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Approximate isometries on bounded sets with an application to measure theory*

by

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Abstract. Given a $\delta > 0$, an $S \subseteq \boldsymbol{R}^n$ of diameter 1, and a function $g\colon S \to \boldsymbol{R}^n$ which alters distances by no more than δ (i.e. for all $s, s' \in S$, $|\,\|g(s)-g(s')\| - \|s-s'\|\,| \leqslant \delta$) we show how to alter g to obtain a true isometry $f\colon S \to \boldsymbol{R}^n$ with $\|f-g\|_\infty \leqslant 27\,\delta^{1/2^n}$. D. H. Hyers and S. M. Ulam proved a similar result, but starting with an approximate isometry g from $\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$.

We use our theorem and an idea of J. Mycielski's to show that two Borel subsets of the Hilbert cube $[0,1]^\omega$ which are isometric under one of the metrics $d_a(x,y) = (\sum a_i^2(x_i-y_i)^2)^{1/2}$ must have the same product measure, provided that the a_i tend to 0 fast enough so that $a_i^{1/2^i}/a_{i-1} \to 0$ as $i \to \infty$.

§ 0. Introduction. In [2] S. M. Ulam and D. H. Hyers proved that if $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is surjective and preserves distances to within $\delta > 0$ (i.e. for all $x, y \in \boldsymbol{R}^n$, $|\,\|x-y\| - \|g(x)-g(y)\|\,| \leqslant \delta$), then there is an isometry $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ which differs from g (sup norm) by no more than 10δ.

Following Ulam and Hyers, several people have considered the problem of finding an isometry near to an approximate isometry in very general contexts (see [1] and references therein), but to our knowledge no one has yet considered the problem when the approximate isometry is not defined on a full Banach space.

In §2 we give a construction which alters an approximate isometry g defined on a bounded subset of $\boldsymbol{R}^n$ to give an isometry f; in Theorem 2.2 we show that the constructed f is near to g.

In §1 we develop the methods for proving this result.

In §3 we apply Theorem 2.2 to partly prove the following conjecture of Ulam's: If any two Borel subsets of the Hilbert cube $[0,1]^\omega$ are isometric under one of the metrics

$$d_a(x,y) = \Big(\sum_{i=0}^{\infty} a_i^2(x_i-y_i)^2\Big)^{1/2}$$

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(where $a_i > 0$ and $\sum a_i^2 < \infty$), then the standard product probability measure in $[0,1]^\omega$ gives these two sets equal measure. J. Mycielski proved (in [3]) the conjecture under the additional assumption that the sets are open. He also gave a condition on a which ensures the truth of the conjecture for pairs of Borel sets isometric under d_a. In Theorem 3.1 Mycielski's ideas and Theorem 2.2 are applied to show that if the a_i tend to 0 very rapidly, namely $a_i^{1/2^i}/a_{i-1} \to 0$ as $i \to \infty$, then Borel sets isometric under d_a have the same measure.

In §4 we pose two further problems on relationships between isometries, approximate isometries, and measure.

I am indebted to J. Mycielski for interesting me in these problems and for criticizing early versions of my results.

§1. Preliminary geometry. For $E \subseteq \mathbf{R}^m$ we write $H(E)$ for the smallest flat containing E, i.e.

$$H(E) = \{\lambda_0 p_0 + \ldots + \lambda_m p_m \mid p_0, \ldots, p_m \in E;\ \lambda_0, \ldots, \lambda_m \in \mathbf{R} \text{ and } \lambda_0 + \ldots + \lambda_m = 1\}.$$

We write $H(p_0, \ldots, p_k)$ instead of $H(\{p_0, \ldots, p_k\})$. $p_0, \ldots, p_k \in \mathbf{R}^m$ are said to be *independent* if $H(p_0, \ldots, p_k)$ is k-dimensional. E. g. three points are independent iff they are not collinear.

We write $d(\cdot, \cdot)$ for the usual Euclidean distance function in $\mathbf{R}^m$.

It is easy to check that if $p_0, \ldots, p_m$ are independent in $\mathbf{R}^m$ and $d_0, \ldots, d_m \geqslant 0$, then the equations $d(x, p_i) = d_i$, $0 \leqslant i \leqslant m$, have at most one solution $x \in \mathbf{R}^m$. Also easy to prove is the following:

1.1. Proposition. *If $p_0, \ldots, p_{m-2}$ are independent in $\mathbf{R}^m$ and $d_0, \ldots, d_{m-2} \geqslant 0$, then the set*

$$S = \{x \in \mathbf{R}^m \mid d(x, p_i) = d_i,\ 0 \leqslant i \leqslant m-2\}$$

is a (possibly degenerate) circle with center in $H(p_0, \ldots, p_{m-2})$. If S is non-degenerate, then $H(S)$ is perpendicular to $H(p_0, \ldots, p_{m-2})$.

Lemma 1.2 is a special case of Lemma 1.3. The latter is the main tool for the proof of Theorem 2.2.

1.2. Lemma. *Let S and c be a circle and point, respectively, in $\mathbf{R}^3$. Suppose that the perpendicular projection of c on the plane of S lies outside of or on S. Let $d \in (0, 1]$ be given, such that the sphere of radius d about c intersects S. Let $\varepsilon \in (0, 1]$ be given, and a point p on S with $|d - d(p, c)| \leqslant \varepsilon$.*

Then there is a point q on S with $d(c, q) = d$ and $d(p, q) \leqslant 2\sqrt{\varepsilon}$.

Proof. Take the sphere about c of radius d, and also the spherical shell about c of radii $d \pm \varepsilon$, and intersect them with the plane of S. The sphere and spherical shell become a circle S' and an annulus about S', respectively. S' has its center c' outside of or on S. Let q be the nearer of the two points of $S \cap S'$ to p. The worst case is when p lies on the outer boundary of the annulus and the segment $\overline{qp}$ is tangent to S'. For this case we apply the pythagorean theorem to the triangle Δcpq to get the desired result. ■

1.3. Lemma. *Let $p_0, \ldots, p_{m-1}$ be independent points in $\mathbf{R}^m$. Define $H = H(p_0, \ldots, p_{m-2})$ and let p_m be any point of $\mathbf{R}^m$ no further from H than is p_{m-1}. Define $d_i = d(p_m, p_i)$, $0 \leqslant i \leqslant m-2$. Suppose that $d_i \leqslant 1$, $0 \leqslant i \leqslant m-2$, and that d_{m-1} and $\varepsilon \in (0, 1]$ are given with $|d_{m-1} - d(p_m, p_{m-1})| \leqslant \varepsilon$.*

If there is a point q with $d(q, p_i) = d_i$ for $0 \leqslant i \leqslant m-1$, then there is such a q with $d(p_m, q) \leqslant 2\sqrt{\varepsilon}$.

Proof. By 1.1 the set of x such that $d(x, p_i) = d_i$, $0 \leqslant i \leqslant m-2$, is a circle S, say of radius r.

If $r = 0$, then $p_m = q$ and we are done. So suppose $r > 0$. Then H is perpendicular to S and passes through the center of S. Let K be a three dimensional flat containing S and p_{m-1}. $H \cap K$ is a line through the center of S, perpendicular to the plane of S. p_{m-1} is at least r from this line. p_m is on S. The sphere of radius d_{m-1} about p_{m-1} intersects S.

So we may apply Lemma 1.2 with S, p, c, d, and q there equal to S, p_m, p_{m-1}, d_{m-1}, and q here, respectively, to get the desired result. ■

§2. Construction of an isometry near to an approximate isometry. Let (M, ϱ) and (N, σ) be metric spaces, and let $\delta > 0$. A function $g\colon M \to N$ is called a *δ-isometry* if for all $x, y \in M$

$$|\sigma(g(x), g(y)) - \varrho(x, y)| \leqslant \delta.$$

That is, a δ-isometry is a function preserving distances to within δ.

In this section we first give a construction which alters a δ-isometry mapping a bounded subset of $\mathbf{R}^n$ into $\mathbf{R}^n$ to give an isometry. Then we prove a theorem which shows that the constructed isometry is near to the original approximate isometry. Some corollaries, comments, and problems follow.

2.1. Construction. Let S be a bounded subset of $\mathbf{R}^n$ and $g\colon S \to \mathbf{R}^n$ a δ-isometry, for some $\delta > 0$.

Step 1. We extend g to $\bar{S}$, the closure of S. For each $x \in \bar{S} \setminus S$ define $g_1(x)$ to be any member of the set

$$\bigcap_n \overline{g\{y \in S \mid d(x, y) < 1/n\}},$$

and for $s \in S$ let $g_1(s) = g(s)$. Then it is easy to verify that g_1 is again a δ-isometry.

Step 2. Let k be the dimension of $H(S)$ and pick $s_0, \ldots, s_k \in \bar{S}$ to satisfy

$$(1)\quad \begin{aligned} &d(s_0, s_1) = \text{diameter of } \bar{S},\\ &d(s_i, H(s_0, \ldots, s_{i-1})) = \sup_{s\in S} d(s, H(s_0, \ldots, s_{i-1})), \quad 2 \leqslant i \leqslant k. \end{aligned}$$

Note that (1) forces $s_0, \ldots, s_k$ to be independent.

Step 3. Define inductively

$$(2a)\quad \begin{aligned} &f(s_0) = g_1(s_0),\\ &f(s_i) = \text{a point as close as possible to } g_1(s_i) \text{ satisfying}\\ &\qquad d(f(s_i), f(s_j)) = d(s_i, s_j), \quad 0 \leqslant j \leqslant i \leqslant k. \end{aligned}$$

And finally, for all $s \in \bar{S}$,

$$(2b)\quad f(s) = \text{the unique point in } H(f(s_0), \ldots, f(s_k)) \text{ satisfying } d(f(s), f(s_i)) = d(s, s_i), \quad 0 \leqslant i \leqslant k.$$

For $S \subseteq \boldsymbol{R}^n$ we write $\operatorname{diam} S$ for the diameter of S. For $\delta \geqslant 0$ define $K_0(\delta) = K_1(\delta) = \delta$, $K_2(\delta) = 3(3\delta)^{1/2}$ and for $i \geqslant 3$, $K_i(\delta) = 27\,\delta^{2^{1-i}}$.

2.2. THEOREM. *Let S be a bounded subset of $\boldsymbol{R}^n$ and $g\colon S \to \boldsymbol{R}^n$ a δ-isometry, where $0 \leqslant \delta$, $3K_n(\delta/\operatorname{diam} S) \leqslant 1$. Then the isometry $f\colon S \to \boldsymbol{R}^n$ gotten by applying Construction* 2.1 *to g satisfies*

$$\sup_{s\in S} d(f(s), g(s)) \leqslant K_{n+1}(\delta/\operatorname{diam} S)\cdot \operatorname{diam} S.$$

Proof. By an easy homothety argument one reduces the general case to the case where $\operatorname{diam} S = 1$. Apply Construction 2.1 to g, and let $g_1, s_0, \ldots, s_k$, and f be as given there.

We prove, by induction on m, that if $t_0, \ldots, t_m \in \bar{S}$, with $t_0, \ldots, t_{m-1}$ independent, satisfy

$$(3)\quad d(t_i, H(t_0, \ldots, t_{i-1})) \geqslant d(t_j, H(t_0, \ldots, t_{i-1})), \quad 1 \leqslant i \leqslant j \leqslant m,$$

and if $h\colon \{t_0, \ldots, t_m\} \to \boldsymbol{R}^n$ is defined inductively by

$$\begin{aligned} &h(t_i) \text{ is a point as close as possible to } g_1(t_i) \text{ satisfying}\\ &\qquad d(h(t_i), h(t_j)) = d(t_i, t_j), \quad 0 \leqslant j \leqslant i \leqslant m, \end{aligned}$$

then

$$d(h(t_i), g_1(t_i)) \leqslant K_m(\delta), \quad 0 \leqslant i \leqslant m.$$

This claim is trivial for $m = 0, 1$. Let $m \geqslant 2$, assume the truth of the claim for m-point sets, and let $t_0, \ldots, t_m$, and h be as described. Then $t_0, \ldots, t_{m-2}, t_{m-1}$ and $t_0, \ldots, t_{m-2}, t_m$ each satisfy an m-point version of (3).

Define

$$h_1(t_i) = \begin{cases} h(t_i), & 0 \leqslant i \leqslant m-1,\\ \text{a point as near as possible to } g_1(t_m) & \\ \text{satisfying } d(h_1(t_m), h_1(t_i)) = d(t_m, t_i) & \\ \text{for } 0 \leqslant i \leqslant m-2, & i = m. \end{cases}$$

Note that it can easily be proven (for example by induction) that $h_1(t_m)$ is in $H(h(t_0), \ldots, h(t_m))$.

By the induction hypothesis

$$d(h_1(t_i), g_1(t_i)) \leqslant K_{m-1}(\delta), \quad 0 \leqslant i \leqslant m.$$

Hence also

$$|d(h_1(t_{m-1}), h_1(t_m)) - d(t_{m-1}, t_m)| \leqslant \delta + 2K_{m-1}(\delta) \leqslant 3K_{m-1}(\delta).$$

Thus we may apply Lemma 1.3 with $p_0, \ldots, p_m$, d_{m-1}, ε, and $\boldsymbol{R}^m$ there equal to $h_1(t_0), \ldots, h_1(t_m)$, $d(t_m, t_{m-1})$, $3K_{m-1}(\delta)$, and $H(h(t_0), \ldots, h(t_m))$ to get a $q \in H(h(t_0), \ldots, h(t_m))$ with $d(q, h(t_i)) = d(t_m, t_i)$ for $0 \leqslant i \leqslant m-2$ and $d(q, h(t_m)) \leqslant 2(3K_{m-1}(\delta))^{1/2}$. Thus

$$\begin{aligned} d(g_1(t_m), h(t_m)) &\leqslant d(g_1(t_m), q)\\ &\leqslant d(g_1(t_m), h_1(t_m)) + d(h_1(t_m), q)\\ &\leqslant K_{m-1}(\delta) + 2(3K_{m-1}(\delta))^{1/2} \leqslant K_m(\delta), \end{aligned}$$

which finishes the proof of our first claim.

Now let $s \in S$ be arbitrary. By (1) and (2) we may apply the preceeding with $m = k+1$, $t_0 = s_0, \ldots, t_k = s_k$, $t_m = s$, and $h = f$ to conclude that

$$d(f(s), g_1(s)) \leqslant K_{k+1}(\delta) \leqslant K_{n+1}(\delta), \quad s \in S. \ \blacksquare$$

2.3. COROLLARY. *Let H be a Hilbert space, with distance d induced by the inner product, and let S be a subset of H with compact closure. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $g\colon S \to H$ is a δ-isometry, there is an isometry $f\colon S \to H$ with $\sup_{s\in S} d(f(s), g(s)) \leqslant \varepsilon$.*

Proof. Since $\bar{S}$ is compact, we may choose a finite number of points $s_1, \ldots, s_n \in S$ so that for any $s \in S$ there is an i with $d(s, s_i) < \varepsilon/4$. Set

$$D = \max_{1\leqslant i,j\leqslant n} d(s_i, s_j).$$

The s_i lie in a finite dimensional subspace of H isometric to some $\boldsymbol{R}^m$ ($m = n$ if H is real, $m = 2n$ if H is complex). Choose $\delta > 0$ so that δ, $K_{m+1}(\delta/D)D < \varepsilon/4$ and $3K_m(\delta/D)D \leqslant 1$. Then by Theorem 2.2 find an isometry $f_1\colon \{s_1, \ldots, s_n\} \to H$ with $\max_{1\leqslant i\leqslant n} d(f_1(s_i), g(s_i)) \leqslant K_{m+1}(\delta/D)D$.

Let f_2 be an isometry of H agreeing with f_1 on $\{s_1, \ldots, s_n\}$ (for example f_2 will be completely specified if we require it to be the identity on the orthogonal complement of the space spanned by $s_1, \ldots, s_n$). Define f to be the restriction of f_2 to S.

Let s be any element of S and pick i, $1 \leqslant i \leqslant n$, so that $d(s, s_i) < \varepsilon/4$. Then $d(g(s), g(s_i)) < \varepsilon/4+\delta$, so

$$d(f(s), g(s)) \leqslant d(f(s), f(s_i)) + d(f(s_i), g(s_i)) + d(g(s_i), g(s)) < \varepsilon/4 + K_{m+1}(\delta/D)D + \varepsilon/4 + \delta < \varepsilon. \ \blacksquare$$

2.4. Corollary. *Let $S \subseteq \boldsymbol{R}^n$ have diameter $D < \infty$; $g\colon S \to \boldsymbol{R}^n$ be a δ-isometry for some $\delta \geqslant 0$ with $3K_n(\delta/D) \leqslant 1$. Then there exists an extension G of g, $G\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$, which is a $K_{n+1}(\delta/D)D$-isometry.*

Proof. With S and g as stated, apply Theorem 2.2 to get an isometry $f\colon S \to \boldsymbol{R}^n$ with $d(f(s), g(s)) \leqslant K_{n+1}(\delta/D)D$ for all s in S. Let F be an isometry from $\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$ which agrees with f on S. Define

$$G(x) = \begin{cases} g(x), & x \in S, \\ F(x), & x \in \boldsymbol{R}^n \setminus S. \end{cases} \ \blacksquare$$

An analogous result follows from 2.3.

2.5. Remarks and problems.

1. The condition that S be bounded is necessary for the validity of the results of this section. In [2] examples are given, for any $\delta > 0$, of a δ-isometry $G\colon \boldsymbol{R} \to \boldsymbol{R}^2$ such that if f is any isometry $\boldsymbol{R} \to \boldsymbol{R}^2$, then $\sup_{r \in \boldsymbol{R}} d(f(r), g(r)) = \infty$.

2. It is easy to find, for any positive δ and M, compact, convex subsets S and T of the plane and a δ-isometry $g\colon S \to T$ such that if $f\colon S \to T$ is any isometry, then $\sup_{s \in S} d(f(s), g(s)) > M$.

3 (J. Mycielski). For sets S having a fixed finite number of points there is a simple (though ineffective) proof of the following qualitative version of Theorem 2.2: For every $n \in \boldsymbol{N}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for each $S \subseteq \boldsymbol{R}$ of diameter 1 (and of the given cardinality) and each δ-isometry $g\colon S \to \boldsymbol{R}^n$ there is an isometry $f\colon S \to \boldsymbol{R}^n$ with $\|f(s) - g(s)\| \leqslant \varepsilon$ for all $s \in S$. This result can be extended to all S of diameter less than some fixed bound by the argument used in the proof of 2.3.

4. It is easy to see that a square root in the functions K_n is necessary for the validity of Theorem 2.2. Problem: Are there constants $a_n > 0$ for $n > 2$ such that Theorem 2.2 remains valid when $K_n(\delta)$ is replaced by $a_n\sqrt{\delta}$?

5. Problem. In Corollary 2.4 is the dependence of ε on S necessary, or could ε be made to depend only on the diameter of S?

§ 3. Application to a conjecture of Ulam on the invariance of measure in the Hilbert cube. Let I be the closed unit interval $[0, 1]$, I^ω the Hilbert cube, and μ the standard product probability measure on I^ω.

Given a sequence $a = (a_0, a_1, \ldots)$ of positive real numbers with $\sum a_i^2 < \infty$, let d_a be the metric in I^ω defined by

$$d_a(x, y) = \Big[\sum a_i^2 (x_i - y_i)^2\Big]^{1/2}, \quad x, y \in I^\omega.$$

$E, F \subseteq I^\omega$ are said to be *d_a-isometric* if there is a bijection between E and F preserving the distance d_a. μ is *d_a-invariant* if d_a-isometric Borel sets have the same μ-measure.

Thus Ulam's conjecture is that for every d_a as above μ is d_a-invariant.

3.1. Theorem. *If the a_n tend to 0 rapidly enough so that*

$$(4) \qquad a_n^{1/2^n}/a_{n-1} \to 0 \quad \text{as} \quad n \to \infty,$$

then μ is d_a-invariant.

Proof. The proof uses Theorem 2.2 and a modification of the reduction of Ulam's conjecture given by Mycielski in Theorem 5 of [3].

Fix a, satisfying (4). Set

$$r_n = \Big(\sum_{i=n}^{\infty} a_i^2\Big)^{1/2} \quad \text{and} \quad \varepsilon_n = 27 r_n^{1/2^n} r_0^{1-1/2^n}$$

($\varepsilon_n = r_0 K_{n+1}(r_n/r_0)$, where K_{n+1} is the function appearing in Theorem 2.2).

Define the parallelepiped in $\boldsymbol{R}^n$

$$C_n = [0, a_0] \times [0, a_1] \times \ldots \times [0, a_{n-1}].$$

Let

$$C_n' = \{x \in \boldsymbol{R}^n \mid d(x, C_n) \leqslant \varepsilon_n\}$$

(where d is again the ordinary Euclidean distance) and

$$\delta_n = \lambda_n(C_n')/\lambda_n(C_n) - 1$$

(λ_n is n-dimensional Lebesgue measure). We claim that

$$(5) \qquad \delta_n \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, from (4) it follows easily that for sufficiently large n

$$a_n < 2^{-n} a_{n-1}$$

and hence

$$r_n < 2a_n.$$

So

$$\varepsilon_n / a_{n-1} \to 0 \quad \text{as} \quad n \to \infty$$

also. Thus for n greater than some n_0

$$\lambda_n(C'_n)/\lambda_n(C_n) < \prod_{i=0}^{n-1} (1+2\varepsilon_n/a_i)$$
$$< (1+2\varepsilon_n/a_{n-1}) \prod_{i=0}^{n_0} (1+2\varepsilon_n/a_i) \prod_{i=n_0+1}^{n-2} (1+a_{n-1}/a_i)$$
$$< (1+2\varepsilon_n/a_{n-1}) \Big[\prod_{i=0}^{n_0} (1+2\varepsilon_n/a_i)\Big] (1+2^{1-n})^{n-1}$$

which tends to 1 as n tends to ∞, and so (5) holds.

Define $\pi_n\colon I^\omega \to C_n$ by

$$\pi_n(x) = (a_0x_0, a_1x_1, \ldots, a_{n-1}x_{n-1})$$

and for any Borel set $E \subseteq C_n$

$$\mu_n(E) = \lambda_n(E)/a_0 \cdot \ldots \cdot a_{n-1}$$

(so $\mu_n = \pi_n^*(\mu)$).

For $E \subseteq I^\omega$ and $t \geqslant 0$ define

$$E^{(t)} = \{x \in I^\omega \mid d_a(x, E) \leqslant t\}$$

and for $E \subseteq C_n$

$$E^{(t)} = \{x \in C_n \mid d(x, E) \leqslant t\}.$$

Thus for $E \subseteq I^\omega$

(6) $$(\pi_n(E))^{(t)} = \pi_n(E^{(t)}).$$

We will need that for compact $S \subseteq I^\omega$

(7) $$\mu_n(\pi_n(S^{(\varepsilon_n)})) \to \mu(S) \quad \text{as} \quad n \to \infty.$$

Let $\eta > 0$ be given. Since μ is regular, pick $t > 0$ so that

$$\mu(S^{(t)}) < \mu(S) + \eta/2.$$

Since μ is a product measure pick N so that for $n > N$

$$\mu_n(\pi_n(S^{(t)})) < \mu(S^{(t)}) + \eta/2 < \mu(S) + \eta.$$

The condition (4) implies that $\varepsilon_n \to 0$, so pick $M > N$ such that for $n > M$ $\varepsilon_n < t$. Then for $n > M$

$$\mu(S) < \mu_n(\pi_n(S^{(\varepsilon_n)})) < \mu_n(\pi_n(S^{(t)})) < \mu(S) + \eta,$$

which proves (7).

Fix now a compact $S \subseteq I^\omega$ and a d_a-isometry $f\colon S \to I^\omega$. Let $q_n\colon \pi_n(S) \to S$ be any function satisfying $q_n(x) \in \pi_n^{-1}(x)$ for all $x \in \pi_n(S)$, and define $F_n\colon \pi_n(S) \to C_n$ by

$$F_n = \pi_n \circ f \circ q_n.$$

Then

(8) $$F_n(\pi_n(S)) \subseteq \pi_n(f(S)).$$

From

$$d(x, y) \leqslant d_a(q_n(x), q_n(y)) \leqslant d(x, y) + r_n, \qquad x, y \in \pi_n(S)$$

and

$$d_a(q_n(x), q_n(y)) - r_n = d_a(f(q_n(x)), f(q_n(y))) - r_n$$
$$\leqslant d(F_n(x), F_n(y)) \leqslant d_a(f(q_n(x)), f(q_n(y)))$$
$$= d_a(q_n(x), q_n(y))$$

it follows that F_n is an r_n-isometry. So by Theorem 2.2 we can find an isometry $f_n\colon \pi_n(S) \to \boldsymbol{R}^n$ within ε_n of F_n. Thus in fact $f_n(\pi_n(S)) \subseteq C'_n$.

We have

$$\lambda_n(f_n(\pi_n(S)) \cap C_n) \geqslant \lambda_n(f_n(\pi_n(S))) - \lambda_n(C'_n \setminus C_n) \geqslant \lambda_n(\pi_n(S)) - \delta_n\lambda_n(C_n),$$

so

$$\lambda_n([F_n(\pi_n(S))]^{(\varepsilon_n)}) \geqslant \lambda_n(\pi_n(S)) - \delta_n\lambda_n(C_n),$$

or

(9) $$\mu_n([F_n(\pi_n(S))]^{(\varepsilon_n)}) \geqslant \mu_n(\pi_n(S)) - \delta_n.$$

Hence, given $\varepsilon > 0$ pick N so that for $n \geqslant N$

$$\begin{aligned} \mu(f(S)) + \varepsilon &\geqslant \mu_n(\pi_n(f(S)^{(\varepsilon_n)})) && \text{by (7)} \\ &= \mu_n([\pi_n(f(S))]^{(\varepsilon_n)}) && \text{by (6)} \\ &\geqslant \mu_n([F_n(\pi_n(S))]^{(\varepsilon_n)}) && \text{by (8)} \\ &\geqslant \mu_n(\pi_n(S)) - \delta_n && \text{by (9)} \\ &\geqslant \mu(S) - \delta_n. \end{aligned}$$

By (5) $\delta_n \to 0$, so $\mu(f(S)) \geqslant \mu(S)$. By symmetry $\mu(f(S)) = \mu(S)$. Thus d_a-isometric compact sets have the same μ-measure, and so, by regularity of μ, μ is d_a-invariant. ■

3.2. Remarks.

1. It would clearly suffice to require instead of (4) the existence of $n_1 < n_2 < n_3 < \ldots$ such that $\delta_{n_i} \to 0$ as $i \to \infty$.

2. From 2.5.3 and the proof of 3.1 would again follow a weakened, qualitative, result: There exists an a such that μ is d_a-invariant.

§ 4. Further problems. In the proof of 3.1 we needed to compare the measure of the domain of an approximate isometry with the measure of an open neighborhood of its range, all inside a fixed parallelepiped. Essentially nothing seems to be known about relations between approximate isometries and measure. In this section we propose two problems related to, but simpler than the problem above, which we feel are interesting even without their relation to Ulam's conjecture.

4.1. *Approximate isometries and measure.* We move the problem of §3 to a more amenable environment, hoping that a solution to the new problem would help in the old situation.

CONJECTURE. There is a constant $c \geqslant 1$ such that if S is a subset of some $\boldsymbol{R}^n$, $g\colon S \to \boldsymbol{R}^n$ is a δ-isometry, and $T = \{x \in \boldsymbol{R}^n \mid d(x, g(S)) \leqslant c\delta\}$, then $\lambda_n(T) \geqslant \lambda_n(S)$.

4.2. *Neighborhoods in bricks.* Let B be a brick (rectangular parallelepiped) in $\boldsymbol{R}^n$, S and T isometric subsets of B. It would seem that not too much more of S than of T can be near the boundary of B. Precisely:

CONJECTURE. There is a $c_n > 0$ depending only on n such that for any $\delta > 0$

$$\lambda_n(\{x \in B \mid d(x, S) \leqslant \delta\}) \leqslant \lambda_n(\{x \in B \mid d(x, T) \leqslant c_n \delta\}).$$

In fact we guess that the best value for c_n is $1+\sqrt{n}$, corresponding to S a small corner of the brick B.

To see the connection between this conjecture and the problem described at the beginning of the section, let S' be a subset of the brick B and $g\colon S' \to B$ an approximate isometry. Let $f\colon S' \to \boldsymbol{R}^n$ be an isometry within, say, δ of g. Set $T = f(S') \cap B$ and $S = f^{-1}(T)$. Then the above conjecture implies that the $(1+c_n)\delta$ neighborhood of $g(S')$ in B has measure at least equal to the measure of S'.

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A characterization of BMO and BMO_ϱ

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Abstract. The purpose of this paper is to use probabilistic characterizations of BMO and BMO_ϱ to solve two problems. Our first result is a characterization of BMO_ϱ in terms of Carleson measures when ϱ is regular. This result was conjectured by Sarason [9], and is similar to Fefferman and Stein's characterization of BMO [2]. Secondly, we give a probabilistic proof of a criterion for BMO due to Hayman and Pommerenke [4].

1. Introduction. Probability has recently become an important tool in complex analysis. Using Brownian motion, Burkholder, Gundy, and Silverstein have characterized the H^p spaces in terms of maximal functions, thereby solving a longstanding problem of Hardy and Littlewood.

Currently, the BMO functions are receiving attention. Fefferman and Stein [2] have shown that BMO is the dual of H^1. An important step in their proof was the characterization of BMO in terms of Carleson measures. The BMO_ϱ spaces were introduced by Spanne [10] as generalizations of BMO. These have also aroused interest (see Sarason [9]).

The purpose of this paper is to use probabilistic characterizations of BMO and BMO_ϱ to solve two problems. Our first result is a characterization of BMO_ϱ in terms of Carleson measures when ϱ is regular. This result was conjectured by Sarason [9], and is similar to, Fefferman and Stein's characterization of BMO [2]. Secondly, we give a probabilistic proof of a criterion for BMO due to Hayman and Pommerenke [4].

Let D be the unit disc in the complex plane, and let I be a subarc of ∂D. We will consider holomorphic functions f in H^1, which are the Poisson integrals of their boundary values. Let

$$I(f) = \frac{1}{|I|}\int_I f(x)\,dx.$$

The space BMO consists of all functions f for which the BMO norm

$$\|f\|_* = \sup_{I \subseteq \partial D} I(|f - I(f)|)$$