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Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions

by

KENNETH F. ANDERSEN[1] (Edmonton, Alta) and
BENJAMIN MUCKENHOUPT[2] (New Brunswick, N.J.)

Abstract. The pairs of nonnegative weight functions (U, V) for which the modified Hardy operator $P_\eta f(x) = x^{-\eta}\int_0^x f(t)\,dt$, η real, is of weak type (p, q) are characterized. Dual results for the operator $Q_\eta f(x) = x^{-\eta}\int_x^\infty f(t)\,dt$ are given. These results complement the classical (strong) Hardy inequalities and their generalizations considered by Artola, Talenti, Tomaselli and Muckenhoupt. New weighted weak type inequalities for Hilbert transforms and maximal functions are derived as applications of these results.

1. Introduction. Let $1 \leqslant p, q < \infty$ and suppose $U(x)$, $V(x)$ are nonnegative extended real valued functions on $(0, \infty)$. We say that (U, V) is a *strong type* (p, q) *weight pair* for the linear operator T if there is a finite constant C independent of f such that

$$\left(\int_0^\infty |Tf(x)|^q U(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty |f(x)|^p V(x)\,dx\right)^{1/p}, \tag{1.1}$$

and we say that (U, V) is a *weak type* (p, q) *weight pair* for T if there is a finite constant C independent of f such that for all $y > 0$

$$\left(\int_{\{x:\,|Tf(x)|>y\}} U(x)\,dx\right)^{1/q} \leqslant Cy^{-1}\left(\int_0^\infty |f(x)|^p V(x)\,dx\right)^{1/p}. \tag{1.2}$$

The smallest choice of constants C in (1.1) and (1.2), called the strong and weak norms of T, are denoted $\|T\|_s$, $\|T\|_w$, respectively. It is well known that (1.1) implies (1.2); moreover, $\|T\|_w \leqslant \|T\|_s$.

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In this paper we shall be concerned with norm inequalities for operators T of the form P_η or Q_η where for real η,

$$P_\eta f(x) = x^{-\eta}\int_0^x f(t)\,dt, \qquad Q_\eta f(x) = x^{-\eta}\int_x^\infty f(t)\,dt.$$

These operators are important in analysis and have been widely studied.

Hardy [7], p. 244, first studied inequalities of the form (1.1) with $U(x) = x^{a-1}$, $V(x) = x^{a+p-1}$, $p = q > 1$, for the operator P_0 and its dual Q_0. His results, known as *Hardy's inequalities* state that (x^{a-1}, x^{a+p-1}) is a strong type (p, p) weight pair for P_0 if and only if $a < 0$, and dually, (x^{a-1}, x^{a+p-1}) is a strong type (p, p) weight pair for Q_0 if and only if $a > 0$. Moreover, the norms are given by $\|P_0\|_s = -p/a$ and $\|Q_0\|_s = p/a$.

The problem of determining those pairs (U, V) for which P_0 and Q_0 are of strong type (p, p) was solved by Artola [3], Talenti [12] and Tomaselli [13]. Recently a new proof of their results was given by Muckenhoupt [9]. Combining the idea of that proof with a technique used by Flett [6] results in the following theorems which have also been proved by Bradley [4].

THEOREM A. *If $1 \leqslant p \leqslant q < \infty$, then (U, V) is a strong type (p, q) weight pair for P_0 if and only if there is a constant B such that for all $r > 0$*

$$\Big(\int_r^\infty U(x)\,dx\Big)^{1/q}\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \leqslant B. \tag{1.3}$$

Moreover, if B denotes the smallest constant in (1.3), then $B \leqslant \|P_0\|_s \leqslant q^{1/q}(q')^{1/p'}B$.

THEOREM B. *If $1 \leqslant p \leqslant q < \infty$, then (U, V) is a strong type (p, q) weight pair for Q_0 if and only if there is a constant B such that for all $r > 0$*

$$\Big(\int_0^r U(x)\,dx\Big)^{1/q}\Big(\int_r^\infty V(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \leqslant B. \tag{1.4}$$

Moreover, if B denotes the smallest constant in (1.4), then $B \leqslant \|Q_0\|_s \leqslant q^{1/q}(q')^{1/p'}B$.

Here and throughout the paper, $1/p + 1/p' = 1$, $0\cdot\infty$ is taken as 0, and for $p = 1$ integrals of the form appearing in (1.3) and (1.4) have the usual interpretation, for example, the second factor in (1.3) is taken as $\operatorname{ess\,sup}_{[0,r]}[1/V(x)]$ when $p = 1$.

The corresponding weak type problems for P_η and Q_η are treated in this paper. As an application of our results we derive new weighted weak type inequalities for the Hilbert transformation and the Hardy–Littlewood maximal function.

From (1.1) we see that $(U(x), V(x))$ is a strong type (p, q) weight pair for P_η if and only if $(x^{-\eta q}U(x), V(x))$ is a strong type (p, q) weight pair for P_0. Thus Theorem A also contains the characterization of strong type (p, q) weight pairs for P_η, η arbitrary. The analogue holds for Q_η. The matter is not so simple for weak type however, for no such direct reduction from (1.2) seems possible. Indeed, as we shall see, weak inequalities for P_η, $\eta \leqslant 0$ are more easily derived and the conditions on the weights have a different appearance than those for $\eta > 0$; a similar situation prevails for Q_η with the roles $\eta < 0$ and $\eta > 0$ reversed.

We now state our main results for P_η which will be proved in Sections 2, 3 and 4.

THEOREM 1. *Suppose $1 \leqslant p \leqslant q < \infty$ and $\eta \leqslant 0$. Then (U, V) is a weak type (p, q) weight pair for P_η if and only if*

$$B(\eta) = \sup_{r>0} r^{-\eta}\Big(\int_r^\infty U(x)\,dx\Big)^{1/q}\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \tag{1.5}$$

is finite; indeed, $\|P_\eta\|_w = B(\eta)$.

THEOREM 2. *Suppose $1 \leqslant p \leqslant q < \infty$, $\eta > 0$ and let*

$$B(\eta; a) = \sup_{r>0}\Big(\int_r^\infty (r/x)^a\big(U(x)/x^{\eta q}\big)\,dx\Big)^{1/q}\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{1/p'}. \tag{1.6}$$

If $B(\eta; a)$ is finite for some $a > 0$, then (U, V) is a weak type (p, q) weight pair for P_η. Conversely, if (U, V) is a weak type (p, q) weight pair for P_η, then $B(\eta; a)$ is finite for all $a > 0$. Furthermore,

$$[a/(\eta q + a)]^{1/q}B(\eta; a) \leqslant \|P_\eta\|_w \leqslant [(\eta q + a)/\eta]^{1/q}(q')^{1/p'}B(\eta; a).$$

COROLLARY 1. *Suppose $1 < p \leqslant q < \infty$, $\eta > 0$. The following are equivalent:*

(a) *(x^{a-1}, x^{a-1}) is a weak type (p, q) weight pair for P_η.*

(b) *(x^{a-1}, x^{a-1}) is a strong type (p, q) weight pair for P_η.*

(c) *$a < p$ and $a(p-q) = pq(\eta - 1)$.*

COROLLARY 2. *If $1 \leqslant p < \infty$ and $W(x)$ is nonnegative and nonincreasing on $(0, \infty)$, then (W, W) is a weak type (p, p) weight pair for P_1.*

It is not difficult to see that *weak* and *strong* do not coincide for P_η (or Q_η) in general. For example, (x^{a-1}, x^{a-1}) is a weak type (1,1) pair for P_1 if $a \leqslant 1$, but a strong type (1,1) pair for P_1 only if $a < 1$; the pair $(x/(\log 1 + x), 1 + x)$ is a weak, though not strong, type (2,2) weight pair for P_1. Other examples can be given. On the other hand we have the following result which is of particular interest when $\eta = 1$.

THEOREM 3. *If $1 < p < \infty$, $\eta > 0$ and $W(x)$ is nonnegative and measurable on $(0, \infty)$, then $(W(x), x^{p(1-\eta)}W(x))$ is a weak type (p, p) weight pair* *for P_η if and only if $(W(x), x^{p(1-\eta)}W(x))$ is a strong type (p, p) weight pair for P_η.*

Since $(Q_\eta f)(x) = (P_{-\eta}g)(1/x)$ where $g(t) = t^{-2}f(1/t)$ and hence also

$$\int\limits_{\{x:|(Q_\eta f)(x)|>y\}} U(x)\,dx = \int\limits_{\{t:|(P_{-\eta}g)(t)|>y\}} t^{-2}U(1/t)\,dt,$$

it follows that $(U(x), V(x))$ is a weak type (p, q) weight pair for Q_η with norm C if and only if $(x^{-2}U(1/x), x^{2(p-1)}V(1/x))$ is a weak type (p, q) weight pair for $P_{-\eta}$ with norm C. Thus we have the following dual results for Q_η.

THEOREM 4. *Suppose $1 \leqslant p \leqslant q < \infty$ and $\eta \geqslant 0$. Then (U, V) is a weak type (p, q) weight pair for Q_η if and only if*

$$B(\eta) = \sup_{r>0} r^{-\eta}\Big(\int_0^r U(x)\,dx\Big)^{1/q}\Big(\int_r^\infty V(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \tag{1.7}$$

is finite; indeed, $\|Q_\eta\|_w = B(\eta)$.

COROLLARY 3. *If $1 \leqslant p \leqslant q < \infty$, $\eta \geqslant 0$, then (x^{a-1}, x^{a+p-1}) is a weak type (p, q) weight pair for Q_η if and only if $p - q = \eta = 0 < a$.*

COROLLARY 4. *If $1 \leqslant p < \infty$ and $W(x)$ is nonnegative and nondecreasing on $(0, \infty)$, then $(W(x), x^pW(x))$ is a weak type (p, p) weight pair for Q_0.*

THEOREM 5. *Suppose $1 \leqslant p \leqslant q < \infty$, $\eta < 0$ and let*

$$B(\eta; a) = \sup_{r>0}\Big(\int_0^r (x/r)^a\big(U(x)/x^{\eta q}\big)dx\Big)^{1/q}\Big(\int_r^\infty V(x)^{-1/(p-1)}dx\Big)^{1/p'}. \tag{1.8}$$

If $B(\eta; a)$ is finite for some $a > 0$, then (U, V) is a weak type (p, q) weight pair for Q_η. Conversely, if (U, V) is a weak type (p, q) weight pair for Q_η, then $B(\eta; a)$ is finite for all $a > 0$. Furthermore,

$$[a/(a-\eta q)]^{1/q}B(\eta; a) \leqslant \|Q_\eta\|_w \leqslant [(\eta q - a)/\eta]^{1/q}(q')^{1/p'}B(\eta; a).$$

THEOREM 6. *If $1 < p < \infty$, $\eta < 0$, and $W(x)$ is nonnegative and measurable on $(0, \infty)$, then $(W(x), x^{p(1-\eta)}W(x))$ is a weak type (p, p) weight pair for Q_η if and only if $(W(x), x^{p(1-\eta)}W(x))$ is a strong type (p, p) weight pair for Q_η.*

The Hilbert transformation H and the Hardy–Littlewood maximal function operator M are defined for locally integrable functions f by

$$(Hf)(x) = \mathrm{P.V.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,dt$$

and

$$(Mf)(x) = \sup_{y\neq x}\frac{1}{y-x}\int_x^y |f(t)|\,dt.$$

It is known [5], [8], [10] that $(w(x), w(x))$ is a weak type (p, p) weight pair for either of these operators if and only if w satisfies the A_p condition, that is, there exists a constant K_p such that for all $-\infty < a < b < \infty$,

$$\Big(\int_a^b w(x)\,dx\Big)\Big(\int_a^b w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant K_p(b-a)^p.$$

As a first application of Theorems 1 and 2 we will prove, in Section 6, a mixed weak type inequality for H and M. For an operator T, a mixed weak type (p, p) inequality is an inequality of the form

$$\int\limits_{\{x:\,|Q(x)(Tf)(x)|>y\}} R(x)\,dx \leqslant (C/y^p)\int |f(x)|^pV(x)\,dx \tag{1.9}$$

where $Q(x)$, $R(x)$ and $V(x)$ are nonnegative functions and C is independent of f. Inequalities of this type are important since they are of the form needed to apply the interpolation with change of measures Theorem given in [11]. It seems that this approach may be the only way to prove two weight function norm inequalities of the form

$$\int |(Tf)(x)|^pU(x)\,dx \leqslant C\int |f(x)|^pV(x)\,dx$$

for the most general possible $U(x)$ and $V(x)$.

Since the strong type inequality corresponding to (1.9) is

$$\int |(Tf)(x)|^pQ(x)^pR(x)\,dx \leqslant C\int |f(x)|^pV(x)\,dx,$$

it would be natural to conjecture that if (1.9) were true for a triple $(Q(x), R(x), V(x))$ of weight functions, then it would also be true for $(Q_1(x), R_1(x), V(x))$ provided only that $Q_1(x)^pR_1(x) = Q(x)^pR(x)$ and $Q_1(x)$ is reasonably smooth. Theorem 7 shows that even in a simple case this is not true and suggests that general mixed weak type inequality theorems may be quite complicated.

THEOREM 7. *Let $w(x)$ satisfy the A_1 condition and suppose d is a real number, $d \neq 1$. If T denotes either the Hilbert transformation H or the Hardy–Littlewood maximal function operator M, then there is a constant C_d such that for all $y > 0$*

$$\int\limits_{\{x:\,|x|^d|(Tf)(x)|>y\}} |x|^{-d}w(x)\,dx \leqslant (C_d/y)\int_{-\infty}^{\infty}|f(x)|\,w(x)\,dx. \tag{1.10}$$

Theorem 7 fails for $d = 1$, for if f is the characteristic function of $[0,1]$ and $w(x) \equiv 1$, then $|(Tf)(x)| \geqslant 1/x$ for $x \geqslant 1$ so that the integral on the left of (1.10) diverges for every choice of y, $y < 1$.

As a second application of Theorem 2 we will prove, in Section 7, the following theorem concerning the Hilbert transformation of an even function which is given for $x > 0$ by

$$(H_e f)(x) = \text{P.V.}\frac{2}{\pi}\int_0^\infty \frac{xf(t)}{t^2 - x^2}\,dt.$$

THEOREM 8. *Let $w(x)$ be nonnegative and measurable on $(0, \infty)$. If for some $\varepsilon > 0$ there exists a constant K_ε such that*

$$\text{(1.11)}\qquad \Big(\int_a^b (a/x + x/b)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/xw(x)\big)\Big) \leqslant K_\varepsilon\big((b^2 - a^2)/ab\big)$$

for all $0 < a < b < \infty$, then $(w(x), w(x))$ is a weak type $(1,1)$ weight pair for H_e. Conversely, if $(w(x), w(x))$ is a weak type $(1,1)$ weight pair for H_e, then (1.11) holds for all $\varepsilon > 0$.

H_e was studied previously in [1]. It was asserted there (the case $p = 1$ of Theorem 2) that the condition

$$\text{(1.12)}\qquad \Big(\int_a^b xw(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/w(x)\big)\Big) \leqslant K(b^2 - a^2)$$

is necessary for $(w(x), w(x))$ to be a weak type $(1, 1)$ weight pair for H_e. Unfortunately, this is not the case as is shown by the example $w(x) = x^{-2}$ which satisfies (1.11) but not (1.12). That (1.12) is in fact a sufficient condition as conjectured in [1] is already implies by the case $d = 1/2$ of Theorem 7 since (1.12) shows that $w(|x|^{1/2})$ satisfies the A_1 condition while $(H_e f)(x) = x(Hg)(x^2)$, where $g(t) = t^{-1/2} f(t^{1/2})$ if $t > 0$ and $g(t) = 0$ otherwise. Of course, the assertions of [1] and [2] concerning the periodic and discrete analogues of H_e, in the case $p = 1$, require the analogous corrections.

The proofs of Theorem 7 and 8 both require the following result, also of independent interest, for a local analogue of the Hilbert transformation defined for $x > 0$ by

$$(Lf)(x) = \text{P.V.}\frac{1}{\pi}\int_{x/2}^{2x} \frac{f(t)}{t - x}\,dt.$$

If d is any real number, let $(L_d f)(x) = x^d (Lf)(x)$.

LEMMA 1. *Let $w(x)$ be nonnegative and measurable on $(0, \infty)$. If $1 \leqslant p$*

$< \infty$ and d is real, then $(x^{-pd} w(x), w(x))$ is a weak type (p, p) weight pair for L_d if and only if w satisfies the local A_p condition, that is:

There exists a constant K_p such that

$$\text{(1.13)}\qquad \Big(\int_a^b w(x)\,dx\Big)\Big(\int_a^b w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant K_p (b - a)^p$$

for all a, b with $0 < a < b \leqslant 2a$.

If $1 < p < \infty$, then (1.13) *is also a necessary and sufficient condition in order that $(w(x), w(x))$ be a strong type (p, p) weight pair for L.*

Lemma 1 will be proved in Section 5.

2. Proof of Theorem 1. We begin with the necessity part. Let $r > 0$ and put

$$h(r) = \Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{1/p'},$$

$$B_r(\eta) = r^{-\eta}\Big(\int_r^\infty U(x)\,dx\Big)^{1/q}\Big(\int_0^r V(x)^{-1(p-1)}\,dx\Big)^{1/p'}.$$

If $h(r) = 0$, then $B_r(\eta) = 0$ by convention. If $h(r) = \infty$, $V(x)^{-1/p}$ is not in $L^{p'}$ on $(0, r)$. Then there is a nonnegative $g(x)$ in L^p on $(0, r)$ with $g(x)V(x)^{-1/p}$ nonintegrable on $(0, r)$. If $f(x) = g(x)V(x)^{-1/p}$ on $(0, r)$ and $f(x) = 0$ elsewhere, then $P_\eta f(x) = \infty$ for $x \geqslant r$ so that the weak type inequality (1.2) forces $\int_r^\infty U(x)\,dx = 0$. Thus $B_r(\eta) = 0$ in this case also. Suppose then that $0 < h(r) < \infty$. If $p = 1$, let $\varepsilon > 0$ and select a set E of positive measure $|E|$, $E \subset (0, r)$, such that $V(x) \leqslant \varepsilon + \operatorname*{ess\,inf}_{[0,r]} V(t)$ for all $x \in E$. If f is the characteristic function of E and $x > r$, then $P_\eta f(x) \geqslant |E| r^{-\eta}$. Hence the weak type inequality (1.2) implies

$$\int_r^\infty U(x)\,dx \leqslant \Big(\|P_\eta\|_w r^\eta \int_E V(x)\,dx\Big)^q$$
$$\leqslant \|P_\eta\|_w^q r^{\eta q}\big(\varepsilon + \operatorname*{ess\,inf}_{[0,r]} V(t)\big)^q,$$

and since $\varepsilon > 0$ was arbitrary, we obtain

$$\text{(2.1)}\qquad \Big(\int_r^\infty U(x)\,dx\Big)\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{q/p'} \leqslant \|P_\eta\|_w^q r^{\eta q}.$$

If $p > 1$, (2.1) may be derived similarly by taking $f(x) = V(x)^{-1/(p-1)}$ on $(0, r)$ and $f(x) = 0$ elsewhere. Thus, in any case, we have $B_r(\eta) \leqslant \|P_\eta\|_w$ which yields $B(\eta) \leqslant \|P_\eta\|_w$ as required.

For the sufficiency part, it suffices to consider $f(x) \geqslant 0$. In this case $P_\eta f(x)$ is nondecreasing so that $\{x: P_\eta f(x) > y\} = (r, \infty)$, where r satisfies $r^{-\eta}\int_0^r f(x)\,dx = y$. Hence, the definition of $B(\eta)$ and Hölder's inequality yields

$$\int\limits_{\{x: P_\eta f(x)>y\}} U(x)\,dx \leqslant \big(B(\eta)r^\eta\big)^q\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{-q/p'}$$
$$= \big(B(\eta)/y\big)^q\Big(\int_0^r f(x)\,dx\Big)^q\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{-q/p'}$$
$$\leqslant \big(B(\eta)/y\big)^q\Big(\int_0^r f(x)^p V(x)\,dx\Big)^{q/p}$$

which shows $\|P_\eta\|_w \leqslant B(\eta)$ as required. The proof of Theorem 1 is complete.

3. Proof of Theorem 2. Suppose first that (U, V) is a weak type weight pair for P_η. An argument similar to that which lead to (2.1) in the proof of Theorem 1 shows that in the present case we have instead

$$\Big(\int_r^y U(x)\,dx\Big)\Big(\int_0^r V(x)^{-1/(p-1)}\,dx\Big)^{q/p'} \leqslant \|P_\eta\|_w^q y^{\eta q} \qquad (r < y).$$

Multiplying this by $y^{-\eta q-a-1}$, $a > 0$, integrating the result over $y \in (r, \infty)$ and applying Fubini's theorem on the left side yields $B(\eta; a) \leqslant [(\eta q + a)/a]^{1/q}\|P_\eta\|_w$ as required. This proves the necessity part.

For the sufficiency part, fix $a > 0$ such that $B(\eta; a) < \infty$ and to simplify the notation let $B = B(\eta; a)$,

$$H(x) = \int_x^\infty (1/t)^a\big(U(t)/t^{\eta q}\big)\,dt \quad \text{and} \quad h(x) = \Big(\int_0^x V(t)^{-1/(p-1)}\,dt\Big)^{1/p'},$$

(with the usual interpretation if $p = 1$). If $x_1 = \sup\{x: h(x) < \infty\}$, then $B < \infty$ shows that $U(x) = 0$ a.e. for $x \geqslant x_1$. We first prove the desired weak type inequality for nonnegative step functions $f(x)$ with compact support in $(0, \infty)$. For such f, $P_\eta f(x)$ is nonnegative, continuous and if $E(y) = \{x: P_\eta f(x) > y\}$, then

$$E(y) \cap (0, x_1) = \bigcup_{k=1}^n (a_k, b_k)$$

where

$$0 < a_1 < \ldots < b_k \leqslant a_{k+1} < \ldots < b_n < \infty, \quad k = 1, \ldots, n-1.$$

We begin by deriving an estimate to be used later. For $x \in [a_k, b_k]$, $x^\eta \leqslant y^{-1}\int_0^x f(t)\,dt$, hence if $p > 1$ and $0 < h(t)V(t) < \infty$ a.e. on $[0, x]$, then Hölder's inequality yields

$$x^{\eta q} \leqslant y^{-q}\Big(\int_0^x f(t)^p V(t)\,dt\Big)^{(q-p)/p}\Big(\int_0^x f(t)^p h(t)V(t)\,dt\Big)\Big(\int_0^x h(t)^{-p'/q}V(t)^{-1/(p-1)}\,dt\Big)^{q/p'}.$$

If the range of integration is enlarged in the first integral and the integration carried out in the third we obtain

$$(3.1) \qquad x^{\eta q} \leqslant y^{-q}\Big(\int_0^{b_n} f(t)^p V(t)\,dt\Big)^{(q-p)/p}\Big(\int_0^x f(t)^p h(t)V(t)\,dt\Big)\big((q')^{q/p'}h(x)^{q-1}\big);$$

moreover, it is easily seen that this inequality also holds for $p = 1$, with $(q')^{q/p'}$ taken to be 1 in this case. The following observations show that (3.1) holds without the assumption that $0 < h(t)V(t) < \infty$ a.e. On the subset E of $[0, x]$ where $V(t) = \infty$, $f(t) = 0$ a.e. and the set E may therefore be dropped from the region of integration before Hölder's inequality is applied. By the definition of x_1, $V(t) > 0$ and $h(t) < \infty$ a.e. on $[0, x]$ and the set where $h(t) = 0$ is a subset of E.

We now proceed to estimate $\int_{E(y)} U(x)\,dx$. Integrating by parts, using the definition of B and integrating by parts again yields

$$\int_{a_k}^{b_k} U(x)\,dx = -x^{\eta q+a}H(x)\Big|_{a_k}^{b_k} + (\eta q + a)\int_{a_k}^{b_k} x^{\eta q+a-1}H(x)\,dx$$
$$\leqslant -x^{\eta q+a}H(x)\Big|_{a_k}^{b_k} + (\eta q + a)B^q\int_{a_k}^{b_k} x^{\eta q-1}h(x)^{-q}\,dx$$
$$= \big[-x^{\eta q+a}H(x) + B^q\big((\eta q + a)/\eta q\big)x^{\eta q}h(x)^{-q}\big]_{a_k}^{b_k} +$$
$$+ B^q\big((\eta q + a)/\eta\big)\int_{a_k}^{b_k} x^{\eta q}h(x)^{-q-1}\,dh(x).$$

Upon summing over k, it follows that $\int_{E(y)} U(x)\,dx$ is bounded by the sum of

$$(3.2) \qquad \sum_{k=1}^n \big[-x^{\eta q+a}H(x) + B^q\big((\eta q + a)/\eta q\big)x^{\eta q}h(x)^{-q}\big]_{a_k}^{b_k}$$

and

$$(3.3) \qquad B^q\big((\eta q + a)/\eta\big)\sum_{k=1}^n \int_{a_k}^{b_k} x^{\eta q}h(x)^{-q-1}\,dh(x).$$

The sum (3.2) is estimated as follows. Since $H(x)$ is nonincreasing,

$$\sum_{k=1}^{n} -x^{\eta q+a}H(x)\Big|_{a_k}^{b_k} = a_1^{\eta q+a}H(a_1) + \sum_{k=1}^{n-1}[a_{k+1}^{\eta q+a}H(a_{k+1}) - b_k^{\eta q+a}H(b_k)] -$$
$$- b_n^{\eta q+a}H(b_n) \leqslant a_1^{\eta q+a}H(a_1) + \sum_{k=1}^{n-1}[a_{k+1}^{\eta q+a} - b_k^{\eta q+a}]H(a_{k+1}),$$

where we have discarded the last (negative) term, and from the inequality

$$a_{k+1}^{\eta q+a} - b_k^{\eta q+a} \leqslant \big((\eta q+a)/\eta q\big)a_{k+1}^{a}(a_{k+1}^{\eta q} - b_k^{\eta q}),$$

we then obtain

$$\sum_{k=1}^{n} -x^{\eta q+a}H(x)\Big|_{a_k}^{b_k} \leqslant B^q\Big\{a_1^{\eta q}h(a_1)^{-q} + \big((\eta q+a)/\eta q\big)\sum_{k=1}^{n-1}[a_{k+1}^{\eta q} - b_k^{\eta q}]h(a_{k+1})^{-q}\Big\}.$$

Hence, (3.2) is bounded by

$$B^q\Big\{-(a/\eta q)a_1^{\eta q}h(a_1)^{-q} + \big((\eta q+a)/\eta q\big)\Big(\sum_{k=1}^{n-1} b_k^{\eta q}[h(b_k)^{-q} -$$
$$- h(a_{k+1})^{-q}] + b_n^{\eta q}h(b_n)^{-q}\Big)\Big\}.$$

Deleting the first (negative) term, dominating $b_k^{\eta q}$ by (3.1) and using the inequality

$$h(b_k)^{-q} - h(a_{k+1})^{-q} \leqslant qh(b_k)^{1-q}\big(h(b_k)^{-1} - h(a_{k+1})^{-1}\big)$$

shows that (3.2) is bounded by

$$I\Big\{\sum_{k=1}^{n-1}\Big(\int_0^{b_k} f(t)^p V(t)\,dt\Big)\big(h(b_k)^{-1} - h(a_{k+1})^{-1}\big) + \tag{3.4}$$
$$+ q^{-1}\Big(\int_0^{b_n} f(t)^p V(t)\,dt\Big)h(b_n)^{-1}\Big\},$$

where for convenience we have set

$$I = B^q\big((\eta q+a)\eta\big)/(q')^{q/p'}y^{-q}\Big(\int_0^{b_n} f(t)^p V(t)\,dt\Big)^{(q-p)/p}. \tag{3.5}$$

Returning to (3.3), this may be estimated by dominating $x^{\eta q}$ by (3.1) and then integrating by parts to obtain the bound

$$I\Big\{\sum_{k=1}^{n}\Big(-h(x)^{-1}\int_0^x f(t)^p h(t) V(t)\,dt\Big|_{a_k}^{b_k} + \int_{a_k}^{b_k} f(x)^p V(x)\,dx\Big)\Big\}. \tag{3.6}$$

Summing (3.4) and (3.6) then shows that $\int_{E(y)} U(x)\,dx$ is bounded by

$$I\Big\{\sum_{k=1}^{n-1}\Big(\int_{b_k}^{a_{k+1}} f(t)^p h(t)V(t)\,dt\Big)h(a_{k+1})^{-1} + \sum_{k=1}^{n}\int_{a_k}^{b_k} f(t)^p V(t)\,dt +$$
$$+\Big(\int_0^{a_1} f(t)^p h(t)V(t)\,dt\Big)h(a_1)^{-1} + (q^{-1}-1)\Big(\int_0^{b_n} f(t)^p h(t)V(t)\,dt\Big)h(b_n)^{-1}\Big\}$$

and since $h(x)$ is nondecreasing this is clearly bounded by

$$I\int_0^{b_n} f(t)^p V(t)\,dt = B^q\big((\eta q+a)/\eta\big)(q')^{q/p'}y^{-q}\Big(\int_0^{b_n} f(t)^p V(t)\,dt\Big)^{q/p}.$$

This completes the proof for nonnegative, compactly supported step functions $f(x)$.

Suppose now only that $\int_0^\infty |f(x)|^p V(x)\,dx < \infty$. Let $x_0 = \sup\{x\colon h(x) = 0\}$ and for each positive integer n let $V_n(x)$ be defined by

$$V_n(x) = \begin{cases} V(x) & \text{if} \quad x \leqslant x_0, \\ [V(x)^{-1/(p-1)} + h(x)^{p'}/n(1+x^2)]^{-(p-1)} & \text{if} \quad x > x_0. \end{cases}$$

Then $V_n(x) \uparrow V(x)$ as $n \to \infty$ and for each $\varepsilon > 0$, $V_n(x)$ is bounded on every finite subinterval of $[x_0+\varepsilon, \infty)$; moreover,

$$\sup_{r>0}\Big(\int_r^\infty (r/x)^a\big(U(x)/x\big)^{\eta q}dx\Big)^{1/q}\Big(\int_0^r V_n(x)^{-1/(p-1)}dx\Big)^{1/p'} \leqslant (1+\pi/2n)^{1/p'}B.$$

Since standard arguments show that step functions are dense in $L^p\big(V_n; [x_0+\varepsilon, \infty)\big)$ it follows from what we proved above that

$$\int_{\{x:(P_\eta g)(x)>y\}} U(x)\,dx$$
$$\leqslant [(1+\pi/2n)^{1/p'}B]^q\big((\eta q+a)/\eta\big)(q')^{q/p'}y^{-q}\Big(\int_{x_0+\varepsilon}^\infty |f(x)|^p V_n(x)\,dx\Big)^{q/p},$$

where $g(t) = |f(t)|$ for $t \geqslant x_0+\varepsilon$ and $g(t) = 0$ elsewhere. The desired inequality now follows from the monotone convergence theorem upon taking the limit as $\varepsilon \to 0+$ and $n \to \infty$. This completes the proof of Theorem 2.

4. Proof of Theorem 3. Theorem 3 follows immediately from Theorem A, Theorem 2 and the following lemma.

LEMMA 2. *Suppose* $1 < p < \infty$, $a > 0$ *and* $W(x)$ *is nonnegative and measurable on* $(0, \infty)$. *If*

$$(4.1)\qquad \Big(\int_r^\infty (r/x)^a \big(W(x)/x^{\eta p}\big)dx\Big)\Big(\int_0^r [x^{p(1-\eta)}W(x)]^{-1/(p-1)}dx\Big)^{p-1} \leqslant C$$

for all $r > 0$, then

$$(4.2)\qquad \Big(\int_r^\infty \big(W(x)/x^{\eta p}\big)dx\Big)\Big(\int_0^r [x^{p(1-\eta)}W(x)]^{-1/(p-1)}dx\Big)^{p-1} \leqslant 2^a C/(1-[1+A^{-1}]^{1-p})$$

for all $r > 0$ where $A = \big(2^a(\ln 2)^{-p}C\big)^{1/(p-1)}$.

To prove the Lemma, let $r > 0$ and put

$$b_k = \int_{2^k r}^{2^{k+1}r} [x^{p(1-\eta)}W(x)]^{-1/(p-1)}dx.$$

Hölder's inequality then yields

$$(4.3)\qquad \int_{2^k r}^{2^{k+1}r} (2^k r/x)^a\big(W(x)/x^{\eta p}\big)dx \geqslant 2^{-a}\int_{2^k r}^{2^{k+1}r}\big(W(x)/x^{\eta p}\big)dx$$
$$\geqslant 2^{-a}\Big(\int_{2^k r}^{2^{k+1}r} dx/x\Big)^p\Big(\int_{2^k r}^{2^{k+1}r}[x^{p(1-\eta)}W(x)]^{-1/(p-1)}dx\Big)^{1-p}$$
$$= 2^{-a}(\ln 2)^p b_k^{1-p}.$$

Replacing r by $2^k r$ in (4.1) leads to

$$(4.4)\qquad \Big(\int_{2^k r}^{2^{k+1}r} (2^k r/x)^a\big(W(x)/x^{\eta p}\big)dx\Big)\Big(\int_0^{2^k r}[x^{p(1-\eta)}W(x)]^{-1/(p-1)}dx\Big)^{p-1} \leqslant C$$

and upon using (4.3) and the definition of b_j this yields

$$(4.5)\qquad \big(2^{-a}(\ln 2)^p b_k^{1-p}\big)\Big(\sum_{j=-\infty}^{k-1} b_j\Big)^{p-1} \leqslant C.$$

Raising (4.5) to the $1/(p-1)$ power and writing $b_k = \sum_{j=-\infty}^{k} b_j - \sum_{j=-\infty}^{k-1} b_j$ gives

$$\sum_{j=-\infty}^{k-1} b_j \leqslant [2^a(\ln 2)^{-p}C]^{1/(p-1)}\Big(\sum_{j=-\infty}^{k} b_j - \sum_{j=-\infty}^{k-1} b_j\Big)$$

and hence also

$$(4.6)\qquad \sum_{j=-\infty}^{k-1} b_j \leqslant \big(A/(1+A)\big)\sum_{j=-\infty}^{k} b_j$$

where we have put $A = [2^a(\ln 2)^{-p}C]^{1/(p-1)}$. By induction it follows that for nonnegative integers k

$$(4.7)\qquad \sum_{j=-\infty}^{-1} b_j \leqslant \big(A/(1+A)\big)^k \sum_{j=-\infty}^{k-1} b_j.$$

Now multiplying (4.4) by the $p-1$ power of (4.7) and deleting the common factor shows that

$$\Big(\int_{2^k r}^{2^{k+1}r} (2^k r/x)^a\big(W(x)/x^{\eta p}\big)dx\Big)\Big(\int_0^r [x^{p(1-\eta)}W(x)]^{-1/(p-1)}dx\Big)^{p-1} \leqslant C\big(A/(1+A)\big)^{k(p-1)}.$$

If x is replaced by $2^{k+1}r$ in the first integral and the resulting inequality summed over all nonnegative integers k we obtain (4.2). The proof of Lemma 2 is complete.

5. Proof of Lemma 1. The necessity of (1.13) is proved in the same way that the A_p condition is proved to be necessary for H, see [5] or [8].

For the sufficiency, observe first that (1.13) and Hölder's inequality leads to

$$\int_{\sqrt{2}a}^{2a} w(x)dx \leqslant \int_a^{2a} w(x)dx \leqslant K_p\cdot a^p\Big(\int_a^{2a} w(x)^{-1/(p-1)}dx\Big)^{1-p}$$
$$\leqslant K_p\cdot a^p\Big(\int_a^{\sqrt{2}a} w(x)^{-1/(p-1)}dx\Big)^{1-p}$$
$$\leqslant K_p\cdot(\sqrt{2}-1)^{-p}\int_a^{\sqrt{2}a} w(x)dx$$

and by iterating this process it follows that

$$\int_a^b w(x)dx \leqslant C_p\int_a^{\sqrt{2}a} w(x)dx$$

whenever $2a \leqslant b \leqslant 8a$. A similar inequality holds for $\int_a^b w(x)^{-1/(p-1)}dx$ so that in fact

(5.1) (1.13) holds (with a new constant K_p) whenever $0 < a < b \leqslant 8a$.

Now, for each integer n define $w_n(x)$ for $-\infty < x < \infty$ by the requirement that w_n be periodic of period $7\cdot 2^n$ and

$$w_n(x) = \begin{cases} w(x) & \text{if } x \in (2^{n-1}, 2^{n+2}],\\ w(2^{n+3}-x) & \text{if } x \in (2^{n+2}, 15\cdot 2^{n-1}].\end{cases}$$

Simple calculations now show that w_n satisfies the A_p condition with a constant K_p independent of n. Let $m = \min(2^{-nd}, 2^{-(n+1)d})$,

$$E(y) = \Big\{x\colon \Big|\int_{2^{n-1}}^{2^{n+2}} \big(f(t)/(t-x)\big)\,dt\Big| > y\Big\}$$

and

$$E_n(y) = \Big\{x \in (2^n, 2^{n+1}]\colon x^d\Big|\int_{2^{n-1}}^{2^{n+2}} \big(f(t)/(t-x)\big)\,dt\Big| > y\Big\}.$$

The results of Hunt, Muckenhoupt and Wheeden [8] then show that

$$\begin{aligned}(5.2)\qquad \int_{E_n(y)} x^{-pd} w(x)\,dx &\leqslant \max(2^{-npd}, 2^{-(n+1)pd}) \int_{E(ym)} w_n(x)\,dx\\ &\leqslant C_p 2^{p|d|} y^{-p} \int_{2^{n-1}}^{2^{n+2}} |f(x)|^p w_n(x)\,dx\\ &= C_{p,d} y^{-d} \int_{2^{n-1}}^{2^{n+2}} |f(x)|^p w(x)\,dx\end{aligned}$$

and moreover, if $1 < p < \infty$, (5.2) may be replaced by the corresponding strong type inequality. If $x \in (2^n, 2^{n+1}]$, Hölder's inequality shows

$$\begin{aligned}\Big|\int_{2^{n-1}}^{x/2} \big(f(t)/(t-x)\big)\,dt\Big|^p &\leqslant \Big((1/2^{n-1}) \int_{2^{n-1}}^{2^n} |f(t)|\,dt\Big)^p\\ &\leqslant 2^{-(n-1)p}\Big(\int_{2^{n-1}}^{2^n} w(t)^{-1/(p-1)}\,dt\Big)^{p-1} \int_{2^{n-1}}^{2^n} |f(t)|^p w(t)\,dt\end{aligned}$$

so that (5.1) yields

$$\begin{aligned}(5.3)\qquad \int_{2^n}^{2^{n+1}} w(x)\Big|\int_{2^{n-1}}^{x/2} \big(f(t)/(t-x)\big)\,dt\Big|^p dx &\leqslant 2^{-(n-1)p}\Big(\int_{2^{n-1}}^{2^{n+1}} w(x)\,dx\Big)\Big(\int_{2^{n-1}}^{2^{n+1}} w(t)^{-1/(p-1)}\,dt\Big)^{p-1} \int_{2^{n-1}}^{2^n} |f(t)|^p w(t)\,dt\\ &\leqslant C_p \int_{2^{n-1}}^{2^n} |f(t)|^p w(t)\,dt\end{aligned}$$

and similarly,

$$(5.4)\qquad \int_{2^n}^{2^{n+1}} w(x)\Big|\int_{2x}^{2^{n+2}} \big(f(t)/(t-x)\big)\,dt\Big|^p dx \leqslant C_p \int_{2^{n+1}}^{2^{n+2}} |f(t)|^p w(t)\,dt.$$

But then, since

$$(Lf)(x) = \frac{1}{\pi}\Big(\int_{2^{n-1}}^{2^{n+2}} - \int_{2^{n-1}}^{x/2} - \int_{2x}^{2^{n+2}}\Big) f(t)/(t-x)\,dt, \qquad x \in (2^n, 2^{n+1}]$$

the desired results follow upon summing (5.2), (5.3) and (5.4) over all integers n. This proves Lemma 1.

6. Proof of Theorem 7. We give the details only for the case $T = H$, the case $T = M$ being similar and in fact slightly simpler.

It suffices to prove

$$(6.1)\qquad \int_{\{x>0\colon x^d|(Hf)(x)|>y\}} x^{-d} w(x)\,dx \leqslant Cy^{-1} \int_{-\infty}^{\infty} |f(x)|\,w(x)\,dx$$

for the contribution over $x<0$ may be handled similary. For convenience, write $V(x) = \min\big(w(x), w(-x)\big)$. Since for $x > 0$

$$(Hf)(x) \leqslant |(Lf)(x)| + \frac{1}{\pi x}\int_0^x \big(|f(t)| + |f(-t)|\big)\,dt + \frac{1}{\pi}\int_x^\infty \big(|f(t)| + |f(-t)|\big)\frac{dt}{t},$$

Lemma 1 shows that (6.1) will follow if we show that

(6.2) $\big(x^{-d}w(x), V(x)\big)$ is a weak type (1,1) weight pair for P_{1-d}

and

(6.3) $\big(x^{d}w(x), xV(x)\big)$ is a strong (hence also weak) type (1, 1) weight pair for Q_{-d}.

Since w satisfies the A_1 condition, if $t > 0$, we have

$$(6.4)\qquad \frac{1}{t}\int_0^t w(x)\,dx \leqslant \frac{1}{t}\int_{-t}^t w(x)\,dx \leqslant 2K \operatorname*{ess\,inf}_{[-t,t]} w(x) = 2K \operatorname*{ess\,inf}_{[0,t]} V(x)$$

and therefore if $r > 0$ and $a = \max(2-d, d)$, (6.4) yields

$$\begin{aligned}\int_r^\infty \big(w(x)/ax^a\big)\,dx &= \int_r^\infty dt/t^{1+a} \int_0^t w(x)\,dx \leqslant 2K\big(\operatorname*{ess\,inf}_{[0,r]} V(x)\big)\int_r^\infty dt/t^a\\ &= \frac{2K}{(a-1)r^{a-1}} \operatorname*{ess\,inf}_{[0,r]} V(x)\end{aligned}$$

so that (6.2) follows from Theorem 1 if $d > 1$ and from Theorem 2 if $d < 1$.

Finally, Fubini's Theorem and (6.4) show that

$$\int_0^\infty w(x)\Big|\int_x^\infty f(t)\,dt\Big|\,dx \leqslant \int_0^\infty |f(t)|\,dt\int_0^t w(x)\,dx$$

$$\leqslant 2K\int_0^\infty |f(t)|\,tV(t)\,dt$$

which yields (6.3). This completes the proof of Theorem 7.

7. Proof of Theorem 8. We prove the sufficiency of (1.11) first. Elementary estimates show that

$$|(H_\varepsilon f)(x)| \leqslant |(Lf)(x)| + c\big((P_1|f|)(x) + (Q_{-1}g)(x)\big)$$

where $g(t) = |f(t)|/t^2$. Since (1.11) shows that w satisfies the local A_1 condition, Lemma 1 then shows that it suffices to prove

(7.1) $\big(w(x),\, w(x)\big)$ is a weak type (1,1) weight pair for P_1

and

(7.2) $\big(w(x),\, x^2 w(x)\big)$ is a weak type (1,1) weight pair for Q_{-1}.

Since (7.2) is equivalent to (7.1) with $w(x)$ replaced by $x^{-2}w(1/x)$ and since $x^{-2}w(1/x)$ satisfies (1.11) if (any only if) $w(x)$ does, it further suffices to prove (7.1). Now (1.11) with $b \geqslant 2a$ leads to

$$\frac{2}{b}\int_{b/2}^{b} w(x)\,dx \leqslant K_\varepsilon\big((b^2-a^2)/b^2\big)\operatorname*{ess\,inf}_{[a,2a]}\big(xw(x)/a\big) \leqslant K_\varepsilon(1/a)\int_a^{2a} w(x)\,dx,$$

that is,

$$(1/b)\int_b^{2b} w(x)\,dx \leqslant K_\varepsilon(1/a)\int_a^{2a} w(x)\,dx$$

whenever $b \geqslant a$. Hence, if $\delta > 0$ and $0 < a \leqslant r$

$$\int_r^\infty (r/x)^\delta\big(w(x)/x\big)\,dx = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} (r/x)^\delta\big(w(x)/x\big)\,dx$$

$$\leqslant K_\varepsilon\Big(\sum_{k=0}^\infty 2^{-k\delta}\Big)(1/a)\int_a^{2a} w(x)\,dx$$

and since w satisfies the local A_1 condition we obtain

$$\int_r^\infty (r/x)^\delta\big(w(x)/x\big)\,dx \leqslant K_{\varepsilon,\delta}\operatorname*{ess\,inf}_{[a,2a]} w(x)$$

for every $0 < a \leqslant r$ so that

$$\int_r^\infty (r/x)^\delta\big(w(x)/x\big)\,dx \leqslant K_{\varepsilon,\delta}\operatorname*{ess\,inf}_{[0,r]} w(x)$$

which yields (7.1) by an application of Theorem 2. This proves the sufficiency of (1.11).

Suppose now that $\big(w(x), w(x)\big)$ is a weak type (1,1) weight pair for H_ε and let $0 < a < b < \infty$ be given. If $b \leqslant 2a$, the necessity proof of [8] is easily adapted to show that

$$\Big(\int_a^b w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/w(x)\big)\Big) \leqslant K(b-a)$$

which is equivalent to (1.11) in this case. Now fix $r > 0$. The estimate

$$|(H_\varepsilon f)(x)| \geqslant c(1/x)\int_0^r f(t)\,dt, \qquad x \geqslant 2r$$

holds for all nonnegative $f(t)$ supported in $(0, r]$ so that the weak type inequality for H_ε leads, just as in the proof of Theorem 2, to

(7.3) $$\Big(\int_{2r}^\infty (2r/x)^\varepsilon\big(w(x)/x\big)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[0,r]}\big(1/w(x)\big)\Big) \leqslant B_\varepsilon \qquad (\varepsilon > 0).$$

Similarly,

$$|(H_\varepsilon f)(x)| \geqslant cx\int_{2r}^\infty f(t)\,dt/t^2 \qquad (0 < x \leqslant r)$$

for all nonnegative $f(t)$ supported on $[2r, \infty]$ leads to

(7.4) $$\Big(\int_0^r (x/r)^\varepsilon xw(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[2r,\infty]}\big(1/x^2w(x)\big)\Big) \leqslant B_\varepsilon \qquad (\varepsilon > 0).$$

Multiplying (7.3) and (7.4) together and using Hölder's inequality in two obvious ways yields

(7.5) $$\Big(\int_0^r (x/r)^\varepsilon xw(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[0,r]}\big(1/w(x)\big)\Big) \leqslant B_\varepsilon r^2$$

and

(7.6) $$\Big(\int_r^\infty (r/x)^\varepsilon\big(w(x)/x\big)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[r,\infty]}\big(1/x^2w(x)\big)\Big) \leqslant B_\varepsilon r^{-2}.$$

Taking $r = b$ in (7.5) and $r = a$ in (7.6) leads to

$$(7.7)\qquad \Big(\int_a^b (x/b)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/w(x)\big)\Big) \leqslant B_\varepsilon b$$

and

$$(7.8)\qquad \Big(\int_a^b (a/x)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/x^2 w(x)\big)\Big) \leqslant B_\varepsilon a^{-1}$$

and therefore

$$(7.9)\qquad \Big(\int_a^b (x/b)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/xw(x)\big)\Big) \leqslant B_\varepsilon (b/a)$$

and

$$(7.10)\qquad \Big(\int_a^b (a/x)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}\big(1/xw(x)\big)\Big) \leqslant B_\varepsilon (b/a).$$

Adding (7.9) and (7.10) then yields (1.11) since $b/a \leqslant (4/3)\,(b^2-a^2)/ab$ when $b \geqslant 2a$. This completes the proof of necessity, and with it, the proof of Theorem 8.

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A generalization of Wiener's criteria for the continuity of a Borel measure

by

JEAN-MARC BELLEY* and PEDRO MORALES (Sherbrooke)

Abstract. An identity is derived for the discrete part of a bounded complex-valued finitely additive set function defined on the Borel sets of an Abelian locally compact Hausdorff topological group. This allows us to establish a generalization of Wiener's necessary and sufficient condition for the continuity of a complex-valued bounded regular measure [16].

1. Introduction. Let $T = \{z \in \mathbf{C}\colon |z| = 1\}$. Then T with the multiplication operation and the topology induced by the usual topology on $\mathbf{C}$ is a compact Abelian topological group. Let $\mathscr{B}(T)$ be the σ-algebra of Borel sets in T. Let $M(T) = \{\mu\colon \mathscr{B}(T)\to \mathbf{C} \mid \mu$ is a bounded regular measure$\}$. The Fourier coefficients of a measure $\mu \in M(T)$ are $\hat{\mu}(n) = \int_T z^{-n}\,d\mu(z)$ for all $n \in \mathbf{Z}$. Recall that a measure $\mu \in M(T)$ is continuous if $\mu(\{z\}) = 0$ for any point z in T. A classical result of Wiener ([16]; [17], Theorem 9.6, p. 108; [8], Corollary, p. 42) states:

1.1. THEOREM. *Let* $\mu \in M(T)$. *Then*

$$\sum_{z\in T} |\mu(\{z\})|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |\hat{\mu}(n)|^2.$$

In particular, μ is continuous if and only if

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |\hat{\mu}(n)|^2 = 0.$$

In this paper, it is shown that this theorem follows from a general result for bounded complex-valued finitely additive set functions defined on the Borel sets $\mathscr{B}(G)$ of an arbitrary locally compact Abelian Hausdorff

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