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# Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction

by

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**Abstract.** An example is given of a reflexive Banach space which fails to have either an equivalent norm that is uniformly convex in every direction or an equivalent norm that is uniformly differentiable in every direction.

**1.** It is shown in [4] that every reflexive Banach space has an equivalent norm that is locally uniformly convex and Fréchet differentiable. This led (see [2]) to the question whether every reflexive Banach space has an equivalent norm which is uniformly convex in every direction or uniformly differentiable in every direction. We give a negative answer to this question:

THEOREM 1.1. *There is a reflexive Banach space $Y$ such that:*

(i) *$Y$ has no equivalent norm which is uniformly convex in every direction.*

(ii) *$Y$ has no equivalent norm which is uniformly differentiable in every direction.*

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**2. Definitions and notations.** The norm of a Banach space $X$ is *uniformly convex in every direction* if $\|x_n-y_n\|\to 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\|x_n\| = \|y_n\| = 1$, $\|x_n+y_n\|\to 2$, and there is a $z$ and $\{\lambda_n\}$ with $x_n-y_n = \lambda_n z$.

The norm of a Banach space $X$ is *uniformly differentiable in every direction* if, for every $y\in X$ with $\|y\| = 1$,

$$\lim_{\tau\to 0}\tau^{-1}\sup\{\|x+\tau y\|+\|x-\tau y\|-2:\ x\in X,\ \|x\| = 1\} = 0.$$

Let $\Phi_\Gamma$ be a family of finite subsets of $\Gamma$ such that $\Phi_\Gamma$ contains all one-point subsets of $\Gamma$ and all subsets of members of $\Phi_\Gamma$. We denote by $\lambda_{12}(\Phi_\Gamma)$ the space of all real-valued functions $x$ on $\Gamma$ such that

$$\|x\| = \sup\Big\{\Big[\sum_{i\in I}\Big(\sum_{\gamma\in A_i}|x(\gamma)|\Big)^2\Big]^{1/2}\Big\} < \infty, \tag{1}$$

where the supremum is taken over all finite systems $\{A_i\}_{i\in I}$ with each $A_i \in \Phi_\Gamma$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

Since each $A_i$ in (1) can be contained in the support of $x$, we have $\|x+y\| \geqslant (\|x\|^2+\|y\|^2)^{1/2}$, if $x$ and $y$ have disjoint supports. Therefore $\lambda_{12}(\Phi_\Gamma)$ is the completion of the space of functions on $\Gamma$ with finite support. Also, if $\{e_\gamma\}_{\gamma\in\Gamma}$ is the natural unconditional basis for $\lambda_{12}(\Phi_\Gamma)$ defined by $e_\gamma(\beta) = \delta_{\gamma\beta}$, then this basis is boundedly complete, meaning that the series $\sum_{\gamma\in\Gamma} a_\gamma e_\gamma$ is convergent unconditionally whenever

$$\sup\Big\{\Big\|\sum_{\gamma\in A} a_\gamma e_\gamma\Big\|:\ A \subset \Gamma,\ |A| < \infty\Big\} < \infty.$$

### 3. Reflexivity of $\lambda_{12}(\Phi_\Gamma)$.

LEMMA 3.1. *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be the natural basis in $\lambda_{12}(\Phi_\Gamma)$ with biorthogonal functionals $\{e_\gamma^*\}_{\gamma\in\Gamma}$. If $A \in \Phi_\Gamma$, then*

$$\Big\|\sum_{\gamma\in A} e_\gamma\Big\| = |A|, \qquad \Big\|\sum_{\gamma\in A} e_\gamma^*\Big\| = 1.$$

The proof is straightforward.

LEMMA 3.2. *Let $\Phi_\Gamma$ have the property that, for every $A \in \Phi_\Gamma$ with $|A| \geqslant 2$, there is a positive integer $k(A)$ such that $|B| \leqslant k(A)$ if $B \in \Phi_\Gamma$ and $B \supset A$. Then $\lambda_{12}(\Phi_\Gamma)$ is reflexive.*

Proof. Recall that uniform convexity of $l_2$ implies that if $\|x+y+z\|$ is "nearly" 3 for $x, y$, and $z$ in the unit ball of $l_2$, then $x, y$, and $z$ are nearly equal. Thus there is a positive number $\Delta$ such that, if $\omega_1$, $\omega_2$, and $\omega_3$ are in the unit ball of $\lambda_{12}(\Phi_\Gamma)$ and have disjoint supports in $\Gamma$, and if $\{B_j\}$ is a finite system with each $B_j \in \Phi_\Gamma$, $B_i \cap B_j = \emptyset$ if $i \neq j$, and $\Big\{\sum_{j\in J}\Big[\sum_{\gamma\in B_j}(\omega_1+\omega_2+\omega_3)(\gamma)\Big]^2\Big\}^{1/2} > 3(1-\Delta)$, then

$$\Big\{\sum_{i\in I}\Big[\sum_{\gamma\in A_i}(\omega_1+\omega_2+\omega_3)(\gamma)\Big]^2\Big\}^{1/2} > 5/2 \tag{2}$$

if $\{A_i\}$ is obtained from $\{B_j\}$ by deleting each $B_j$ that either contains no point in the support of $\omega_1$ or contains one point in the support of $\omega_1$ and no point in the support of $\omega_2$.

Suppose $\lambda_{12}(\Phi_\Gamma)$ is not reflexive. Then there is (see [3]) a bounded sequence $\{x_i\}$ and a positive number $\theta$ such that

$$\operatorname{dist}(\operatorname{conv}\{x_1, \ldots, x_n\}, \operatorname{conv}\{x_i:\ i > n\}) > \theta \quad \text{if} \quad n \geqslant 1. \tag{3}$$

Since each $x_i$ can have finite support, we can replace $\{x_i\}$ by a subsequence $\{y_i\}$ for which all followers of $y_p$ are approximately equal on the union of the supports of $x_p$ and its predecessors, for each $p$. Since the basis $\{e_\gamma\}$ is boundedly complete, there is no loss of generality if we assume that the sequence $\{x_i\}$ which satisfies (3) also has the property that all followers of $x_p$ are zero on the union of the supports of $x_p$ and its predecessors, for each $p$. Let

$$\tau(1-\tfrac{1}{2}\Delta) = \liminf_{n\to\infty}\{\|x\|:\ x \in \operatorname{conv}\{x_i:\ i > n\}\},$$

and choose $M$ so that $\|w\| > \tau(1-\Delta)$ if $w \in \operatorname{conv}\{x_i:\ i > M\}$. Since $\tau > 0$, there is a sequence $\{w_i\}$ that satisfies (3), with $w_1 \in \operatorname{conv}\{x_i:\ M < i \leqslant p_1\}$, $w_2 \in \operatorname{conv}\{x_i:\ p_1 < i \leqslant p_2\}$, etc., and

$$\|w_i\| < \tau \quad \text{for each } i.$$

Let $S_1$ and $S_2$ be the supports of $w_1$ and $w_2$, and let $K$ be an upper bound for $\sum|A_i|$ for all finite systems $\{A_i\}$ with each $A_i \in \Phi_\Gamma$, $A_i \cap A_j = \emptyset$ if $i \neq j$, and each $A_i$ containing at least two points of $S_1 \cup S_2$. Choose any $q > 1$ and let

$$w = (1/2K)\sum_{i=q+1}^{q+2K} w_i.$$

Then $w(\gamma) < \frac{1}{2}\tau/K$ for each $\gamma \in \Gamma$. Because of (2), there is a finite system $\{A_i\}$ with each $A_i \in \Phi_\Gamma$, $A_i \cap A_j = \emptyset$ if $i \neq j$, each $A_i$ containing at least two points of $S_1 \cup S_2$, and

$$\Big\{\sum_{i\in I}\Big[\sum_{\gamma\in A_i}(w_1+w_2+w)(\gamma)\Big]^2\Big\}^{1/2} > 5\tau/2. \tag{4}$$

This is false since $\|w_1\| < \tau$, $\|w_2\| < \tau$ and

$$\sum_{i\in I}\Big[\sum_{\gamma\in A_i}|w(\gamma)|\Big] < K(\tfrac{1}{2}\tau/K) = \tfrac{1}{2}\tau$$

imply the left member of (4) is less than $5\tau/2$.

### 4. Proof of Theorem 1.1.

Let $\Delta = \prod_{i=2}^{\infty}\{1, 2, \ldots, i\}$. That is, $\Delta$ is the family of all sequences $\gamma = \{\gamma^i\}_{i=1}^\infty$ of positive integers such that $1 \leqslant \gamma^i \leqslant i+1$. We denote by $\Psi_\Delta$ the family of all finite subsets of $\Delta$ which have the property that, if $A \in \Psi_\Delta$, then there is a positive integer $m$ such that, if $\gamma_k = \{\gamma_k^i\}_{i=1}^\infty$ and $\gamma_j = \{\gamma_j^i\}_{j=1}^\infty$ are different members of $A$, then $\gamma_k^m \neq \gamma_j^m$ and $\gamma_k^i = \gamma_j^i$ for $1 \leqslant i \leqslant m-1$. (A family similar to $\Psi_\Delta$ is considered in [1].) In the sequel, we shall use the following.

LEMMA 4.1. *Let $\Delta = \bigcup_{l=1}^{\infty}\Delta_l$. Then for some $\Delta_r$,*

$$\sup\{|A|:\ A \subset \Delta_r,\ A \in \Psi_\Delta\} = \infty.$$

Proof. For $\delta = \{\delta^i\}_{i=1}^\infty$, we define $\pi_n(\delta) = \delta^n$, $P_n(\delta) = \{\delta^i\}_{i=1}^n$.

Suppose that for every $l$

$$m_l = \sup\{|A|:\ A \subset \Delta_l,\ A \in \Psi_\Delta\} < \infty. \tag{5}$$

We shall define inductively an increasing sequence of nonnegative integers $\{n_j\}_{j=0}^{\infty}$, and a sequence of finite systems $\{\delta^i\}_{i=n_{j-1}+1}^{n_j}$ with $\delta^i \in \{1, 2, \ldots, i+1\}$, such that

$$p_{n_j}^{-1}(\{\delta^i\}_{i=1}^{n_j}) \cap \Big(\bigcup_{s=1}^{j} \Delta_s\Big) = \emptyset.$$

Let $n_0 = 0$. Suppose $\{n_j\}_{j=1}^{k}, \{\delta^i\}_{i=1}^{n_k}$ with the above property have been found. If $p_{n_k}^{-1}(\{\delta^i\}_{i=1}^{n_k}) \cap \Big(\bigcup_{s=1}^{k+1} \Delta_s\Big) = \emptyset$, we take $n_{k+1} = n_k$. If $p_{n_k}^{-1}(\{\delta^i\}_{i=1}^{n_k}) \cap \Big(\bigcup_{s=1}^{k+1} \Delta_s\Big) \neq \emptyset$, we put $n_{k+1} = \max(n_k+1, m_{k+1})$. There is $\delta_{k+1} \in \Delta_{k+1}$ such that $\delta^i = \pi_i(\delta_{k+1})$ if $1 \leqslant i \leqslant n_k$. Put $\delta^i = \pi_i(\delta_{k+1})$ for $n_k+1 \leqslant i \leqslant n_{k+1}-1$. From (5) it follows that

$$\Big|\Big\{\pi_{n_{k+1}}\big(p_{n_{k+1}-1}^{-1}(\{\delta^i\}_{i=1}^{n_{k+1}-1}) \cap \Delta_{k+1}\big)\Big\}\Big| \leqslant m_{k+1} \leqslant n_{k+1}.$$

Consequently there is a positive integer $\delta^{n_{k+1}}$, $1 \leqslant \delta^{n_{k+1}} \leqslant n_{k+1}+1$, such that $p_{n_{k+1}}^{-1}(\{\delta^i\}_{i=1}^{n_{k+1}}) \cap \Delta_{k+1} = \emptyset$. From the construction, it is clear that $n_k \to \infty$. For $\delta = \{\delta^i\}_{i=1}^{\infty}$, we have $\delta \in \Delta$, $\delta \notin \Delta_l$, $l = 1, 2, \ldots$, which contradicts the assumption $\Delta = \bigcup_{l=1}^{\infty} \Delta_l$.

It follows from Lemma 3.2. that both $\lambda_{12}(\Psi_\Delta)$ and $\lambda_{12}^*(\Psi_\Delta)$ are reflexive.

Denote by $Y$ the cartesian product of $\lambda_{12}(\Psi_\Delta)$ and $\lambda_{12}^*(\Psi_\Delta)$. It is simple to show that $Y$ has the properties of Theorem 1.1, using Lemma 3.1, Lemma 4.1, and the following

PROPOSITION 4.2 (1) [5]. *Let $X$ be a Banach space with unconditional basis $\{u_\gamma\}_{\gamma\in\Gamma}$. Then*

(i) *if the norm in $X$ is uniformly convex in every direction, then for every $\varepsilon > 0$ there is a decomposition $\Gamma = \bigcup_{l=1}^{\infty} \Gamma_l^{(\varepsilon)}$ such that*

$$\inf\Big(\Big\|\sum_{\gamma\in A} u_\gamma\Big\|:\ A \subset \Gamma_l^{(\varepsilon)},\ |A| = l\Big) \geqslant \varepsilon^{-1};$$

(ii) *if the norm in $X$ is uniformly differentiable in every direction, then for every $\varepsilon > 0$ there is a decomposition $\Gamma = \bigcup_{l=1}^{\infty} \Gamma_l^{(\varepsilon)}$ such that*

$$\sup\Big(\Big\|\sum_{\gamma\in A} u_\gamma\Big\|:\ A \subset \Gamma_l^{(\varepsilon)},\ |A| = l\Big) \leqslant l\varepsilon.$$

(1) In [5] it is shown that the conditions in Proposition 4.2 are not only necessary but sufficient as well.



### References


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