$$(7.7)\qquad \Big(\int_a^b (x/b)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}(1/w(x))\Big) \leqslant B_\varepsilon b$$

and

$$(7.8)\qquad \Big(\int_a^b (a/x)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}(1/x^2 w(x))\Big) \leqslant B_\varepsilon a^{-1}$$

and therefore

$$(7.9)\qquad \Big(\int_a^b (x/b)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}(1/xw(x))\Big) \leqslant B_\varepsilon (b/a)$$

and

$$(7.10)\qquad \Big(\int_a^b (a/x)^{1+\varepsilon} w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{[a,b]}(1/xw(x))\Big) \leqslant B_\varepsilon (b/a).$$

Adding (7.9) and (7.10) then yields (1.11) since $b/a \leqslant (4/3)\,(b^2-a^2)/ab$ when $b \geqslant 2a$. This completes the proof of necessity, and with it, the proof of Theorem 8.

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A generalization of Wiener's criteria for the continuity of a Borel measure

by

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Abstract. An identity is derived for the discrete part of a bounded complex-valued finitely additive set function defined on the Borel sets of an Abelian locally compact Hausdorff topological group. This allows us to establish a generalization of Wiener's necessary and sufficient condition for the continuity of a complex-valued bounded regular measure [16].

1. Introduction. Let $T = \{z \in \boldsymbol{C}\colon |z| = 1\}$. Then T with the multiplication operation and the topology induced by the usual topology on $\boldsymbol{C}$ is a compact Abelian topological group. Let $\mathscr{B}(T)$ be the σ-algebra of Borel sets in T. Let $M(T) = \{\mu\colon \mathscr{B}(T)\to\boldsymbol{C} \mid \mu$ is a bounded regular measure$\}$. The Fourier coefficients of a measure $\mu \in M(T)$ are $\hat{\mu}(n) = \int_T z^{-n} d\mu(z)$ for all $n \in \boldsymbol{Z}$. Recall that a measure $\mu \in M(T)$ is continuous if $\mu(\{z\}) = 0$ for any point z in T. A classical result of Wiener ([16]; [17], Theorem 9.6, p. 108; [8], Corollary, p. 42) states:

1.1. THEOREM. *Let $\mu \in M(T)$. Then*

$$\sum_{z\in T} |\mu(\{z\})|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |\hat{\mu}(n)|^2.$$

In particular, μ is continuous if and only if

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |\hat{\mu}(n)|^2 = 0.$$

In this paper, it is shown that this theorem follows from a general result for bounded complex-valued finitely additive set functions defined on the Borel sets $\mathscr{B}(G)$ of an arbitrary locally compact Abelian Hausdorff

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topological group G. This result also generalizes that of W.F. Eberlein ([3], Theorem 1, p. 310) in the case of Radon measures on $\mathcal{B}(G)$.

2. Preliminaries. In this section we introduce a slight generalization of a theorem of Sinclair ([14], p. 363) which is an essential tool for obtaining our main result. Let $\mathcal{A}$ be an algebra of subsets of a set X. A *charge* on $\mathcal{A}$ is a complex-valued bounded finitely additive set function defined on $\mathcal{A}$. For every bounded $\mathcal{A}$-measurable complex-valued function on X, we can define the integral $\int_X f\,d\mu$ by the usual Moore–Smith method ([12], pp. 183–191; [15], pp. 401–404) or, equivalently [9], by the Dunford–Schwartz method ([2], pp. 101–125). A function $f\colon X\to C$ is called $\mathcal{A}$-*continuous* if, for every $\varepsilon>0$, there exists a finite partition $\{E_i\}_{1\leqslant i\leqslant n}$ of X such that $E_i\in\mathcal{A}$ and $\sup_{x,y\in E_i}|f(x)-f(y)|<\varepsilon$ for all $i=1,2,\ldots,n$. It is clear that if X is a topological space and if $f\colon X\to C$ is bounded and continuous, then f is $\mathcal{B}(X)$-continuous.

Let X and Y be arbitrary sets and let $f\colon X\times Y\to C$. We say that f satisfies the *double limit condition* or f is a DLC *function* if, whenever $\{x_i\}$, $\{y_j\}$ are sequences in X and Y, respectively, such that the iterated limits

$$\alpha=\lim_{i\to\infty}\lim_{j\to\infty}f(x_i,y_j)$$

and

$$\beta=\lim_{j\to\infty}\lim_{i\to\infty}f(x_i,y_j)$$

exist, then $\alpha=\beta$. This notion was introduced by Banach ([1], p. 222) to give a criteria for the weak convergence to 0 of a sequence in a Banach space, and used extensively by Grothendieck ([5], pp. 182–186) in his search for more general weak convergence criteria. In the case where X and Y are completely regular spaces and f a real-valued bounded separately continuous function on $X\times Y$, this notion was used by Pták ([11], p. 573) to obtain extension criteria for f.

It is said that $\mathcal{A}$ *separates points* on X, or $\mathcal{A}$ is an SP *algebra* on X if, whenever $x,y\in E$, $x\neq y$, there are disjoint sets $A,B\in\mathcal{A}$ such that $x\in A$ and $y\in B$. It is clear that if X is a Hausdorff topological space, then $\mathcal{B}(X)$ is an SP algebra on X.

The following result is a trivial generalization of Theorem 4.4 of Sinclair ([14], p. 363):

2.1. THEOREM. *Let $\mathcal{A}$ and $\mathcal{B}$ be* SP *algebras on the sets X and Y, respectively. Let μ and ν be charges on $\mathcal{A}$ and $\mathcal{B}$, respectively. If $f\colon X\times Y\to C$ is bounded and satisfies the additional conditions*:

(i) *$f(\cdot,y)$ is $\mathcal{A}$-continuous for all $y\in Y$,*
(ii) *$f(x,\cdot)$ is $\mathcal{B}$-continuous for all $x\in X$, and*
(iii) *f is a* DLC *function,*

then

(1) *$\int_X f(\cdot,y)\,d\mu$ is $\mathcal{B}$-continuous,*

(2) *$\int_Y f(x,\cdot)\,d\nu$ is $\mathcal{A}$-continuous, and*

(3) *$\int_Y\int_X f\,d\mu\,d\nu=\int_X\int_Y f\,d\nu\,d\mu$.*

3. Preparatory propositions. Henceforth G will denote an Abelian Hausdorff locally compact group. When we need to give G a topology τ different from the original, we will write G_τ. In particular, we shall consider the discrete topology (τ_d), the pointwise topology (τ_p) and the topology of uniform convergence on compacta (τ_{uc}).

A *character* on G is a continuous homomorphism on G to T. The set of all character on G is an Abelian group under addition and, with the τ_{uc} topology, it is a locally compact topological group ([10], p. 137). The topological group so obtained is called the *dual group* of G, denoted $G^\wedge$. The value of an element $\hat z\in G^\wedge$ at the point $z\in G$ will be denoted by $\langle z,\hat z\rangle$ and its complex conjugate by $\overline{\langle z,\hat z\rangle}$. For all $z\in G$ consider the function $u_z\colon G^\wedge\to T$ given by $u_z(\hat z)=\langle z,\hat z\rangle$. Then u_z is a character on $G^\wedge$. The Pontryagin duality theorem ([7], p. 378) states that the mapping $z\to u_z$ is a topological isomorphism of G onto $G^{\wedge\wedge}$. This result permits us to identify G with its own second character group $G^{\wedge\wedge}$. With G can be associated its *Bohr compactification* ([13], p. 30) $G^*=\big((G^\wedge)_{\tau_d}\big)^\wedge$ which is an Abelian Hausdorff compact topological group whose topology is τ_p. It is well known that G can be embedded into G^* as a dense subgroup ([13], Theorem 1.3.2, p. 30). It is easy to show that the continuous function $(z,\hat z)\to\langle z,\hat z\rangle$ can be extended to a continuous function on $G_{\tau_d}\times G^{\wedge *}$. We note that every element of the Bohr compactification of an Abelian Hausdorff locally compact topological group is a character on some discrete topological group.

3.1. LEMMA. *The restriction to G of an element of $G^{*\wedge *}$ belongs to $G^{\wedge *}$.*

Proof. We note the two trivial facts: (a) If $\hat z\in G^{*\wedge *}$, then the domain of $\hat z$ contains the set G; (b) Let K be a topological subgroup of an Abelian Hausdorff locally compact group H. If $\hat z\in H^\wedge$, then $\hat z|_K\in K^\wedge$. The lemma now follows by taking $K=G_{\tau_d}$ and $H=(G^*)_{\tau_d}$.

Remarks. (1) From fact (a) and the property $G^{*\wedge *}=(G^{*\wedge *\wedge})^\wedge$ follows that G_{τ_d} is a topological subgroup of $G^{*\wedge *\wedge}$.

(2) From Remark (1), it follows that $(G^\wedge)_{\tau_d}$ is a topological subgroup of $G^{*\wedge *\wedge *\wedge}$.

(3) From the lemma it follows that the restriction to $G^{*\wedge}$ of an element of $G^{*\wedge *\wedge *}$ belongs to G^*.

3.2. Lemma. *If* $\{z_i\}$ and $\{\hat{z}_j\}$ *are sequences in* G *and* $G\hat{}$, *respectively, then there are subsequences* $\{z_{im}\}$ *and* $\{\hat{z}_{jn}\}$ *of* $\{z_i\}$ *and* $\{\hat{z}_j\}$, *respectively, such that* $\lim_{m\to\infty}\lim_{n\to\infty}\langle z_{im}, \hat{z}_{jn}\rangle = \lim_{n\to\infty}\lim_{m\to\infty}\langle z_{im}, \hat{z}_{jn}\rangle$.

Proof. Since $\hat{z}_j \in (G\hat{})_{\tau_d} = G^{*\hat{}} \subseteq G^{*\hat{}*}$, there exists a subsequence $\{\hat{z}_{jn}\}$ of $\{\hat{z}_j\}$ converging to an element $\hat{z} \in G^{*\hat{}*}$. Hence, by Remark (1), $\lim_{n\to\infty}\langle z_i, \hat{z}_{jn}\rangle = \langle z_i, \hat{z}\rangle$ $(i = 1, 2, 3, \ldots)$. Also, since $G_{\tau_d} \subset G^{*\hat{}*\hat{}}$, there exists a subsequence $\{z_{im}\}$ of $\{z_i\}$ converging to an element $z \in G^{*\hat{}*\hat{}*}$. Therefore $\lim_{m\to\infty}\lim_{n\to\infty}\langle z_{im}, \hat{z}_{jn}\rangle = \lim_{m\to\infty}\langle z_{im}, \hat{z}\rangle$ and, by duality and the fact that the topology on $G^{*\hat{}*\hat{}*}$ is τ_p, we have $\lim_{m\to\infty}\langle z_{im}, \hat{z}\rangle = \langle z, \hat{z}\rangle$. So $\lim_{m\to\infty}\lim_{n\to\infty}\langle z_{im}, \hat{z}_{jn}\rangle = \langle z, \hat{z}\rangle$.

On the other hand, since $\{\hat{z}_{jn}\} \subset G^{*\hat{}*\hat{}*\hat{}}$ by Remark (2), the continuity of $(w, \hat{w}) \to \langle w, \hat{w}\rangle$ on $G^{*\hat{}*\hat{}*} \times G^{*\hat{}*\hat{}*\hat{}}$ yields $\lim_{m\to\infty}\langle z_{im}, \hat{z}_{jn}\rangle = \langle z, \hat{z}_{jn}\rangle$ $(n = 1, 2, 3\ldots)$. Since $z \in G^{*\hat{}*\hat{}*}$, by Remark (3), $z|(G\hat{})_{\tau_d} \in G^*$ and by the fact that the topology on G^* is τ_p, we have $\lim_{n\to\infty}\langle z, \hat{z}_{jn}\rangle = \langle z, \hat{z}\rangle$. This completes the proof.

3.3. Proposition. *The function* $f\colon G \times G\hat{} \to T$ *given by* $f(z, \hat{z}) = \langle z, \hat{z}\rangle$ *satisfies the double limit condition.*

Proof. This follows immediately from Lemma 3.2 and the continuity of f.

Let m be the normalized Haar measure on the Borel subsets $\mathscr{B}(G\hat{}*)$ of $G\hat{}*$. For a given $z \in G$ and $t \in T$, denote by $z^{-1}(t)$ the set $\{\hat{z} \in G\hat{}*\colon \langle z, \hat{z}\rangle = t\}$.

3.4. Proposition. *Given* $z \in G$, *there are but a finite number of points* $t \in T$ *for which* $z^{-1}(t)$ *has positive Haar measure.*

Proof. Let t and t' be two points on T for which $m(z^{-1}(t)) > 0$ and $m(z^{-1}(t')) > 0$. (Then $z^{-1}(t)$ and $z^{-1}(t')$ are not empty.) Let us show that $z^{-1}(t) = \hat{z} + z^{-1}(t')$ for some $\hat{z} \in G\hat{}*$.

Choose an arbitrary $\hat{z}_t \in z^{-1}(t)$ and an arbitrary $\hat{z}_{t'} \in z^{-1}(t')$. Let $\hat{z} = \hat{z}_t - \hat{z}_{t'}$. If we choose a $\hat{z}'$ in $z^{-1}(t)$, we have $\langle z, \hat{z}'\rangle = \langle z, \hat{z} + (\hat{z}' - \hat{z})\rangle$ where $(\hat{z}' - \hat{z}) \in z^{-1}(t')$, since

$$\langle z, \hat{z}' - \hat{z}\rangle = \langle z, \hat{z}' - (\hat{z}_t - \hat{z}_{t'})\rangle = \langle z, \hat{z}'\rangle\langle z, \hat{z}_{t'} - \hat{z}_t\rangle$$
$$= \langle z, \hat{z}'\rangle\langle z, \hat{z}_{t'}\rangle\overline{\langle z, \hat{z}_t\rangle} = tt'\bar{t} = t'.$$

Thus $z^{-1}(t) \subseteq \hat{z} + z^{-1}(t')$. Similarly we can show the inclusion $\hat{z} + z^{-1}(t') \subseteq z^{-1}(t)$. Since m is translation invariant,

$$m(z^{-1}(t')) = m(\hat{z} + z^{-1}(t')) = m(z^{-1}(t)).$$

The proposition now follows from $m(G\hat{}*) = 1$ and the disjointness of the sets $\{z^{-1}(t)\colon t \in T\}$.

Let $\mathring{V}$ denote the interior of V and $\bar{V}$ its closure.

3.5. Proposition. *Given* $z \in G$ *and* $n = 1, 2, 3\ldots$, *there exists a partition* $R_{z,n}$ *of* $G\hat{}*$ *into Borel subsets for which*

(i) $m(\mathring{V}) = m(\bar{V})$ *whenever* $V \in R_{z,n}$,

(ii) *given* $\hat{z} \in G\hat{}*$, *there exists a* $V \in R_{z,n}$ *for which* (a) $\hat{z} \in V$ *and* (b) $|\langle z, \hat{z}\rangle - \langle z, \hat{z}'\rangle| < 2\pi/n$ *for all* $\hat{z}' \in V$.

Proof. Let $I_1, I_2, \ldots, I_n$ be disjoint half open arcs of T of length $2\pi/n$ each. By the proposition, it follows that we can rotate these arcs along T if necessary until they are such that none of the end points t has $m(z^{-1}(t)) > 0$. Then let $R_{z,n} = \{z^{-1}(I_i)\colon i = 1, 2, \ldots, n\}$. Thus if $V_i = z^{-1}(I_i)$, $i = 1, 2, \ldots, n$, then $m(\bar{V}_i \setminus \mathring{V}_i) = m(z^{-1}(\bar{I}_i \setminus \mathring{I}_i))$ and this vanishes since $\bar{I}_i \setminus \mathring{I}_i$ consists of two end points each with Haar measure of their inverses under z equal to zero. This proves part (i).

Part (ii) follows by construction.

Let $R = \{A \in \mathscr{B}(G\hat{}*)\colon m(\mathring{A}) = m(\bar{A})\}$. By Proposition 3.5, R is not empty.

3.6. Proposition. R *is a subalgebra of* $\mathscr{B}$ $(G\hat{}*)$.

Proof. Let U be an element of R. Then $U^c \in R$ since $\mathring{U} = (\overline{U^c})^c$ and $\bar{U} = ((U^c)^\circ)^c$ imply $m((U^c)^\circ) = m(\overline{U^c}) = 1 - m(\bar{U}) = 1 - m(U) = m(U^c) = 1 - m(U) = 1 - m(\mathring{U}) = m(\mathring{U}^c) = m(\overline{U^c})$ and so $m((U^c)^\circ) = m(U^c) = m(\overline{U^c})$.

Choose $U, V \in R$, Since $\mathring{U} \cup \mathring{V} \subseteq (U \cup V)^\circ$, we have

$$m\big((\overline{U \cup V}) \setminus (U \cup V)^\circ\big) \leqslant m\big((\bar{U} \cup \bar{V}) \setminus (\mathring{U} \cup \mathring{V})\big)$$
$$\leqslant m\big((\bar{U} \setminus \mathring{U}) \cup (\bar{V} \setminus \mathring{V})\big) \leqslant m(\bar{U} \setminus \mathring{U}) + m(\bar{V} \setminus \mathring{V}) = 0$$

and so $m((U \cup V)^\circ) = m(U \cup V) = m(\overline{U \cup V})$, i.e. $U \cup V \in R$. Since $G\hat{}* \in R$, this completes the proof.

Let $M = \{A \cap G\hat{}\colon A \in R\}$.

3.7. Proposition. M *is an* SP *algebra on* $G\hat{}$.

Proof. A compact Hausdorff space is normal. Hence, by Urysohn's lemma, there exists a continuous function $f\colon G\hat{}* \to [0, 1]$ such that $f = 0$ on the closed set $\{\hat{x}\}$ and $f = 1$ on $\{\hat{y}\}$. By the Stone–Weierstrass theorem ([10], p. 9) $f(\hat{z})$ can be uniformly approximated by polynomials $\sum_{k=1}^{n} c_k\langle z_k, \hat{z}\rangle$ where $z_k \in G$, $k = 1, 2, \ldots, n$. If $\langle z, \hat{x}\rangle = \langle z, \hat{y}\rangle$ for all $z \in G$, then $f(\hat{x}) = f(\hat{y})$, which is a contradiction. Hence, for some $z_0 \in G$, $\langle z_0, \hat{x}\rangle \neq \langle z_0, \hat{y}\rangle$. Thus, there exist disjoint half open intervals $I_{\hat{x}}$ and $I_{\hat{y}}$ on the unit circle containing $\langle z_0, \hat{x}\rangle$ and $\langle z_0, \hat{y}\rangle$, respectively, for which

$$z_0^{-1}(I_{\hat{x}}) = \{\hat{z} \in G\hat{}\colon \langle z_0, \hat{z}\rangle \in I_{\hat{x}}\}$$

and

$$z_0^{-1}(I_{\hat{y}}) = \{\hat{z} \in G\hat{\ }: \langle z_0, \hat{z}\rangle \in I_{\hat{y}}\}$$

are disjoint elements of M containing $\hat{x}$ and $\hat{y}$, respectively.

3.8. PROPOSITION. *For all $z \in G$, the function $\langle z, \cdot\rangle: G\hat{\ } \to T$ is M-continuous.*

Proof. This follows immediately from Proposition 3.5 (ii).

3.9. PROPOSITION. *If for some $A, B \in R$, $A \cap G\hat{\ } = B \cap G\hat{\ }$, then $m(A \triangle B) = 0$.*

Proof. Since $A \setminus B \subseteq (A \setminus \mathring{A}) \cup (\mathring{A} \setminus \bar{B}) \cup (\bar{B} \setminus B)$, then $m(A \setminus B) \leqslant m(A \setminus \mathring{A}) + m(\mathring{A} \setminus \bar{B}) + m(\bar{B} \setminus B) = 0 + m(\mathring{A} \setminus \bar{B}) + 0$. But $\mathring{A} \setminus \bar{B}$ is open and $G\hat{\ }$ is dense in $G\hat{\ }^*$ and so $\mathring{A} \setminus \bar{B} = \emptyset$; for otherwise $(A \setminus B) \cap G\hat{\ } \supseteq (\mathring{A} \setminus \bar{B}) \cap G\hat{\ } \neq \emptyset$, contradicting $A \cap G\hat{\ } = B \cap G\hat{\ }$. Hence $m(A \setminus B) = 0$. Similarly $m(B \setminus A) = 0$, and so $m(A \triangle B) = 0$. The proposition is proved.

Remark. It is clear from Proposition 3.9 that the set function $\nu(A \cap G\hat{\ }) = m(A)$, where $A \in R$, is well defined. The following proposition is trivial.

3.10. PROPOSITION. *The set function ν is a non-negative charge on M.*

Remark. Since the product of two M-continuous functions is M-continuous, from Proposition 3.8 it follows that the integral $\int_{G\hat{\ }} \langle z, \hat{z}\rangle \overline{\langle w, \hat{z}\rangle} d\nu(\hat{z})$ exists for all $z, w \in G$.

3.11. PROPOSITION.

$$\int_{G\hat{\ }} \langle z, \hat{z}\rangle \overline{\langle w, \hat{z}\rangle} d\nu(\hat{z}) = \begin{cases} 0 & \text{if} \quad z \neq w, \\ 1 & \text{if} \quad z = w. \end{cases}$$

Proof. Note that $\langle z, \hat{z}\rangle \overline{\langle w, \hat{z}\rangle} = \langle z-w, \hat{z}\rangle$ is a character. Let n be a positive integer. As in the proof of Proposition 3.5, consider disjoint half open arcs $I_{i,n}$, $i = 1, \ldots, n$ of equal length for which $(z-w)^{-1}(I_{i,n}) = V_{i,n} \in R_{n,z-w} \subset R$ and points $\hat{z}_{i,n} \in V_{i,n} \cap G\hat{\ }$. Then

$$\begin{aligned}\int_{G\hat{\ }*} \langle z-w, \hat{z}\rangle dm &= \lim_{n\to\infty} \sum_{i=1}^{n} \langle z-w, \hat{z}_{i,n}\rangle m(V_{i,n}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \langle z-w, \hat{z}_{i,n}\rangle \nu(V_{i,n} \cap G\hat{\ }) \\ &= \int_{G\hat{\ }} \langle z-w, \hat{z}\rangle d\nu(\hat{z}).\end{aligned}$$

Since ([8], exercise 6, p. 193)

$$\int_{G\hat{\ }*} \langle z-w, \hat{z}\rangle dm(\hat{z}) = \begin{cases} 0 & z \neq w, \\ 1 & z = w, \end{cases}$$

the result follows.

4. The main result. Let μ be a charge on $\mathscr{B}(G)$. We define the Fourier transform $\hat{\mu}$ of μ by

$$\hat{\mu}(\hat{z}) = \int_G \overline{\langle z, \hat{z}\rangle} d\mu(z).$$

Then $\hat{\mu}$ is a bounded complex-valued function on $G\hat{\ }$. Taking $f(z, \hat{z}) = \overline{\langle z, \hat{z}\rangle}$ it is clear, by Propositions 3.3, 3.7 and 3.8, that the hypotheses of Theorem 2.1 are verified. Then part (1) of that theorem assures that $\hat{\mu}$ is M-continuous. We are now in a position to establish our principal result:

4.1. THEOREM. *There exist an algebra M of subsets of $G\hat{\ }$ and a non-negative charge ν on M satisfying the following properties:*

(1) *For all $z \in G$, the function $\langle z, \cdot\rangle$ is M-continuous.*

(2) *For any charge μ on $\mathscr{B}(G)$ we have*

(a) *the Fourier transform $\hat{\mu}$ of μ is M-continuous, and*

(b) *for all $z \in G$, $\mu(\{z\}) = \int_{G\hat{\ }} \hat{\mu}(\hat{z}) \langle z, \hat{z}\rangle d\nu(\hat{z})$.*

Proof. It remains to prove (b). Let μ be a charge on $\mathscr{B}(G)$ and let $z \in G$. Then

$$\begin{aligned}\int_{G\hat{\ }} \hat{\mu}(\hat{z}) \langle z, \hat{z}\rangle d\nu(\hat{z}) &= \int_{G\hat{\ }} \Big(\int_G \overline{\langle w, \hat{z}\rangle} d\mu(w)\Big) \langle z, \hat{z}\rangle d\nu(\hat{z}) \\ &= \int_{G\hat{\ }} \int_G \overline{\langle w, \hat{z}\rangle} \langle z, \hat{z}\rangle d\mu(w) d\nu(\hat{z}).\end{aligned}$$

Let $f(w, \hat{z}) = \overline{\langle w, \hat{z}\rangle} \langle z, \hat{z}\rangle = \langle z-w, \hat{z}\rangle$. Then, by Propositions 3.3 and 3.8, the hypotheses of Theorem 2.1 are verified. Then part (3) of that theorem allows us to write:

$$\begin{aligned}\int_{G\hat{\ }} \hat{\mu}(\hat{z}) \langle z, \hat{z}\rangle d\nu(\hat{z}) &= \int_G \int_{G\hat{\ }} \overline{\langle w, \hat{z}\rangle} \langle z, \hat{z}\rangle d\nu(\hat{z}) d\mu(w) \\ &= \mu(\{z\}) \qquad \text{(Proposition 3.11).}\end{aligned}$$

4.2. COROLLARY. *For any charge μ on $\mathscr{B}(G)$, we have*

$$\sum_{z\in G} |\mu(\{z\})|^2 = \int_{G\hat{\ }} |\hat{\mu}(\hat{z})|^2 d\nu(\hat{z}).$$

In particular, μ is continuous if and only if $\int_{G\hat{\ }} |\hat{\mu}(\hat{z})|^2 d\nu(\hat{z}) = 0$.

Proof. Applying Theorem 2.1 twice, we obtain

$$\begin{aligned}\int_{G\hat{\ }} |\hat{\mu}(\hat{z})|^2 d\nu(\hat{z}) &= \int_{G\hat{\ }} \Big(\int_G \langle w, \hat{z}\rangle d\mu(w)\Big) \Big(\int_G \overline{\langle z, \hat{z}\rangle} d\mu(z)\Big) d\nu(\hat{z}) \\ &= \int_G \int_G \Big(\int_{G\hat{\ }} \langle w, \hat{z}\rangle \overline{\langle z, \hat{z}\rangle} d\nu(\hat{z})\Big) d\mu(w) d\bar{\mu}(z) = \sum_{z\in G} |\mu(\{z\})|^2.\end{aligned}$$

By the standard methods used in [8], pp. 34–42, it is clear that the following corollary contains Theorem 1.1.

4.3. Corollary. *Let μ be a charge on $\mathscr{B}(T)$. For all $z \in T$,*

$$\mu(\{z\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \hat{\mu}(n) z^n.$$

Proof. By Theorem 4.1,

$$\mu(\{z\}) = \int_{\boldsymbol{Z}} \hat{\mu}(n) z^n d\nu(n)$$

for all z in T. Thus, it is sufficient to show that for any M-continuous complex-valued function f on $\boldsymbol{Z}$,

$$\int_{\boldsymbol{Z}} f(n) d\nu(n) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} f(n).$$

The result will then follow by taking $f(n) = \hat{\mu}(n) z^n$. Since f can be uniformly approximated by M-measurable step functions, it is sufficient to take $f(n) = \chi_E(n)$ where χ_E is the characteristic function for $E \in M$. Choose $F \in R$ such that $F \cap \boldsymbol{Z} = E$. Let $\varepsilon > 0$. By the regularity of m, there exist a compact set K and an open set V in $\boldsymbol{Z}^*$ such that $K \subset \mathring{F} \subset \overline{F} \subset V$, $m(\mathring{F} \setminus K) < \varepsilon$ and $m(V \setminus \overline{F}) < \varepsilon$. By Urysohn's lemma, there exist two continuous real functions g_K, g_V on $\boldsymbol{Z}^*$ such that $g_K | K = 1$, $g_K | (\boldsymbol{Z}^* \setminus \mathring{F}) = 0$, and $g_V | \overline{F} = 1$, $g_V | (\boldsymbol{Z}^* \setminus V) = 0$. Then $g_K \leqslant \chi_E \leqslant g_V$. Since the restriction to $\boldsymbol{Z}$ of a continuous function on $\boldsymbol{Z}^*$ is almost periodic, we can write ([7], p. 256)

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} g_K(n) \leqslant \liminf_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \chi_E(n)$$
$$\leqslant \limsup_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \chi_E(n) \leqslant \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} g_V(n).$$

Thus

$$\limsup_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \chi_E(n) - \liminf_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \chi_E(n)$$
$$\leqslant \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \big(g_V(n) - g_K(n)\big).$$

But, the last term yields ([7], p. 256 and [10], pp. 169, 170)

$$= \int_{\boldsymbol{Z}^*} \big(g_V(n) - g_K(n)\big) dm(n)$$

$$= \int_{V \setminus K} \big(g_V(n) - g_K(n)\big) dm(n)$$
$$\leqslant 2m(V \setminus K) < 4\varepsilon.$$

So $\lim\limits_{N\to\infty} \frac{1}{2N+1} \sum\limits_{-N}^{N} \chi_E(n)$ exists and is equal to $\int_{\boldsymbol{Z}} \chi_E(n) d\nu(n)$ since

$$\left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \chi_E(n) - \int_{\boldsymbol{Z}} \chi_E(n) d\nu(n) \right|$$
$$\leqslant \left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \big(\chi_F(n) - g_V(n)\big) \right| + \left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} g_V(n) - \right.$$
$$\left. - \int_{\boldsymbol{Z}} \chi_E(n) d\nu(n) \right| \leqslant \left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \big(g_K(n) - g_V(n)\big) \right| +$$
$$+ \left| \int_{\boldsymbol{Z}^*} \big(g_V(n) - \chi_F(n)\big) dm(n) \right| \leqslant \int_{\boldsymbol{Z}^*} \big(g_V(n) - g_K(n) dm(n) +$$
$$\leqslant \int_{\boldsymbol{Z}^*} \big(g_V(n) - g_K(n)\big) dm(n) \leqslant 2m(V \setminus K) \leqslant 4\varepsilon.$$

From Corollary 4.2 follows immediately the following generalization of a result of Helson ([6], Theorem 2, p. 481, see also [4]).

4.4. Corollary. *Let μ be a charge on $\mathscr{B}(G)$. If $|\hat{\mu}| = 1$, then* $\sum_{z\in G} |\mu(\{z\})|^2 = 1$.

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Approximate isometries on bounded sets with an application to measure theory*

by

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Abstract. Given a $\delta > 0$, an $S \subseteq \boldsymbol{R}^n$ of diameter 1, and a function $g\colon S \to \boldsymbol{R}^n$ which alters distances by no more than δ (i.e. for all $s, s' \in S$, $|\,\|g(s)-g(s')\| - \|s-s'\|\,| \leqslant \delta$) we show how to alter g to obtain a true isometry $f\colon S \to \boldsymbol{R}^n$ with $\|f-g\|_\infty \leqslant 27\,\delta^{1/2^n}$. D. H. Hyers and S. M. Ulam proved a similar result, but starting with an approximate isometry g from $\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$.

We use our theorem and an idea of J. Mycielski's to show that two Borel subsets of the Hilbert cube $[0,1]^\omega$ which are isometric under one of the metrics $d_a(x,y) = (\sum a_i^2(x_i-y_i)^2)^{1/2}$ must have the same product measure, provided that the a_i tend to 0 fast enough so that $a_i^{1/2^i}/a_{i-1} \to 0$ as $i \to \infty$.

§ 0. Introduction. In [2] S. M. Ulam and D. H. Hyers proved that if $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is surjective and preserves distances to within $\delta > 0$ (i.e. for all $x, y \in \boldsymbol{R}^n$, $|\,\|x-y\| - \|g(x)-g(y)\|\,| \leqslant \delta$), then there is an isometry $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ which differs from g (sup norm) by no more than 10δ.

Following Ulam and Hyers, several people have considered the problem of finding an isometry near to an approximate isometry in very general contexts (see [1] and references therein), but to our knowledge no one has yet considered the problem when the approximate isometry is not defined on a full Banach space.

In §2 we give a construction which alters an approximate isometry g defined on a bounded subset of $\boldsymbol{R}^n$ to give an isometry f; in Theorem 2.2 we show that the constructed f is near to g.

In §1 we develop the methods for proving this result.

In §3 we apply Theorem 2.2 to partly prove the following conjecture of Ulam's: If any two Borel subsets of the Hilbert cube $[0,1]^\omega$ are isometric under one of the metrics

$$d_a(x, y) = \Big(\sum_{i=0}^{\infty} a_i^2 (x_i - y_i)^2\Big)^{1/2}$$

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