\substack{|\alpha|=k\\|\beta|=1}}\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k}}\int (x-y)^{\alpha+\beta}h(x-y)\times$$

$$\times\varphi\left(\frac{|x-y|}{\varepsilon}\right)\int R_{\alpha+\beta}(y-t)\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)g(t)\,dt\,dy+$$

$$+\frac{(-1)^{k+1}(M-k-1)!}{M!}\sum_{\substack{|\alpha|=k\\|\beta|=1}}\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k}}\int (x-y)^{\alpha+\beta}h(x-y)\varphi\left(\frac{|x-y|}{\varepsilon}\right)\times$$

$$\times\int R_\alpha(y-t)\cdot\frac{\partial}{\partial y^\beta}\Big\{\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)\Big\}g(t)\,dt\,dy+$$

$$+\frac{(-1)^{k+1}(M-k-1)!}{M!}\sum_{|\alpha|=k}\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k}}\int (x-y)^{\alpha}h(x-y)\left(\frac{|x-y|}{\varepsilon}\right)\times$$

$$\times\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\int R_\alpha(y-t)\cdot\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)g(t)\,dt\,dy$$

$$\equiv F_\varepsilon^{4k+1}f(x) + F_\varepsilon^{4k+2}f(x) + F_\varepsilon^{4k+3}f(x).$$

As always we consider $F_\varepsilon^{4k+3}f$ with all the error terms. To handle F_ε^{4k+2} we must calculate

$$\frac{\partial}{\partial y^\beta}\Big\{\sum_{\substack{J\in\mathcal{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k}}\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J;\sigma(J')](y, t)\Big\} \tag{6.2}$$

where $|\beta| = 1$. To handle (6.2) it is convenient to introduce one additional piece of notation. Given an integer $1\leqslant i\leqslant m$ we let $\mathcal{J}(i)$ denote the collection of all subsets of $\{1, 2, \ldots, i-1, i+1, \ldots, m\}$. Given $J\in\mathcal{J}(i)$ we let $J'(i)$ denote the complement of J with respect to $\{1, \ldots, i-1, i+1, \ldots \ldots, m\}$. We then have

$$(6.3)\quad \frac{\partial}{\partial y^\beta}\Big\{\sum_{\substack{J\in\mathscr{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k}}\prod_{j\in J}P_{m_j}(b^j; x, y)Q[J; \sigma(J')](y, t)\Big\}$$

$$= \sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q\\|Q|>k}}\frac{\partial}{\partial y^\beta}\{P_{m_i}(b^i; x, y)\}\prod_{j\in J}P_{m_j}(b^j; x, y)Q\big[J; \sigma\big(J'(i)\big)\big](y, t)+$$

$$+\sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q\\|Q|>k}}\frac{\partial}{\partial y^\beta}\Big\{\sum_{|\gamma|<m_i}P_{m_i-|\gamma|}(b^i_\gamma; y, t)(x-y)^\gamma\Big\}\times$$

$$\times\prod_{j\in J}P_{m_j}(b^j; x, y)\times Q\big[J; \sigma\big(J'(i)\big)\big](y, t)+$$

$$+\sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q\\|Q|\leqslant k}}\frac{\partial}{\partial y^\beta}\Big\{\sum_{0\leqslant|\gamma|<m_i-k+|Q|}P_{m_i-|\gamma|}(b^i_\gamma; y, t)(x-y)^\gamma\Big\}\times$$

$$\times\prod_{j\in J}P_{m_j}(b^j; x, y)Q\big[J; \sigma\big(J'(i)\big)\big](y, t)$$

$$= \sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q\\|Q|>k}}\frac{\partial}{\partial y^\beta}\{P_{m_i}(b^i; x, t)\}\prod_{j\in J}P_{m_j}(b^j; x, y)Q\big[J; \sigma\big(J'(i)\big)\big](y, t)+$$

$$+\sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q\\|Q|\leqslant k}}\Big\{\sum_{|\gamma|=m_i-k+|Q|-1}P_{m_i-|\gamma|}(b^i_{\gamma+\beta}; y, t)(x-y)^\gamma\Big\}\times$$

$$\times\prod_{j\in J}P_{m_j}(b^j; x, y)Q\big[J; \sigma\big(J'(i)\big)\big](y, t)$$

$$= \sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q\\|Q|\leqslant k}}\sum_{|\gamma|=m_i-k+|Q|-1}\{P_{m_i-|\gamma|}(b^i_{\gamma+\beta}; y, t)(x-y)^\gamma\}\times$$

$$\times\prod_{j\in J}P_{m_j}(b^j; x, y)Q\big[J; \sigma J'\big((i)\big)\big](y, t)$$

since $P_{m_i}(b^i; x, t)$ is independent of y. We now return to our estimate of $F_\varepsilon^{4k+2}f$. Using (6.3) we have

$$(6.4)\quad F_\varepsilon^{4k+2}f(x) = \frac{(-1)^{k+1}(M-k-1)!}{M!}\times$$

$$\times\sum_{\substack{|\alpha|=k\\|\beta|=1}}\sum_{i=1}^m\sum_{J\in\mathscr{J}(i)}\sum_{\substack{Q[J;\sigma(J'(i))]\\|Q|\leqslant k}}\times\sum_{|\gamma|=m_i-k+|Q|-1}\int(x-y)^\eta h(x-y)\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\cdot$$

$$\cdot\prod_{j\in J}P_{m_j}(b^j; x, y)\int R_\alpha(y-t)\times P_{m_i-|\gamma|}(b^i_{\gamma+\beta}; y, t)\cdot$$

$$\cdot\prod_{p\in J'(i)}P_{m_p-|\sigma(p)|}(b^p_{\sigma(p)}; y, t)g(t)\,dt\,dy$$

where $\eta = \alpha+\beta+\gamma+\sum_{p\in J'(i)}\sigma(p)$. Then $|\eta| = k+1+m_i-k+|Q|-1-$ $-\sum_{p\in J'(i)}(m_p-|\sigma(p)|)+\sum_{p\in J'(i)}|\sigma(p)| = m_i+\sum_{p\in J'(i)}m_p = M-\sum_{j\in J}m_j$. Thus $x^\eta h(x)$ is homogeneous of degree $-n-M+\big(M-\sum_{j\in J}m_j\big) = -n-\sum_{j\in J}m_j$. Also we note that $m_i-|\gamma|+\sum_{p\in J'(i)}m_p-|\sigma(p)| = k = |\alpha|$. It follows that each term in (6.4) is the composition of a commutator of order $|J|$ and a commutator of order $|J'(i)|+1$. Since $|J|<m$, we may use the induction hypothesis on the outer integral, the "odd" result on the inner integral, and the boundedness of the Riesz transform to conclude

$$\|\sup_\varepsilon|F_\varepsilon^{4k+2}f(\cdot)|\,\|_q\leqslant C\prod_{j=1}^m\Big(\sum_{|\alpha|=m_j}\|b^j_\alpha\|_{r_j}\Big)\|f\|_p\|\Omega\|_{L\log^+L(\Sigma)}\quad\text{for } k<M-1.$$

For the terms $F_\varepsilon^{4k+1}f$, $k\geqslant 1$, we write $F_\varepsilon^{4k+1}f(x) = G_\varepsilon^{4k+1}f(x)+$ $+F_\varepsilon^{4(k+1)}f(x)$ where

$$G_\varepsilon^{4k+1}f(x) = \frac{(-1)^{k+1}(M-k-1)!}{M!}\sum_{\substack{|\alpha|=k\\|\beta|=1}}\sum_{\substack{J\in\mathscr{J}\\|J|<m}}\sum_{\substack{Q\\|Q|=k+1}}\int(x-y)^{\alpha+\beta}h(x-y)\times$$

$$\times\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\prod_{j\in J}P_{m_j}(b^j; x, y)\int R_{\alpha+\beta}(y-t)\cdot Q[J; \sigma(J')](y, t)g(t)\,dt\,dy,$$

$$F_\varepsilon^{4(k+1)}f(x) = \frac{(-1)^{k+1}(M-k-1)!}{M!}\sum_{|\alpha|=k+1}\sum_{\substack{J\in\mathscr{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k+1}}\int(x-y)^\alpha h(x-y)\times$$

$$\times\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\prod_{j\in J}P_{m_j}(b^j; x, y)\int R_\alpha(y-t)\cdot Q[J; \sigma(J')](y, t)g(t)\,dt\,dy.$$

We have

$$(6.5)\quad G_\varepsilon^{4k+1}f(x) = \frac{(-1)^{k+1}(M-k-1)!}{M!}\sum_{\substack{|\alpha|=k\\|\beta|=1}}\sum_{\substack{J\in\mathscr{J}\\|J|<m}}\sum_{\substack{Q\\|Q|=k+1}}\int(x-y)^\eta h(x-y)\times$$

$$\times\varphi\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j;x,y)\int R_{\alpha+\beta}(y-t)\cdot\prod_{p\in J'}P_{m_p-|\sigma(p)|}(b^p_{\sigma(p)};y,t)\times$$
$$\times g(t)\,dt\,dy$$

where $\eta=\alpha+\beta+\sum_{s\in J'}\sigma(p_s)$ and $|\eta|=k+1-\sum_{p\in J'}(m_p-|\sigma(p)|)+\sum_{p\in J'}m_p=k+1-(k+1)+\sum_{p\in J'}m_p=\sum_{p\in J'}m_p$. Thus $x^\eta h(x)$ is homogeneous of degree $-n-\sum_{j\in J}m_j$. It follows that each term in (6.5) is the composition of a commutator of order $|J|$ and a commutator of order $|J'|$. By induction on the outer integrals, the result in the "odd" case on the inner integrals, and the boundedness of the Riesz transform, we may conclude that for $1\leqslant k\leqslant M-1$,

$$\|\sup_\varepsilon|G_\varepsilon^{k+1}f(\cdot)|\|_q\leqslant C\prod_{j=1}^m\Big(\sum_{|\alpha|=m_j}\|b^j_\alpha\|_{r_j}\Big)\|f\|_p\|\Omega\|_{L\log^+L(\Sigma)}.$$

We now write $F_\varepsilon^{4(k+1)}f=F_\varepsilon^{4(k+1)+1}f+F_\varepsilon^{4(k+1)+2}f+F_\varepsilon^{4(k+1)+3}f$ and iterate. The process terminates when $k=M-1$, because in this case $F_\varepsilon^{4M}f=G_\varepsilon^Mf$ It now follows that in order to obtain the desired estimate for $F_\varepsilon f$ we need only estimate the error terms $F_\varepsilon^{4k+3}f$ for $0\leqslant k<m-1$.

7. The error terms. To finish the proof we need only show that the terms involving φ' satisfy the appropriate L^q estimates. Before estimating these error terms we note that φ' is supported on the interval $[\frac14,\frac34]$. Consequently, the region of integration in y in all terms with $\varphi'(|x-y|/\varepsilon)$ is the annulus $\frac14\varepsilon\leqslant|x-y|\leqslant\frac34\varepsilon$. In this region we have the estimates $|x-y|/\varepsilon\leqslant 1$ and $|H(x-y)|\leqslant 4\varepsilon^N\big|H\big((x-y)/(|x-y|)\big)\big|$ for H homogeneous of degree N.

The first error term we estimate is

$$F_\varepsilon^3f(x)=\int h(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x-y|}{\varepsilon}\prod_{j=1}^mP_{m_j}(b^j;x,y)f(y)\,dy-$$
$$-\int h(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x-y|}{\varepsilon}\int R(y-t)\cdot\prod_{j=1}^mP_{m_j}(b^j;x,t)g(t)\,dt\,dy$$
$$=I_\varepsilon^1f(x)+I_\varepsilon^2f(x).$$

For the first term,

$$|I_\varepsilon^1f(x)|\leqslant\frac{1}{\varepsilon^{M+n}}\int\left|\Omega(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j=1}^mP_{m_j}(b^j;x,y)f(y)\right|dy.$$

Applying Lemma 1 with $\Phi(s)=\varphi'(s)$ and $N(x)=\Omega(x)$ we get the estimate

$$\|\sup_{\varepsilon>0}|I_\varepsilon^1f(\cdot)|\|_q\leqslant C\prod_{j=1}^m\Big(\sum_{|\alpha|=m_j}\|b^j_\alpha\|_{r_j}\Big)\|f\|_p\|\Omega\|_{L\log^+L(\Sigma)}.\tag{7.1}$$

To evaluate I_ε^2f we note that

$$I_\varepsilon^2f(x)=\int N_\varepsilon(x-t)\prod_{j=1}^mP_{m_j}(b^j;x,t)g(t)\,dt,$$

where

$$N_\varepsilon(x)=\int R(x-y)\varphi'\left(\frac{|y|}{\varepsilon}\right)\frac{|y|}{\varepsilon}h(y)\,dy.$$

Letting $y=u\varepsilon$ and applying results in § 5 of [4] we have

$$|N_\varepsilon(x)|\leqslant\begin{cases}\dfrac{\varepsilon}{|x|^{M+n+1}}, & |x|\geqslant\varepsilon,\\ \varepsilon^{-M-n}G(x), & |x|\leqslant\varepsilon\end{cases}$$

where $G(x)$ is homogeneous of degree zero and $\|G\|_{L^1(\Sigma)}\leqslant C\|\Omega\|_{L\log^+L(\Sigma)}$.

Splitting I_ε^2f into an integral over $|x-t|\leqslant\varepsilon$ and $|x-t|\geqslant\varepsilon$ we can apply lemma 1 to each piece and get an L^q estimate of the type (7.1) for the maximal operator associated with I_ε^2f.

To estimate the remaining error terms we need only show that the maximal operator associated with the general term $F_\varepsilon^{4k+3}f$ satisfies an estimate of the type (7.1). Splitting up the integration in t, we can write

$$F_\varepsilon^{4k+3}f(x)=\sum_{|\alpha|=k}\sum_{\substack{J\in\mathscr{J}\\|J|<m}}\sum_{\substack{Q\\|Q|>k}}\int(x-y)^\alpha h(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x-y|}{\varepsilon}\times\tag{7.2}$$
$$\times\prod_{j\in J}P_{m_j}(b^j;x,y)\Big\{\Big(\int_{|y-t|\leqslant5\varepsilon}+\int_{|y-t|>5\varepsilon}\Big)R_\alpha(y-t)\cdot Q[J;\sigma(J')](y,t)g(t)\,dt\Big\}dy$$
$$=I_\varepsilon^{1,k}f(x)+I_\varepsilon^{2,k}f(x).$$

To evaluate $I_\varepsilon^{1,k}f$ we look at the single term corresponding to a fixed α, J and Q. We estimate its associated maximal function and sum over the index sets given in equation (7.2).

If we take the absolute value of a single term, it is clearly bounded by

$$C\frac{1}{\varepsilon^{n+|m(J)|}}\int\left|\Omega(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j;x,y)\right|\times$$

$$\times\left\{\sup_{\varepsilon>0}\frac{1}{\varepsilon^{n+|Q|}}\int_{|y-t|\leqslant 5\varepsilon}\left|R_\alpha\left(\frac{y-t}{|y-t|}\right)\right|\left|\frac{y-t}{\varepsilon}\right|^{-n-k}\cdot\right.$$

$$\left.\cdot\prod_{p\in J'}P_{m_p-|\sigma(p)|}(b^p_{\sigma(p)};y,t)g(t)\right|dt\Bigg\}dy$$

where $|m(J)|=\sum_{j\in J}m_j$.

The inner operator satisfies Lemma 1 with $\Phi(s)=s^{-n-k}\chi_{[0,5]}(s)$ and $N(y)=R_\alpha(y/|y|)$. The L^r norm of the inner operator is thus bounded by

$$C\prod_{j\in J'}\Big(\sum_{|\alpha|=m_j}\|b^j_\alpha\|_{r_j}\Big)\|f\|_p\quad\text{where}\quad\frac{1}{r}=\frac{1}{p}+\sum_{j\in J'}\frac{1}{r_j}.$$

The outer integral satisfies Lemma 1 with $\Phi(s)=\varphi'(s)$ and $N(x)=\Omega(x)$. Since the inner operator is in L^r, the maximal operator for each α, J, Q fixed satisfies an estimate of the type (7.1).

To estimate $I^{2,k}_\varepsilon f$ we look at a term with α fixed and we sum over J and Q. We are left to evaluate terms of the type

(7.3)

$$\sum_{\substack{J\in\mathscr{J}\\ |J|<m}}\sum_{\substack{Q\\ |Q|>k}}\int(x-y)^\alpha h(x-y)\frac{|x-y|}{\varepsilon}\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j;x,y)\times$$

$$\times\int_{|y-t|\geqslant 5\varepsilon}R_\alpha(y-t)Q[J;\sigma(J')](y,t)g(t)\,dt\,dy$$

$$=\Big\{\sum_{J\in\mathscr{J}}\sum_{Q}-\sum_{J\in\mathscr{J}}\sum_{|Q|\leqslant k}\Big\}\int(\ldots)\,dt\,dy.$$

We estimate these sums separately. For the second sum we first consider a term where $|Q|=k$. Using the notation $x^{\sigma(J')}=\prod_{p\in J'}x^{\sigma(p)}$, writing out Q as a product and taking absolute values, we have,

$$\Bigg|\int(x-y)^{\alpha+\sigma(J')}h(x-y)\frac{|x-y|}{\varepsilon}\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j;x,y)\times$$

$$\times\Big\{\int_{|y-t|\geqslant 5\varepsilon}R_\alpha(y-t)\prod_{p\in J'}P_{m_p-|\sigma(p)|}(b^p_{\sigma(p)};y,t)g(t)\,dt\Big\}dy\Bigg|$$

$$\leqslant C\frac{1}{\varepsilon^{n+|m(J)|}}\int\left|\Omega(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j;x,y)\right|\times$$

$$\times\Big\{\sup_{\varepsilon>0}\Big|\int_{|y-t|>5\varepsilon}R_\alpha(y-t)\cdot\prod_{p\in J'}P_{m_p-|\sigma(p)|}(b^p_{\sigma(p)};y,t)g(t)\,dt\Big|\Big\}dy.$$

The inner operator is in $L^r(1/r=1/p+\sum_{j\in J'}(1/r_j))$, because the derivatives of the Riesz kernel satisfy (i) of our theorem. The maximal operator associated with the outer integral satisfies Lemma 1 with $\Phi(s)=\varphi'(s)$ and $N(x)=\Omega(x)$. Summing over $|\alpha|=k$ and appropriate Q's and J's we get an L^q estimate of the type (7.1).

For the terms where $|Q|<k-1$ and J fixed, we see after taking absolute values that a typical term is bounded by

$$C\frac{1}{\varepsilon^{n+|m(J)|}}\int\left|\Omega(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\prod_{j\in J}P_{m_j}(b^j;x,y)\right|\times$$

$$\times\left\{\sup_{\varepsilon>0}\frac{1}{\varepsilon^{n+|Q|}}\int_{|y-t|\geqslant 5\varepsilon}\left|R_\alpha\left(\frac{y-t}{|y-t|}\right)\right|\left|\frac{y-t}{\varepsilon}\right|^{-n-k}\times\right.$$

$$\left.\times\prod_{p\in J'}P_{m_p-|\sigma(p)|}(b^p_{\sigma(p)};y,t)g(t)\right|dt\Bigg\}dy.$$

The inner operator satisfies Lemma 1 with $\Phi(s)=s^{-n-k}\chi_{[5,\infty)}(s)$ and $N(y)=R_\alpha(y/|y|)$ since $\int\Phi(s)s^{n+|Q|-1}ds<\infty$ for $|Q|<k-1$. The outer integral satisfies Lemma 1 with $\Phi(s)=\varphi'(s)$ and $N(x)=\Omega(x)$. Thus the maximal operators associated with terms where $|Q|<k$ satisfy L^q estimates of the type (7.1).

To estimate the term coming from the first sum in (7.3) we recall that

$$\sum_{J\in\mathscr{J}}\sum_{Q}\prod_{j\in J}P_{m_j}(b_j;x,y)Q[J;\sigma(J')](y,t)=\prod_{j=1}^{m}P_{m_j}(b^j;x,t).$$

So we must estimate the integral

(7.4)
$$\int(x-y)^\alpha h(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x-y|}{\varepsilon}\cdot$$

$$\cdot\int_{|y-t|\geqslant 5\varepsilon}R_\alpha(y-t)\prod_{j=1}^{m}P_{m_j}(b^j;x,t)g(t)\,dt\,dy.$$

Using the facts that

$$\int_{|y-t|>5\varepsilon}(\ldots)\,dt=\int_{|x-t|>5\varepsilon}(\ldots)\,dt+\Big\{\int_{|y-t|>5\varepsilon}-\int_{|x-t|>5\varepsilon}\Big\}(\ldots)\,dt$$

and $|y-t|>5\varepsilon>5|x-y|$, we see that the absolute value of the term in

(7.4) is bounded by

$$\left|\int (x-y)^\alpha h(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x-y|}{\varepsilon}\int\limits_{|x-t|>5\varepsilon} R_\alpha(y-t)\cdot \prod_{j=1}^{m} P_{m_j}(b^j; x, t)g(t)\,dt\,dy\right| + \int\left|(x-y)^\alpha h(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x-y|}{\varepsilon}\times \times\left\{\int\limits_{2\varepsilon\leqslant|y-t|\leqslant 10\varepsilon}\left|R_\alpha(y-t)\cdot\prod P_{m_j}(b^j; x, t)g(t)\right|dt\right\}\right|dy. \tag{7.5}$$

To evaluate the first term we use the cancellation properties of h which allow us to replace $R_\alpha(y-t)$ by

$$R_\alpha(y-t) - \sum_{|\delta|<M-k+1} R_{\alpha+\delta}(x-t)\frac{(y-x)^\delta}{\delta!} = P_{M-k+1}(R_\alpha; y-t, x-t).$$

Since $|x-t| > 5\varepsilon > 5|x-y|$, we can use the estimate

$$|P_{M-k+1}(R_\alpha; y-t, x-t)| \leqslant C\frac{|x-y|^{M-k+1}}{|x-t|^{M+n+1}}.$$

So the first term in (7.5) is bounded by

$$\frac{1}{\varepsilon^n}\int\left|\Omega(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\left\{\frac{1}{\varepsilon^{n+M}}\int\limits_{|x-t|>5\varepsilon}\left(\frac{|x-t|}{\varepsilon}\right)^{-M-n-1}\times \times\left|\sum_{j=1}^{m} P_{m_j}(b^j; x, t)g(t)\right|dt\right\}\right|dy \leqslant C\int\limits_{|x|=1}|\Omega(x)|\,dx\times \times\left\{\sup_{\varepsilon>0}\frac{1}{\varepsilon^{n+M}}\int\limits_{|x-t|>5\varepsilon}\left|\left(\frac{|x-t|}{\varepsilon}\right)^{-M-n-1}\prod_{j=1}^{m} P_{m_j}(b^j; x, t)g(t)\right|dt\right\}.$$

The second integral satisfies Lemma 1 so the first term in (7.5) satisfies and L^q estimate of the type (7.1).

Finally, we use the fact that for $2\varepsilon \leqslant |x-t| \leqslant 10\varepsilon$ and $|x-y| < \varepsilon$ we have $|R_\alpha(y-t)| \leqslant c/\varepsilon^{n+|\alpha|}$. This implies that the second term in (7.5) is bounded by

$$C\frac{1}{\varepsilon^n}\int\left|\Omega(x-y)\varphi'\left(\frac{|x-y|}{\varepsilon}\right)\right|dy\left\{\sup_{\varepsilon>0}\frac{1}{\varepsilon^{n+M}}\int\limits_{2\varepsilon\leqslant|x-t|\leqslant 10\varepsilon}\left|\prod_{j=1}^{m} P_{m_j}(b^j; x, t)\right||g(t)|\,dt\right\}.$$

The first integral is bounded by a constant times the L^1 norm of Ω on the sphere. The second integral satisfies Lemma 1 with $\Phi(s) = \chi_{[2,10]}(s)$ and $N(x) = 1$. So the maximal operator associated with the second term in (7.5) satisfies an L^q estimate of the type (7.1).

This completes the proof of part (ii) of our theorem.

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