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On interpolation of weighted L^p-spaces and Ovchinnikov's theorem

by

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Abstract. We use the interpolation method $\langle \bar{A} \rangle_\varrho$ introduced in [3] to interpolate between weighted L^p-spaces. As an application we get an improvement of the interpolation theorem of Stein–Weiss. We also use the machinary to get a proof of a weak version of Ovchinnikov's interpolation theorem. Finally we prove that the method $\langle \bar{A} \rangle_\varrho$ coincides with Ovchinnikov's method $\varphi_l(\bar{A})$ for dual couples $\bar{A}$.

0. Introduction. The purpose of this paper is to continue the study of the interpolation spaces $\langle \bar{A} \rangle_\varrho$ introduced in [3], p. 45, which we henceforth call the "$\pm$ method".

In [3] the $\pm$ method was used in connection with Orlicz spaces. Now we utilize them to interpolate between $L^p(w_0)$ and $L^p(w_1)$, $0 < p \leqslant \infty$ (weighted L^p-spaces). Our results should be compared with what can be done with other interpolation spaces (cf. [1]): If we use the complex method, we have to take $p \geqslant 1$. (The complex method corresponds to the case $\varrho(t) = t^\theta$.) On the other hand, using the real method, we can allow $p < 1$ but then we have to take different interpolation functors for each p. In a way thus the $\pm$ method is a substitute for the complex method.

All this is done in Section 2.

In Section 3 we apply the results in connection with the remarkable paper by V. I. Ovchinnikov [4]. Ovchinnikov introduces three new interpolation methods: $\varphi_l(\bar{A})$, $\varphi_m(\bar{A})$ and $\varphi_u(\bar{A})$ and he also proves an interesting interpolation theorem for operators from weighted L^∞-spaces into weighted L^1-spaces. A major tool in [4] is Grothendieck's inequality. Here we use the in some respects more elementary tools of [3], notably versions of Khintchine's and Carlson's inequalities. As a result we have to impose the restriction $\varrho \in \mathscr{P}^{+-}$ (see [2], p. 293, and [3], p. 37). In this auxiliary hypothesis we can give a rather simple proof of Ovchinnikov's theorem. We also establish inclusion relations between the spaces $\varphi_l(\bar{A})$ and $\varphi_u(\bar{A})$ and our $\pm$ method.

Finally, I would like to thank Professor Jaak Peetre for his valuable advice and steady interest in my work.

1. Preliminaries. In this section we shall give some fundamental definitions and the basic interpolation result for the $\pm$ method.

DEFINITION 1.1. Let μ be a positive measure on a set M and let w be a positive μ-measurable function on M. A μ-measurable function f is said to *belong to* $L^p(M, \mu, w)$, where $0 < p < \infty$, if the (quasi)-norm

$$\|f\|_{L^p(w)} = \Big(\int_M |fw|^p d\mu\Big)^{1/p} < \infty. \tag{1}$$

If $p = \infty$, then (1) is replaced by

$$\|f\|_{L^\infty(w)} = \sup_{x\in M} |f(x)|\, w(x) < \infty \qquad \mu\text{-a.e.}$$

We shall now define the $\pm$ method and give an interpolation result. However, first we recall the definition of $\mathscr{P}$ in [3], p. 37.

DEFINITION 1.2. A positive function ϱ on $\boldsymbol{R}_+$ is said to be *pseudo-concave* if

$$\varrho(\lambda x) \leqslant \max(1, \lambda)\,\varrho(x)$$

for every positive λ and x. In that case we write: $\varrho \in \mathscr{P}$.

DEFINITION 1.3. Let A_0 and A_1 be two quasi-Banach spaces which are continuously embedded in a Hausdorff topological vector space $\mathscr{A}$. Let $\varrho \in \mathscr{P}$. Then $a \in \langle \bar{A}\rangle_\varrho$ if there is a sequence $\{a_\nu\}_{-\infty}^{\infty}$ with $a_\nu \in A_0 \cap A_1$ and such that

$$a = \sum_{-\infty}^{\infty} a_\nu \quad (\text{convergence in } A_0 + A_1), \tag{2}$$

(3) for every finite subset $F \subset \boldsymbol{Z}$ we have

$$\Big\|\sum_{\nu\in F}\Big(\pm\frac{2^{i\nu}a_\nu}{\varrho(2^\nu)}\Big)\Big\|_{A_i} \leqslant C \quad (i = 0, 1)$$

with C independent of F and the sign combination. As a (quasi)-norm we use

$$\|a\|_{\langle\bar{A}\rangle_\varrho} = \inf_{\{a_\nu\}} C.$$

Remark 1.1. In [3], p. 45, we have the restriction

$$\Big\|\sum_{\nu\in F}\xi_\nu\frac{2^{i\nu}a_\nu}{\varrho(2^\nu)}\Big\|_{A_i} \leqslant C \quad (i = 0, 1) \tag{4}$$

with $|\xi_\nu| \leqslant 1$. But using [5], pp. 327, 328, we get that (4) is a consequence of (3).

Before we take over a basic fact from [3], p. 46, we need a notation for the operator norm. We write

$$\|T\|_{X\to Y} = \sup_{x\neq 0}\frac{\|Tx\|_Y}{\|x\|_X},$$

where T is a continuous linear operator from X into Y.

PROPOSITION 1.1. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be any two quasi-Banach couples. If T is a continuous linear operator from $\bar{A}$ into $\bar{B}$, then T is a continuous operator from $\langle\bar{A}\rangle_\varrho$ into $\langle\bar{B}\rangle_\varrho$. For the operator norms involved we have*

$$\|T\|_{\langle\bar{A}\rangle_\varrho\to\langle\bar{B}\rangle_\varrho} \leqslant \max_{i=0,1}\|T\|_{A_i\to B_i}.$$

DEFINITION 1.4. We denote by $l_1(\sigma)$ the space of sequences $\{x_i\}_{-\infty}^{\infty}$ such that the norm

$$\|\{x_i\}\|_{l_1(\sigma)} = \sum_{-\infty}^{\infty}|x_i|\,\sigma_i < \infty.$$

We also denote by $l_\infty(\sigma)$ the space of sequences such that

$$\|\{x_i\}\|_{l_\infty(\sigma)} = \sup_i |x_i|\,\sigma_i < \infty.$$

In the next definition we give two frequently used notations.

DEFINITION 1.5. Let $\varrho \in \mathscr{P}$. Then we write

$$\varphi(x, y) = x\varrho(y/x) \tag{5}$$

and

$$\varphi^*(x, y) = 1/\varphi(x^{-1}, y^{-1}) = \frac{x}{\varrho(x/y)}. \tag{6}$$

Finally we give two of Ovchinnikov's methods. (See [4], p. 288.)

DEFINITION 1.6. Let $\bar{A} = (A_0, A_1)$ be a couple of Banach spaces which are continuously embedded in a Hausdorff topological vector space $\mathscr{A}$. Then $a \in \varphi_l(\bar{A})$ if there exists a continuous operator T and weights w_0 and w_1 such that

$$T\colon \big(l_\infty(w_0), l_\infty(w_1)\big) \to (A_0, A_1)$$

and $a = Ta_\varphi$ where

$$a = \Big\{\varphi\Big(\frac{1}{w_{0,\nu}}, \frac{1}{w_{1,\nu}}\Big)\Big\}_{-\infty}^{\infty}.$$

As a norm we use

$$\|a\|_{\varphi_l(\bar{A})} = \inf\max_{i=0,1}\|T\|_{l_\infty(w_i)\to A_i},$$

where the inf is taken over w_0, w_1 and T.

DEFINITION 1.7. Let $\bar{A} = (A_0, A_1)$ be as in Defininition 1.6. Then $a \in \varphi_u(\bar{A})$ if

$$\|a\|_{\varphi_u(\bar{A})} = \sup \|Ta\|_{l_1(\varphi^*(\sigma_0, \sigma_1))} < \infty,$$

where T is a continuous operator such that

$$T\colon (A_0, A_1) \to (l_1(\sigma_0), l_1(\sigma_1))$$

and

$$\|T\|_{A_i \to l_1(\sigma_i)} \leqslant 1 \qquad (i = 0, 1).$$

The sup is taken over σ_0, σ_1 and T.

2. Interpolation between weighted L^p-spaces. We start with the case $p_0 = p_1 = p$.

THEOREM 2.1. *We have*

$$L^p(\varphi^*(w_0, w_1)) \subset \langle L^p(w_0), L^p(w_1)\rangle_\varrho$$

where $0 < p < \infty$, $\varrho \in \mathscr{P}$ *and*

(7) $$\varrho(t) = o(\max(1, t)) \quad \textit{as} \quad t \to 0 \ \textit{or} \ \infty.$$

φ^* *is given by* (6).

Proof. Let $a \in L^p(\varphi^*(w_0, w_1))$. Put

$$e_\nu = \left\{x\colon \frac{w_0(x)}{w_1(x)} \in [2^{\nu-1}, 2^\nu[\right\}$$

and

$$a_\nu(x) = \begin{cases} a(x) & \text{if } x \in e^\nu, \\ 0 & \text{otherwise}. \end{cases}$$

First we prove that $a = \sum_{-\infty}^{\infty} a_\nu$ with convergence in $L^p(w_0) + L^p(w_1)$. Put

$$E_0 = \bigcup_{\nu<0} e_\nu, \qquad E_1 = \bigcup_{\nu \geqslant 0} e_\nu.$$

Moreover, put

$$a_{00}(x) = \begin{cases} a(x) & \text{if } x \in E_0, \\ 0 & \text{otherwise}, \end{cases}$$

and $a_{11} = a - a_{00}$. Then $a_{00} \in L^p(w_0)$ and $a_{11} \in L^p(w_1)$. Furthermore,

$$\Big\| a - \sum_{N_1}^{N_2} a_\nu \Big\|_{L^p(w_0)+L^p(w_1)} \leqslant \Big\| a_{00} - \sum_{N_1 \leqslant \nu < 0} a_\nu \Big\|_{L^p(w_0)} + \Big\| a_{11} - \sum_{0 \leqslant \nu \leqslant N_2} a_\nu \Big\|_{L^p(w_1)}$$
$$= \Big(\sum_{\nu<N_1} (\varrho(2^\nu))^p \int_{e_\nu} \Big(\frac{|a(x)|}{\varrho(2^\nu)} w_0(x) \Big)^p d\mu \Big)^{1/p} +$$
$$+ \Big(\sum_{\nu>N_2} (2^{-\nu} \varrho(2^\nu))^p \int_{e_\nu} \Big(\frac{2^\nu |a(x)|}{\varrho(2^\nu)} w_1(x) \Big)^p d\mu \Big)^{1/p}$$
$$\leqslant \big(\varrho(2^{N_1}) + 2^{-N_2+1} \varrho(2^{N_2}) \big) \|a\|_{L^p(\varphi^*(w_0, w_1))}.$$

The convergence as $N_1 \to -\infty$ and $N_2 \to +\infty$ now follows from (7).

We shall now verify (3) in Definition 1.3:

$$\Big\| \sum_{\nu \in F} \Big(\pm \frac{a_\nu}{\varrho(2^\nu)} \Big) \Big\|^p_{L^p(w_0)} = \sum_j \int_{e_j} \Big(\Big| \sum_\nu \Big(\pm \frac{a_\nu(x)}{\varrho(2^\nu)} \Big) \Big| w_0(x) \Big)^p d\mu$$
$$= \sum_j \int_{e_j} \Big(\frac{|a(x)|}{\varrho(2_j)} w_0(x) \Big)^p d\mu \leqslant \|a\|^p_{L^p(\varphi^*(w_0, w_1))}.$$

Analogously,

$$\Big\| \sum_{\nu \in F} \Big(\pm \frac{2^\nu a_\nu}{\varrho(2^\nu)} \Big) \Big\|^p_{L^p(w_1)} \leqslant 2^p \|a\|^p_{L^p(\varphi^*(w_0, w_1))}.$$

Consequently,

$$\|a\|_{\langle L^p(w_0), L^p(w_1)\rangle_\varrho} \leqslant 2 \|a\|_{L^p(\varphi^*(w_0, w_1))}.$$

The proof is complete.

For the proof of the opposite inclusion we need a trivial lemma.

LEMMA 2.1. *If* $|a| \leqslant \varphi(|a_0|, |a_1|)$, *then*

$$\|a\|_{L^p(\varphi^*(w_0, w_1))} \leqslant 2^{1/p} \max_{i=0,1} \|a_i\|_{L^p(w_i)}.$$

Proof. We have

$$(|a| \varphi^*(w_0, w_1))^p \leqslant (\varphi(|a_0|, |a_1|) \varphi^*(w_0, w_1))^p$$
$$= \Big(\varphi\Big(|a_0| w_0 \cdot \frac{1}{w_0}, |a_1| w_1 \cdot \frac{1}{w_1} \Big) \cdot \varphi^*(w_0, w_1) \Big)^p$$
$$\leqslant \max_{i=0,1} (|a_i| w_i)_p \leqslant (|a_0| w_0)^p + (|a_1| w_1)^p.$$

Thus

$$\|a\|^p_{L^p(\varphi^*(w_0, w))} \leqslant 2 \max_{i=0,1} \|a_i\|^p_{L^p(w_i)}.$$

Before we give the next result we define the subset $\mathscr{P}^{+-}$ of $\mathscr{P}$. (Cf. [2], p. 293, and [3], p. 37.)

Definition 2.1. A function ϱ in $\mathscr{P}$ is said to *belong to* $\mathscr{P}^{+-}$ if

$$\sup_x \frac{\varrho(tx)}{\varrho(x)} = o\big(\max(1,t)\big) \quad \text{as} \quad t \to 0 \text{ or } \infty.$$

Theorem 2.2. *If $\varrho \in \mathscr{P}^{+-}$ and $0 < p < \infty$, then the following inclusion holds:*

$$L^p\big(\varphi^*(w_0, w_1)\big) \supset \langle L^p(w_0), L^p(w_1)\rangle_\varrho .$$

Proof. Let $a \in \langle L^p(w_0), L^p(w_1)\rangle_\varrho$ with

$$\|a\|_{\langle L^p(w_0), L^p(w_1)\rangle_\varrho} = 1.$$

We choose a sequence $\{a_\nu\}_{-\infty}^{\infty}$ in $L^p(w_0)\cap L^p(w_1)$ such that

$$a = \sum_{-\infty}^{\infty} a_\nu \quad (\text{convergence in } L^p(w_0)+L^p(w_1)) \tag{8}$$

and

$$\Big\| \sum_{\nu\in F} \pm \frac{2^{i\nu} a_\nu}{\varrho(2^\nu)} \Big\|^p_{L^p(w_i)} \leqslant 2 \quad (i = 0, 1).$$

If we use Fubini's theorem, we obtain

$$\int_M E\Big(\Big| \sum_{\nu\in F} \pm \frac{2^{i\nu} a_\nu(x)}{\varrho(2^\nu)} \Big|^p\Big) w_i^p d\mu \leqslant 2,$$

where E stands for "expectation". Then Khintchine's inequality (see [7], p. 213) implies that

$$\int_M \Big(\sum_{\nu\in F} \Big| \frac{2^{i\nu} a_\nu}{\varrho(2^\nu)} \Big|^2 \Big)^{p/2} w_i^p d\mu \leqslant C \quad (i = 0, 1).$$

Combining Lemma 2.1 and Carlson's inequality [3], pp. 38–39, we get

$$\Big\| \sum_{\nu\in F} a_\nu \Big\|_{L^p(\varphi^*(w_0, w_1))} \leqslant C.$$

But in (8) also pointwise convergence is true μ-a.e. Thus Fatou's lemma finally gives that

$$\|a\|_{L^p(\varphi^*(w_0, w_1))} \leqslant C.$$

The proof is complete.

We shall now study the limiting case $p = \infty$. Notice that here we do not need the restriction $\varrho \in \mathscr{P}^{+-}$.

Theorem 2.3. *If $\varrho = \mathscr{P}$ and ϱ fulfils condition* (7), *then we have that*

$$L^\infty\big(\varphi^*(w_0, w_1)\big) = \langle L^\infty(w_0), L^\infty(w_1)\rangle_\varrho$$

with equivalent norms.

For the proof we need a modified version of Lemma 2.1.

Lemma 2.1′. *If $|a| \leqslant \varphi(|a_0|, |a_1|)$, then*

$$\|a\|_{L^\infty(\varphi^*(w_0, w))} \leqslant \max_{i=0,1} \|a_i\|_{L^\infty(w_i)}.$$

Proof of Theorem 2.3. The inclusion

$$L^\infty\big(\varphi^*(w_0, w_1)\big) \subset \langle L^\infty(w_0), L^\infty(w_1)\rangle_\varrho$$

can be proved in the same way as Theorem 2.1.

For the reverse inclusion let $a \in \langle L^\infty(w_0), L^\infty(w_1)\rangle_\varrho$ with

$$\|a\|_{\langle L^\infty(w_0), L^\infty(w_1)\rangle_\varrho} = 1.$$

Choose a sequence $\{a_\nu\}$ in $L^\infty(w_0)\cap L^\infty(w_1)$ such that

$$a = \sum_{-\infty}^{\infty} a_\nu \quad (\text{convergence in } L^\infty(w_0)+L^\infty(w_1)) \tag{9}$$

and

$$\sup_M \Big| \sum_{\nu\in F} \pm \frac{2^{i\nu} a_\nu(x)}{\varrho(2^\nu)} \Big| w_i(x) \leqslant 2 \quad (i = 0, 1).$$

Then

$$\sup_M \sum_{\nu\in F} \frac{2^{i\nu} |a_\nu(x)|}{\varrho(2^\nu)} w_i(x) \leqslant 2.$$

For $\varrho \in \mathscr{P}$ we can write Carlson's inequality in the following way (see [3], p. 39)

$$\Big| \sum_{\nu\in F} a_\nu \Big| \leqslant C\varphi\Big(\sum_{\nu\in F} \frac{|a_\nu|}{\varrho(2^\nu)}, \sum_{\nu\in F} \frac{2^\nu |a_\nu|}{\varrho(2^\nu)} \Big).$$

According to Lemma 2.1′ we get

$$\Big\| \sum_{\nu\in F} a_\nu \Big\|_{L^\infty(\varphi^*(w_0, w_1))} \leqslant C_1 . \tag{10}$$

In (9) also pointwise convergence is true μ-a.e. Therefore, passing to the limit in (10), we get

$$\|a\|_{L^\infty(\varphi^*(w_0, u_1))} \leqslant C_1 .$$

The proof is complete.

As a corollary of Theorems 2.1–2.3 and Proposition 1.1 we get a generalization of Stein–Weiss' interpolation theorem. (See [1], p. 115 and [6], pp. 163, 164.)

COROLLARY 2.1. *Let $\varrho \in \mathscr{P}^{+-}$ and let T be a continuous operator such that*

$$T\colon \big(L^p(M,\mu,w_0),\, L^p(M,\mu,w_1)\big) \to \big(L^q(N,\nu,\sigma_0),\, L^q(N,\nu,\sigma_1)\big)$$

where $0 < p,\ q \leqslant \infty$. Then

$$T\colon\ L^p\big(M,\mu,\varphi^*(w_0,w_1)\big) \to L^q\big(N,\nu,\varphi^*(\sigma_0,\sigma_1)\big).$$

For the operator norms the following inequality holds:

$$\|T\|_{L^p(M,\mu,\varphi^*(w_0,w_1))\to L^q(N,\nu,\varphi^*(\sigma_0,\sigma_1))} \leqslant C\max_{i=0,1}\|T\|_{L^{p_i}(M,\mu,w_i)\to L^{q_i}(N,\nu,\sigma_i)}.$$

Remark 2.2. The real interpolation functor depends on p. The complex method needs the restrictions $1 \leqslant p,\, q$ and $\varrho(t) = t^\theta$. However, the inequality between the operator norms will be sharper with the complex method than with the $\pm$ method.

We shall now interpolate between $L^{p_0}(w_0)$ and $L^{p_1}(w_1)$. We shall also specialize to $\varrho(t) = t^\theta$, $0 \leqslant \theta \leqslant 1$.

THEOREM 2.4. *With $\varrho(t) = t^\theta$, $0 \leqslant \theta \leqslant 1$, $0 < p_0, p_1 < \infty$ the following*

$$L^p(w) = \langle L^{p_0}(w_0),\, L^{p_1}(w_1)\rangle_\varrho$$

holds, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad w = w_0^{1-\theta} w_1^{\theta}.$$

Proof. Let $a \in L^p(w)$. Put

$$e_\nu = \{x\colon\ |a(x)|^{p/r} w_0^{p/p_1}(x)\cdot w_1^{-p/p_0}(x) \in [2^{\nu-1}, 2^\nu[\}$$

where

$$\frac{1}{r} = \frac{1}{p_1} - \frac{1}{p_0}.$$

Furthermore, put

$$a_\nu(x) = \begin{cases} a(x) & \text{if } x \in e_\nu, \\ 0 & \text{otherwise}. \end{cases}$$

As in the proof of Theorem 2.1 we can prove that

$$a = \sum_{-\infty}^{\infty} a_\nu \quad (\text{convergence in } L^{p_0}(w_0) + L^{p_1}(w_1)).$$

We shall now check (3) in Definition 1.3:

$$\Big\|\sum_{\nu\in F} \pm \frac{a_\nu}{\varrho(2^\nu)}\Big\|_{L^{p_0}(w_0)}^{p_0} = \sum_j \int_{e_j} \Big(\Big|\sum_{\nu\in F} \pm \frac{a_\nu(x)}{2^{\nu\theta}}\Big| w_0(x)\Big)^{p_0} d\mu$$

$$= \sum_j \int_{e_j} \Big(\frac{|a(x)|}{2^{j\theta}} w_0(x)\Big)^{p_0} d\mu \leqslant \sum_j \int_{e_j} \Big(\frac{|a(x)|}{|a(x)|^{\theta p/r}}\Big)^{p_0} w^p(x)\,d\mu = \|a\|_{L^p(w)}^p$$

where the last equality follows from the relation

$$(t\cdot t^{-\theta p/r})^{p_0} = t^p.$$

Analogously,

$$\Big\|\sum_{\nu\in F} \pm \frac{2^\nu a_\nu}{\varrho(2^\nu)}\Big\|_{L^{p_1}(w_1)}^{p_1} \leqslant 2^{p_1}\|a\|_{L^p(w)}^p.$$

Thus we have proved that

$$L^p(w) \subset \langle L^{p_0}(w_0),\, L^{p_1}(w_1)\rangle_\varrho.$$

The converse inclusion can be proved in the same way as in Theorem 2.2.

The following result is a limiting case of Theorem 2.4.

THEOREM 2.5. *With $\varrho(t) = t^\theta$, $0 \leqslant \theta \leqslant 1$, $0 < p_0, p_1 < \infty$, the following*

$$L^p(w) = \langle L^{p_0}(w_0),\, L^\infty(w_1)\rangle_\varrho$$

holds, where

$$\frac{1}{p} = \frac{1-\theta}{p_0}, \quad w = w_0^{1-\theta} w_1^{\theta}.$$

Moreover,

$$L^q(w) = \langle L^\infty(w_0),\, L^{p_1}(w_1)\rangle_\varrho \tag{11}$$

where

$$\frac{1}{q} = \frac{\theta}{p_1}, \quad w = w_0^{1-\theta} w_1^{\theta}.$$

Proof. With

$$e_\nu = \{x\colon\ \big(|a(x)|w_1(x)\big)^{1/(\theta-1)} \in [2^{\nu-1}, 2^\nu[\}$$

the inclusion

$$L^p(w) \subset \langle L^{p_0}(w_0),\, L^\infty(w_1)\rangle_\varrho$$

follows as in the proof of Theorem 2.4. The rest of the proof is essentially a repeat of corresponding parts of the proof of Theorem 2.2. We use Carlson's inequality in the form

$$\Big|\sum_{\nu\in F} a_\nu\Big| \leqslant C\Big(\sum_{\nu\in F}\Big(\frac{|a_\nu|}{\varrho(2^\nu)}\Big)^2\Big)^{(1-\theta)/2}\cdot\Big(\sum_{\nu\in F}\frac{2^\nu|a_\nu|}{\varrho(2^\nu)}\Big)^{\theta}.$$

(Cf. [3], p. 39.)

The proof of (11) is analogous.

We shall now give an improved version of Stein–Weiss' theorem. (Cf. [1], p. 120.)

COROLLARY 2.2. *Assume that* $0 < p_0, p_1, q_0, q_1 \leqslant \infty$ *and that*

$$T\colon \big(L^{p_0}(M,\mu,w_0),\ L^{p_1}(M,\mu,w_1)\big) \to \big(L^{q_0}(N,\nu,\sigma_0),\ L^{q_1}(N,\nu,\sigma_1)\big)$$

is a continuous linear operator. Then T is a continuous operator from $L^p(w)$ into $L^q(\sigma)$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

$$w = w_0^{1-\theta} w_1^{\theta}, \qquad \sigma = \sigma_0^{1-\theta}\sigma_1^{\theta}.$$

For the operator norms the following inequality holds:

$$\|T\|_{L^p(w)\to L^q(\sigma)} \leqslant C \max_{i=0,1} \|T\|_{L^{p_i}(w_i)\to L^{q_i}(\sigma_i)}.$$

Proof. We only have to combine Theorems 2.1–2.5 with Proposition 1.1.

Remark 2.3. Notice that p_0 and p_1 as well as q_0 and q_1 may be equal. We also observe that the restriction $p \leqslant q$ from the real method as well as the restriction $1 \leqslant p_0, p_1, q_0, q_1$ from the complex method are eliminated.

3. On Ovchinnikov's theorem and connections between $\langle \bar{A}\rangle_\varrho$, $\varphi_l(\bar{A})$ and $\varphi_u(\bar{A})$. In this section we shall prove that $\langle \bar{A}\rangle_\varrho = \varphi_l(\bar{A})$ for many couples $\bar{A}$. For $\varrho = \mathscr{P}^{+-}$ we also give a simple proof for the inclusion $\varphi_l(\bar{A}) \subset \varphi_u(\bar{A})$. However, we start with a proof of a weak version of Ovchinnikov's theorem. We need the restriction $\varrho \in \mathscr{P}^{+-}$ but Ovchinnikov needs only the assumption $\varrho \in \mathscr{P}$. On the other hand, our proof does not use Grothendieck's inequality.

THEOREM 3.1. *Assume that* $\varrho \in \mathscr{P}^{+-}$ *and that*

$$T\colon \big(l_\infty(w_0),\ l_\infty(w_1)\big) \to \big(l_1(\sigma_0),\ l_1(\sigma_1)\big)$$

is a continuous linear operator. Then T is a continuous operator from $l_\infty(\varphi(w_0, w_1))$ into $l_1(\varphi(\sigma_0,\sigma_1))$. For the operator norms we have

$$\|T\|_{l_\infty(\varphi(w_0,w_1))\to l_1(\varphi(\sigma_0,\sigma_1))} \leqslant C \max_{i=0,1} \|T\|_{l_\infty(w_i)\to l_1(\sigma_i)}.$$

Proof. Put $\varrho^*(t) = 1/\varrho(t^{-1})$. Clearly, $\varrho \in \mathscr{P}^{+-}$ if and only if $\varrho^* \in \mathscr{P}^{+-}$. According to Proposition 1.1 we get that T is a continuous operator from $\langle l_\infty(w_0), l_\infty(w_1)\rangle_{\varrho^*}$ into $\langle l_1(\sigma_0), l_1(\sigma_1)\rangle_{\varrho^*}$. To complete the proof we just have to combine this with Theorems 2.1–2.3.

Now we compare $\varphi_l(\bar{A})$ and $\varphi_u(\bar{A})$ with $\langle \bar{A}\rangle_\varrho$. (It is also possible to give a direct proof of the inclusion $\varphi_l(\bar{A}) \subset \varphi_u(\bar{A})$.)

THEOREM 3.2. *We have*

$$\text{(12)} \qquad \varphi_l(\bar{A}) \subset \langle \bar{A}\rangle_\varrho \quad \text{if} \quad \varrho \in \mathscr{P},$$

$$\text{(13)} \qquad \langle \bar{A}\rangle_\varrho \subset \varphi_u(\bar{A}) \quad \text{if} \quad \varrho \in \mathscr{P}^{+-}.$$

Proof. We start with (12). Let $a \in \varphi_l(\bar{A})$. Then there are weights w_0 and w_1 and a continuous operator

$$T\colon \big(l_\infty(w_0),\ l_\infty(w_1)\big) \to (A_0, A_1)$$

such that $a = Ta_\varphi$ where

$$a_\varphi = \left\{\varphi\left(\frac{1}{w_{0,\nu}}, \frac{1}{w_{1,\nu}}\right)\right\}.$$

Clearly,

$$\|a_\varphi\|_{l_\infty(\varphi^*(w_0,w_1))} = 1.$$

Thus

$$\text{(14)} \qquad a_\varphi \in l_\infty(\varphi^*(w_0,w_1)) = \langle l_\infty(w_0),\ l_\infty(w_1)\rangle_\varrho,$$

where the last equality follows from Theorem 2.3. Then Proposition 1.1 and (14) imply that

$$a = Ta_\varphi \in \langle \bar{A}\rangle_\varphi$$

and that

$$\|a\|_{\langle \bar{A}\rangle_\varrho} \leqslant C \max_{i=0,1} \|T\|_{l_\infty(w_i)\to A_i}.$$

Taking the inf over w_0, w_1 and T, we get

$$\|a\|_{\langle \bar{A}\rangle_\varrho} \leqslant C \|a\|_{\varphi_l(\bar{A})}.$$

It remains to prove (13). Let $a \in \langle \bar{A}\rangle_\varrho$ and let S be a continuous linear operator from (A_0, A_1) into $\big(l_1(\sigma_0), l_1(\sigma_1)\big)$ with

$$\|S\|_{A_i\to l_1(\sigma_i)} \leqslant 1 \qquad (i = 0, 1).$$

Then Proposition 1.1 implies that

$$\|S\|_{\langle \bar{A}\rangle_\varrho \to \langle l_1(\sigma_0), l_1(\sigma_1)\rangle_\varrho} \leqslant \max_{i=0,1} \|S\|_{A_i\to l_1(\sigma_i)} \leqslant 1.$$

If we take account of Theorem 2.2, we get

$$\|a\|_{\varphi_u(\bar{A})} = \sup \|Sa\|_{l_1(\varphi^*(\sigma_0,\sigma_1))} \leqslant C\|a\|_{\langle \bar{A}\rangle_\varrho}.$$

The proof is complete.

Now we turn to the converse of (12).

THEOREM 3.3. *Assume that $A_i = X_i'$ (duals) $(i = 0, 1)$ where $X_0 \cap X_1$ is dense in both X_0 and X_1. Moreover, assume that ϱ fulfils condition (7). Then*

$$\langle \bar{A}\rangle_\varrho \subset \varphi_l(\bar{A}).$$

Proof. Let $a \in \langle \bar{A} \rangle_\varrho$ with $\|a\|_{\langle \bar{A} \rangle_\varrho} = 1$. We choose a sequence $\{a_\nu\}$ in $A_0 \cap A_1 = (X_0 + X_1)'$ (see [1], p. 32) such that

$$a = \sum_{-\infty}^{\infty} a_\nu \quad \text{(convergence in } A_0 + A_1)$$

and

$$\Big\| \sum_{\nu \in F} \Big(\pm \frac{2^{i\nu} a_\nu}{\varrho(2^\nu)} \Big) \Big\|_{A_i} \leqslant 2 \quad (i = 0, 1).$$

Let $\lambda = \{\lambda_\nu\} \in l_\infty(w_0)$, where $w_{0,\nu} = \varrho(2^\nu)$. Furthermore, let $x_0 \in X_0$ and let $\langle\ ,\ \rangle$ denote duality between A_0 and X_0. With a suitable choice of signs we get

$$\sum_{\nu \in F} |\langle \lambda_\nu a_\nu, x_0 \rangle| = \sum_{\nu \in F} (\pm \langle \lambda_\nu a_\nu, x_0 \rangle) = \Big\langle \sum_{\nu \in F} (\pm \lambda_\nu a_\nu), x_0 \Big\rangle$$
$$\leqslant \Big\| \sum_{\nu \in F} \Big(\pm \lambda_\nu \varrho(2^\nu) \frac{a_\nu}{\varrho(2^\nu)} \Big) \Big\|_{A_0} \|x_0\|_{X_0} \leqslant \|\lambda\|_{l_\infty(w_0)} \cdot 2 \cdot \|x_0\|_{X_0}.$$

Thus $\sum_{|\nu| \leqslant n} \lambda_\nu a_\nu$ is weak* convergent as $n \to \infty$. Denote the limit by $s = \sum \lambda_\nu a_\nu$. Then $s \in A_0$ and

$$|\langle s, x_0 \rangle| \leqslant \|\lambda\|_{l_\infty(w_0)} 2 \|x_0\|_{X_0}.$$

Consequently,

$$\|s\|_{A_0} \leqslant 2 \|\lambda\|_{l\infty(w_0)}. \tag{15}$$

If $\lambda \in l_\infty(w_0)$ we can define a continuous, linear operator T from $l_\infty(w_0)$ into A_0 via

$$T\lambda = s = \sum \lambda_\nu a_\nu.$$

We can make an analogous definition of T on $l_\infty(w_1)$ where $w_{1,\nu} = 2^{-\nu} \varrho(2^\nu)$. Since $X_0 \cap X_1$ is dense in both X_0 and X_1 the two definitions of T will be consistent on $l_\infty(w_0) \cap l_\infty(w_1)$. T is extended to $l_\infty(w_0) + l_\infty(w_1)$ in the standard way. With the weights w_0 and w_1 given above we have $a_\varphi = \{1\}$. We split a_φ as $a_\varphi = \varepsilon + \sigma$, where $\varepsilon = \{\varepsilon_\nu\}$ with

$$\varepsilon_\nu = \begin{cases} 1 & \text{if} \quad \nu < 0, \\ 0 & \text{if} \quad \nu \geqslant 0. \end{cases}$$

Then

$$Ta_\varphi = \sum \varepsilon_\nu a_\nu + \sum \sigma_\nu a_\nu.$$

Since ϱ fulfils condition (7), $\sum_{\nu < 0} a_\nu$ converges (strongly) in A_0 and $\sum_{\nu \geqslant 0} a_\nu$ converges in A_1. Thus

$$Ta_\varphi = a$$

and

$$\|a\|_{\varphi_l(\bar{A})} \leqslant \max_{i=0,1} \|T\|_{l_\infty(w_i) \to A_i} \leqslant 2$$

where the last inequality follows from (15). The proof is complete.

We intend to give a slightly modified version of Theorem 3.3. We do not know if this new setting is more general than Theorem 3.3.

Before we state the theorem we shall introduce vector-valued Banach limits.

We denote by $l_\infty(A)$ all bounded sequences $s = \{s_n\}$ with s_n in the Banach space A. A continuous linear operator L is called a *vector-valued Banach limit* on A if

$$L\colon l_\infty(A) \to A$$

and

$$L(s) = \lim_{n \to \infty} s_n \quad \text{if the sequence } s \text{ is convergent.}$$

THEOREM 3.3′. *Assume that L is a continuous operator from $l_\infty(A_0) + l_\infty(A_1)$ into $A_0 + A_1$. Moreover, assume that the restriction of L to $l_\infty(A_i)$ is a vector-valued Banach limit on A_i $(i = 0, 1)$. Furthermore, assume that ϱ fulfils condition* (7).

Then

$$\langle \bar{A} \rangle_\varrho \subset \varphi_l(\bar{A}).$$

Proof. Let $a \in \langle \bar{A} \rangle_\varrho$ with $\|a\|_{\langle \bar{A} \rangle_\varrho} = 1$. We choose a sequence $\{a_\nu\}$ as in the previous proof. We also choose the same weights w_0 and w_1 as before. Then we can define a continuous linear operator T from $l_\infty(w_0)$ into A_0 via

$$T\lambda = L\Big(\Big\{ \sum_{|\nu| \leqslant n} \lambda_\nu a_\nu \Big\}\Big)$$

where $\lambda \in l_\infty(w_0)$. Notice that $s = \{s_n\}$ where

$$s_n = \sum_{|\nu| \leqslant n} \lambda_\nu a_\nu$$

is bounded in A_0. Analogously we define T on $l_\infty(w_1)$ via

$$T\lambda = L\Big(\Big\{ \sum_{|\nu| \leqslant n} \lambda_\nu a_\nu \Big\}\Big)$$

where $\lambda \in l_\infty(w_1)$. The two definitions are consistent on $l_\infty(w_0)\cap l_\infty(w_1)$. In fact, if $\lambda \in l_\infty(w_0)\cap l_\infty(w_1)$ then

$$\Big\{\sum_{|\nu|\leqslant n} \lambda_\nu a_\nu\Big\}$$

is bounded in A_0+A_1. As in the previous proof, $a_\varphi = \{1\}$ and we also split it in the same way:

$$a_\varphi = \varepsilon+\sigma.$$

We get

$$Ta_\varphi = T\varepsilon+T\sigma = L\Big(\Big\{\sum_{|\nu|\leqslant n} \varepsilon_\nu a_\nu\Big\}\Big)+L\Big(\Big\{\sum_{|\nu|\leqslant n} \sigma_\nu a_\nu\Big\}\Big) = \sum_{-\infty}^{-1} a_\nu+\sum_{0}^{\infty} a_\nu$$

where the last equality follows from the convergence of $\sum\limits_{-\infty}^{-1} a_\nu$ in A_0 and the convergence of $\sum\limits_{0}^{\infty} a_\nu$ in A_1 (see condition (7)). The rest of the proof follows exactly as in the proof of Theorem 3.3.

Remark 3.1. If $\bar{A} = (A_0, A_1) = (X_0', X_1')$ with $X_0\cap X_1$ dense in both X_0 and X_1, then we can find a vector-valued Banach limit L, which fulfils the assumptions in Theorem 3.3′. In fact, let f be a usual scalar-valued Banach limit. Then we first define two vector-valued Banach limits L_i on A_i via

$$\langle L_i(s), x\rangle_i = f(\{\langle s_n, x\rangle_i\}),$$

where $s = \{s_n\}$ is a bounded sequence in A_i and $x \in X_i$ and $\langle\, ,\,\rangle_i$ denotes duality between A_i and X_i ($i = 0, 1$). However, $X_0\cap X_1$ is dense in both X_0 and X_1. Thus L_0 and L_1 coincide if $s_n \in A_0\cap A_1$.

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