Since $\gamma = \nu\otimes(\varrho-\lambda/2)+(\mu-\nu/2)\otimes\lambda$, we see that γ is a probability measure. It is a matter of simple calculations on characteristic functions to check that the characteristic function of γ is of the form $\hat{\gamma}(a) = \exp\big(-\psi(a)\big)\cdot(b, a)$ and that ψ fulfils condition (2) and does not fulfil condition (1). Thus, for each character $a \in (V\oplus T)^*$ the image measure $a(\gamma)$ is Gaussian on T and γ is not a Gaussian distribution on $V\oplus T$. Since the last group is isomorphic to a subgroup of X, it follows that X does not fulfil condition (i). This ends the proof of the implication (i) ⇒ (ii).

Remark. A locally compact abelian group X fulfils condition (ii) if its dual group has a maximal, independent system of infinitely divisible elements, or the component of 0 in X is isomorphic to T.

Acknowledgment. The author wishes to express his deep gratitude to the translator of this article, his work greatly exceeded the usual job of a translator.

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Received December 20, 1979 (1596)

STUDIA MATHEMATICA, T. LXXIII. (1982)

Fractional integration on Hardy spaces

by

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Abstract. The classical fractional integration theorems for Riesz potentials on L^p spaces are extended to the real variable Hardy classes, $0 < p \leqslant 1$. It is further shown that the Riesz potentials can be replaced by a large class of convolution operators. Finally, one obtains results for certain operators which are not of convolution type by using a theory of local Hardy spaces.

§ 0. Introduction.

Let $K_\alpha\colon \boldsymbol{R}^n\setminus\{0\}\to\boldsymbol{R}$ be given by

$$K_\alpha(x) = \gamma_{\alpha,n}|x|^{\alpha-n}, \quad 0<\alpha<n,$$

where $\gamma_{\alpha,n} \equiv \big(\pi^{n/2}2^\alpha\Gamma(\alpha/2)\big)/\Gamma\big((n-\alpha)/2\big)$. It is an old theorem of Hardy and Littlewood (for $n = 1$) and of Sobolev (for $n > 1$) that the operator

$$I_\alpha(f) = f*K_\alpha$$

is a bounded linear map from $L^p(\boldsymbol{R}^n)$ to $L^q(\boldsymbol{R}^n)$, $1<p<n/\alpha$, $1/q = 1/p-\alpha/n$ (see [10]).

For $0<\alpha+\beta<n$, one can compute (see [11]) that $I_{\alpha+\beta} = I_\alpha\circ I_\beta = I_\beta\circ I_\alpha$. This motivates the definition

$$I_\alpha = I_{\alpha/k}\circ I_{\alpha/k}\circ\ldots\circ I_{\alpha/k} \quad (k \text{ times})$$

when $\alpha < kn$, and one checks that the definition is unambiguous.

Now let $H^p(\boldsymbol{R}^n)$ denote the generalized Hardy classes defined and developed in [11], [10], [3]. These will be considered in detail below. In [11] the following result is proved:

$$I_\alpha\colon H^p(\boldsymbol{R}^n)\to H^q(\boldsymbol{R}^n) \tag{0.1}$$

boundedly, provided $(n-1)/n < p \leqslant n/(n+\alpha)$, $\alpha>0$, and $1/q = 1/p-\alpha/n$. Also

$$I_\alpha\colon H^p(\boldsymbol{R}^n)\to L^q(\boldsymbol{R}^n) \tag{0.2}$$

boundedly, provided $n/(n+\alpha)<p<n/\alpha$, $\alpha>0$, and $1/q = 1/p-\alpha/n$. It is known that in case $p = n/\alpha$, the appropriate target space is BMO

* Research supported in part by NSF Grant #MCS 77-02213.

(the functions of bounded mean oscillation). For $p > n/a$, I_a maps L^p to the Lipschitz class $\Lambda^{\text{loc}}_{a-n/p}(\boldsymbol{R}^n)$ (see [10] and references therein).

Moving in another direction, let us note that the question of replacing K_a by a more general sort of kernel has been investigated. In case $1 < p < n/a$, [4] gives essentially the weakest possible conditions on $K(x, y)$ which guarantee that

$$f \mapsto \int f(y) K(x, y)\, dy$$

is bounded from L^p to L^q, $q > p$. Comparably weak conditions for $p \geqslant n/a$ do not seem to be known.

In this note we wish to extend the above results to the case when the domain of I_a is taken to be $H^p(\boldsymbol{R}^n)$, $0 < p \leqslant 1$. We show further that I_a can be replaced by a large class of convolution kernels. Finally, one can replace I_a by certain kernels which are *not* convolution kernels provided one is willing to relax the definition of H^p space in the fashion of Goldberg [6].

The proofs are fairly easy, given the current advanced state of the theory. We make use of the atomic decomposition of $H^p(\boldsymbol{R}^n)$ and of the maximal function characterization of $H^p(\boldsymbol{R}^n)$.

Although we only formulate and prove results on $\boldsymbol{R}^n$, they are true in some generality. An example of particular interest is the unit ball in $\boldsymbol{C}^n$. We will comment on this in Section 3.

We wish to thank S. Ross Barker for a useful conversation.

§ 1. Definitions and background information. In what follows, the letters C, C_j, etc. will be used to denote various constants whose values may be different in different contexts. Let

$$\boldsymbol{R}^{n+1}_+ = \{(x, y) = (x_1, \ldots, x_n, y)\colon\ y > 0\}$$

be the usual upper half space. Let $H^p(\boldsymbol{R}^n)$ be the space of functions harmonic on $\boldsymbol{R}^{n+1}_+$ whose nontangential maximal function $u^*(x) = \sup\limits_{|x-t|<y} |u(t, y)|$ is in $L^p(\boldsymbol{R}^n)$, $0 < p < \infty$. Let $\|u\|_{H^p} \equiv \|u^*\|_{L^p}$. If $0 < p \leqslant 1$, let $N(p) = [n(1-p)/p]$, the integer part of $n(1-p)/p$. Define a p-atom to be a function $a\colon \boldsymbol{R}^n \to \boldsymbol{C}$ so that a is supported on a cube $Q \subseteq \boldsymbol{R}^n$ with sides parallel to the axes and so that

(1.1) $$|a(x)| \leqslant |Q|^{-n/p},$$

where $|Q|$ is the volume of Q;

(1.2) $\int a(x) x^a\, dx = 0$ for all multi-indices a of order $|a| \leqslant N(p)$.

It is known (see [3]) that if $u \in H^p(\boldsymbol{R}^n)$ then $\lim\limits_{t\to 0} u(\cdot, t) = f$ exists in the sense of tempered distributions and that f uniquely determines u.

It will be convenient to also denote this space of boundary distributions on $\boldsymbol{R}^n$ by $H^p(\boldsymbol{R}^n)$.

We wish to recall the following important results about $H^p(\boldsymbol{R}^n)$:

(1.3) Let $1 < p < \infty$. Then $f \in L^p(\boldsymbol{R}^n)$ if and only if $f \in H^p(\boldsymbol{R}^n)$, in the sense that

$$C_1 \|f^*\|_{L^p} \leqslant \|f\|_{L^p} \leqslant C_2 \|f^*\|_{L^p}.$$

(1.4) Let $\varphi \in \mathscr{S}$, the Schwartz testing class, with $\int \varphi\, dx = 1$. Let f be a tempered distribution on $\boldsymbol{R}^n$. Define

$$u^+(x) = \sup_{\varepsilon > 0} |\varphi_\varepsilon * f(x)|$$

where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then $f \in H^p(\boldsymbol{R}^n)$ if and only if $u^+ \in L^p(\boldsymbol{R}^n)$, $0 < p < \infty$, and

$$C_1 \|u^+\|_{L^p} \leqslant \|f\|_{H^p} \leqslant C_2 \|u^+\|_{L^p}.$$

(1.5) A tempered distribution f is in $H^p(\boldsymbol{R}^n)$, $0 < p \leqslant 1$, if and only if there is a sequence $\{a_i\}$ of p-atoms and a sequence $\{\lambda_i\}$ of non-negative real numbers so that

$$f = \sum_{i=1}^{\infty} \lambda_i a_i$$

in the sense of distributions and

$$C_1 \|f\|^p_{H^p} \leqslant \sum_{i=1}^{\infty} \lambda_i^p \leqslant C_2 \|f\|^p_{H^p}.$$

Here C_1, C_2 depend *only* on n and p.

(1.6) Remark. The reader should refer to [3] for proofs of (1.3) and (1.4) and for the history of these ideas. For (1.5), see [7] and references therein.

While the foregoing ideas are rather familiar, those following may be less so. They are due to Goldberg [6]. If u is harmonic on $\boldsymbol{R}^{n+1}_+$, we say that u is in the *local Hardy class* h^p provided that

(1.7) $$u^*_{\text{loc}}(x) \equiv \sup_{|x-t|<y\leqslant 1} |u(t, y)|$$

satisfies $u^*_{\text{loc}} \in L^p(\boldsymbol{R}^n)$, $0 < p < \infty$. Let $\varphi \in \mathscr{S}$ satisfy $\int \varphi\, dx \neq 0$. Then Goldberg has proved that $u \in h^p$ if and only if

(1.8) $$u^+_{\text{loc}}(x) \equiv \sup_{0<\varepsilon\leqslant 1} |u * \varphi_\varepsilon(x)|$$

satisfies $u^+_{\text{loc}} \in L^p(\boldsymbol{R}^n)$. Finally, if we modify our definition of p-atom so that atoms supported on cubes of measure greater than 1 (where the cube is chosen as small as possible) satisfy no moment conditions, then elements of h^p satisfy an atomic decomposition just as in (1.5).

(1.9) Remark. It is a straightforward excercise that if $f \in L^1_{loc}(\boldsymbol{R}^n)$ then

$$f^+(x) \geqslant f^+_{loc}(x) \geqslant |f(x)| \text{ a.e.}, \quad x \in \boldsymbol{R}^n.$$

Therefore, for $1 < p < \infty$, we have that

$$\|f^+_{loc}\|_{L^p} \leqslant \|f^+\|_{L^p} \leqslant C_1 \|Mf\|_{L^p} \leqslant C_2 \|f\|_{L^p} \leqslant C_3 \|f^+_{loc}\|_{L^p}$$

where M is the Hardy–Littlewood maximal operator. So

$$H^p(\boldsymbol{R}^n) \cong h^p(\boldsymbol{R}^n) \cong L^p(\boldsymbol{R}^n), \quad 1 < p < \infty.$$

§ 2. Formulation and proof of the theorems. In what follows, $K(s, t)$ will be a complex valued function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ which is smooth off the diagonal $\Delta = \{s = t\}$. Let $\boldsymbol{Z}^+$ denote the non-negative integers. We let $D_1^\alpha K(a, t) \equiv (\partial/\partial s)^\alpha K(s, t)|_{s=a}$ whenever $\alpha \in (\boldsymbol{Z}^+)^n$, $a, t \in \boldsymbol{R}^n$; likewise $D_2^\alpha K(s, b) \equiv (\partial/\partial t)^\alpha K(s, t)|_{t=b}$.

(2.1) Theorem. *Let $n, N_0 \in \boldsymbol{Z}^+$ be non-zero, let $0 < \alpha < n$, and let $K\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ be in $C^{N_0}(\boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \Delta)$. Suppose that, for all $\mu > 0$, we have*

$$\text{(2.1.1a)} \qquad \int\limits_{|t-w|\leqslant\mu} |K(s, t)|\, dt \leqslant C\mu^\alpha, \quad \textit{all } s, w \in \boldsymbol{R}^n;$$

$$\text{(2.1.1b)} \qquad \int\limits_{|s-w|\leqslant\mu} |K(s, t)|\, ds \leqslant C\mu^\alpha, \quad \textit{all } t, w \in \boldsymbol{R}^n;$$

$$\text{(2.1.2)} \qquad \int\limits_{\substack{|s-t|\geqslant 2|v|\\ |v|\leqslant\mu}} \Big|K(s, t+v) - \sum_{|\beta|\leqslant N} D_2^\beta K(s, t) v^\beta\Big|\, dv \leqslant C|s-t|^{-n-N-1+\alpha}\mu^{n+N+1},$$

$$\textit{all } s, t \in \boldsymbol{R}^n, \text{ all } 0 \leqslant N < N_0;$$

$$\text{(2.1.3)} \qquad \int\limits_{\substack{|s-t|\geqslant 2|v|\\ |v|\leqslant\mu}} |K(s, t+v) - K(s-v, t)|\, dv \leqslant C\mu^{N_0+n}, \quad \textit{all } s, t \in \boldsymbol{R}^n.$$

Let $n/(n+N_0) < p \leqslant 1$ and $1/q = 1/p - \alpha/n$. Then the operator

$$T\colon f \mapsto \int K(x, t) f(t)\, dt, \quad f \text{ a } p\text{-atom},$$

extends to be a bounded linear operator from $H^p(\boldsymbol{R}^n)$ to $h^q(\boldsymbol{R}^n)$.

(2.2) Theorem. *Let all notation be as in Theorem 2.1 and suppose in addition that T is a convolution operator (that is, $K(s, t) = k(s-t)$ for some $k \in C^{N_0}(\boldsymbol{R}^n \setminus \{0\})$). Then T maps $H^p(\boldsymbol{R}^n)$ to $H^q(\boldsymbol{R}^n)$ boundedly.*

(2.3) Corollary. *Let $\alpha > 0$ (α may be equal to or greater than n). Then the operator I_α maps $H^p(\boldsymbol{R}^n)$ to $H^q(\boldsymbol{R}^n)$ boundedly, $0 < p < n/\alpha$, $1/q = 1/p - \alpha/n$. In case $p = n/\alpha$, I_α maps $H^p(\boldsymbol{R}^n)$ to $\mathrm{BMO}(\boldsymbol{R}^n)$ boundedly. For $p > n/\alpha$, I_α maps $H^p(\boldsymbol{R}^n)$ to $\Lambda^{loc}_{\alpha-n/p}(\boldsymbol{R}^n)$.*

(2.2.1) Remark. A version of 2.2 using the theory of molecules [2] was announced by M. Taibleson and G. Weiss at the 1978 AMS Summer Institute on Harmonic Analysis in Williamstown, Massachusetts.

(2.3.1) Remark. The corollary is immediate from (2.2), (0.1), and (0.2) and the comments following (0.2). For one writes $I_\alpha = I_{\alpha/k} \circ \ldots \circ I_{\alpha/k}$ (k times, some $k > \alpha/n$) and bootstraps up to the desired result.

(2.4) Remark. Since $H^q(\boldsymbol{R}^n) \cong L^q(\boldsymbol{R}^n)$, $1 < q < \infty$, the theorems are consistent with the known results stated in Section 0. In the classical setting of the unit disc, Corollary 2.3 is proved in [12]. In case $q = 1$, a version of Corollary 2.3 is proved by S. Ross Barker [1] using a variant of the Carleson measures.

(2.5) Remark. The H^p spaces are closely linked to the convolutions (translation structure) on $\boldsymbol{R}^n$. Therefore it is not too surprising that we must require T to be "nearly" a convolution operator in order to obtain results. Condition (2.1.3) is a quantification of this requirement.

(2.6) Remark. We will prove both theorems simultaneously. In this fashion we can isolate the precise place where it appears to be necessary (for Theorem 2.2) that T be a convolution operator.

(2.7) Remark. Theorems 2.1, 2.2 are false if the domain of the operator is replaced by h^p. Indeed, let $n = 1$, $0 < \alpha < 1$, $K(s, t) = |s-t|^{\alpha-1}$, $f(t) = X_{[0,2]}(t)$. So $f \in h^p$, $0 < p \leqslant 1$. Then $Tf(x) \sim |x|^{\alpha-1}$ for x large. Thus Tf is not pth power integrable at ∞, hence not in h^q, any $0 < q \leqslant 1$.

Proof of theorem. Fix n, N_0, α, and $p > n/(n+N_0)$. To prove Theorem 2.1 (Theorem 2.2), we need only prove that $\|Ta\|_{h^q} \leqslant C (\|Ta\|_{H^q} \leqslant C)$ when a is a p-atom. Fix $\varphi \in C_c^{N_0}(\boldsymbol{R}^n)$, $\operatorname{supp}\varphi \subseteq \{|x| \leqslant 1/4\}$, $\varphi \geqslant 0$, $\int \varphi\, dx = 1$. By (1.8) we see that in order to prove Theorem 2.1 it is necessary to show that $\|A^+_{loc}\|_{L^q} \leqslant C$, where $A = Ta$. For Theorem 2.2, it is necessary to prove that $\|A^+\|_{L^q} \leqslant C$.

We may suppose that

$$\operatorname{supp} a \subseteq B(0, \delta) \equiv \{t \in \boldsymbol{R}^n\colon |t| \leqslant \delta\}$$

and that $|a(x)| \leqslant \delta^{-n/p}$. The proof divides rather naturally into three cases. The proofs are identical until we come to Case III. In this case, in order to obtain our results, we either must assume that T is a convolution operator or that $0 < \varepsilon < 1$.

Case I. $|x| \leqslant 80\delta$. Then

$$\text{(2.8)} \qquad \begin{aligned} |\varphi_s * A(x)| &= \Big|\int \varphi_s(x-s) \int\limits_{|t|\leqslant\delta} K(s, t) a(t)\, dt\, ds\Big| \\ &\leqslant \int \varphi_s(x-s)\, \delta^{-n/p} \int\limits_{|t|\leqslant\delta} |K(s, t)|\, dt\, ds \\ &\leqslant c \int \varphi_s(x-s)\, \delta^{-n/p} \delta^\alpha\, ds \leqslant C\delta^{\alpha-n/p} \qquad \text{(by (2.1.1))}. \end{aligned}$$

Case II. $|x| > 80\delta$, $\varepsilon \leqslant 2|x|$. Then

$$|\varphi_\varepsilon * A(x)| = \Big|\int \varphi_\varepsilon(x-s) \int_{|t|\leqslant\delta} a(t)\Big\{K(s,t) - \sum_{|\beta|\leqslant N(p)} D_2^\beta K(s,0)t^\beta/\beta!\Big\}\,dt\,ds\Big|$$

by the orthogonality of p-atoms to all monomials (in t) of order not exceeding $N(p)$. Now on the set where the integral does not vanish we have, since $\varepsilon < 2|x|$, that $|s| \geqslant |x|/2 > 2|t|$.

Moreover, $|t| \leqslant \delta$ on the domain of integration. Thus we use the hypothesis (2.1.2) to estimate the last line by

$$C\int \varphi_\varepsilon(x-s)\,\delta^{-n/p}\,|s|^{-n-N(p)-1+\alpha}\,\delta^{n+N(p)+1}\,ds \leqslant C|x|^{-n-N(p)-1+\alpha}\,\delta^{-n/p+n+N(p)+1}.$$

Case III. $|x| > 80\delta$, $\varepsilon > 2|x|$. Then we write

$$|\varphi_\varepsilon * A(x)| \leqslant \Big|\int \varphi_\varepsilon(x-s)\int K(s-t,0)\,a(t)\,dt\,ds\Big| + $$
$$+\Big|\int \varphi_\varepsilon(x-s)\int [K(s,t) - K(s-t,0)]\,a(t)\,dt\,ds\Big| \equiv X + Y.$$

Now

$$X = \Big|\int K(s,0)\int \varphi_\varepsilon(x-s-t)\,a(t)\,dt\,ds\Big|$$
$$= \Big|\int K(s,0)\int\Big[\varphi_\varepsilon(x-s-t) - \sum_{|\beta|\leqslant N(p)} (D^\beta\varphi_\varepsilon)(x-s)\,(-t)^\beta/\beta!\Big]\,a(t)\,dt\,ds\Big|$$

since a is orthogonal to all monomials (in t) of degree not exceeding $N(p)$. Using Lagrange's form of the remainder in Taylor's formula, and recalling that $\varphi_\varepsilon(u) = \varepsilon^{-n}\varphi(u/\varepsilon)$, we may majorize the last line by

$$C\int_{|s|<2\varepsilon} |K(s,0)| \int_{|t|\leqslant\delta} \varepsilon^{-n-N(p)-1}\,\|\varphi\|_{C^{N(p)+1}}\,|t|^{N(p)+1}\,\delta^{-n/p}\,dt\,ds$$
$$\leqslant C\varepsilon^{-n-N(p)-1+\alpha}\,\delta^{N(p)+n+1-n/p}$$
$$\leqslant C|x|^{-n-N(p)-1+\alpha}\,\delta^{N(p)+n+1-n/p} \qquad \text{(by (2.1.1))}.$$

It is in order to estimate Y that we need to distinguish cases. In case T is a convolution operator then $Y \equiv 0$ and there is nothing to do. In case T is *not* a convolution operator, we use h^p for the target space, so we restrict ε to $0 < \varepsilon \leqslant 1$. Therefore $80\delta < |x| \leqslant 1/2$ and $\delta \leqslant 1/160$ (otherwise Case III is vacuous). We apply (2.1.1b) and (2.1.3) to estimate

$$Y \leqslant \int_{|s|\leqslant 2\delta} + \int_{|s|>2\delta} \leqslant C\varepsilon^{-n}\int_{|t|\leqslant 2\delta} \delta^{\alpha-n/p}\,dt + C\int \varphi_\varepsilon(x-s)\,\delta^{-n/p}\,\delta^{N_0+n}\,ds$$
$$\leqslant C|x|^{-n}\,\delta^{n+\alpha-n/p} + C\delta^{-n/p+N_0+n} \leqslant C|x|^{-n}\,\delta^{n+\alpha-n/p} + C.$$

To summarize cases II, III, we have the following:

(2.9) In case T is a convolution operator, $\operatorname{supp} a \subseteq B(0,\delta)$, and $|x| > 80\delta$, we have

$$A^+(x) \leqslant C|x|^{-n-N(p)-1+\alpha}\,\delta^{N(p)+n+1-n/p}.$$

(2.10) In case T is not a convolution operator, $\operatorname{supp} a \subseteq B(0,\delta)$, and $|x| > 80\delta$, we have

$$A^+_{\mathrm{loc}}(x) \leqslant \begin{cases} \max\{C|x|^{-n}\,\delta^{n+\alpha-n/p} + C,\ C|x|^{-n-N(p)-1+\alpha}\,\delta^{N(p)+n+1-n/p}\}, & 80\delta < |x| \leqslant 1/2, \\ C|x|^{-n-N(p)-1+\alpha}\,\delta^{N(p)+n+1-n/p}, & |x| \geqslant 1/2, \end{cases}$$

Now we complete the proof in a few strokes. In case T is a convolution operator we have

$$\int [A^+(x)]^q\,dx = \int_{|x|\leqslant 80\delta} A^+(x)^q\,dx + \int_{|x|>80\delta} A^+(x)^q\,dx \equiv S + T.$$

Now, by (2.8),

$$S \leqslant C\delta^n\,\delta^{q(\alpha-n/p)} = C.$$

Thus $S \leqslant C$. Likewise, by (2.9),

$$T \leqslant C\delta^{(N(p)+n+1-n/p)q}\int_{80\delta}^{\infty} r^{(-n-N(p)-1+\alpha)q}\,r^{n-1}\,dr \leqslant C$$

by a simple computation. The integral converges by the choice of $N(p)$ and q. This completes the proof of 2.1.

The computations for 2.2 are nearly identical and we omit them. Theorems 2.1 and 2.2 are proved.

§ 3. Concluding remarks. It is known that non-constant coefficient operators do not preserve H^p. This is one of the chief reasons for developing the local Hardy spaces (see [6]). Conditions (2.1.1) and (2.1.2) are a weak version of the homogeneity of the classical kernels K_α. They cannot be essentially weakened. We do not know whether (2.1.3) can be relaxed.

In the case of convolution operators our proof can be simplified. Indeed one need only show that $\|A\|_{L^q} \leqslant C$ and then invoke the fact that convolution operators commute with the Riesz transforms (see [10], [3]). However, the proof we presented in Section 2 is valid, without change, in more general contexts for which there are no Riesz transforms. That is to say, there are rather more general circumstances under which the following assertion holds:

(3.1) A distribution f satisfies $f^+ \in L^p$ if and only if f has an appropriate atomic decomposition.

The assertion (3.1) can be formulated without reference to translation structure or to Riesz transforms. For p sufficiently near 1, assertion (3.1) holds on a number of "spaces of homogeneous type" (see [2]). We would merely like to comment that 3.1 holds, in particular, on the unit ball in C^n (for a proof, see [5]) and that the proofs of 2.1 and 2.2 transfer, without significant change, to that context.

Finally, we note that the techniques of this paper can be used to give some sufficient conditions that an operator $f \to \int K(x,t) f(t)\,dt$ map $H^p(\boldsymbol{R}^m)$ to $h^q(\boldsymbol{R}^n)$, $m \neq n$, $q > p$.

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Received April 18, 1979 (1534)

STUDIA MATHEMATICA, T. LXXIII (1982)

Inequalities for product operators and vector valued ergodic theorems*

by

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Abstract. The basic setting in this consideration is the function class $\Omega_\mu^\alpha(X;\mathfrak{X})$ consisting of all strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions f defined on X such that $[|||f|||/t][\log^+|||f|||/t]^\alpha$ is integrable on the set where $|||f|||>t$ for every $t>0$, where $(X,\mathscr{B},\mu)$ is a σ-finite measure space and $(\mathfrak{X},|||\cdot|||)$ is a Banach space. In this setting some strong type and weak type inequalities (which are indispensable for studying ergodic theorems) for products of L_∞-bounded quasi-linear operators of weak type (1,1) are proved as generalizations of the maximal and the dominated ergodic theorems for the ergodic maximal operators. Moreover, we demonstrate that these results enable us to obtain some vector valued extensions of the ergodic theorems of Dunford–Schwartz type and further generalizations to functions in the class $\Omega_\mu^\alpha(X;\mathfrak{X})$. Local (mean and pointwise) ergodic theorems are also obtained to add to the above results.

1. Introduction. The ergodic theorems (usually called the mean ergodic theorem and the pointwise ergodic theorem) had received a considerably general operator-theoretic treatment in the case where the underlying space is just the Lebesgue space $L_p = L_p(X,\mathscr{B},\mu)$, where $(X,\mathscr{B},\mu)$ is a σ-finite measure space. After the lapse of time, especially, a new approach to the study of pointwise ergodic theorems has been developed in several recent papers. The contrivance to this is the weak type inequality powerful and indispensable for investigating the convergence almost everywhere of operator averages. The first step in this direction was taken by Fava [4] who extended the so-called "non-commuting ergodic theorems" of Dunford and Schwartz for positive operators to functions in a larger class than the space L_p by means of a weak type inequality for products of maximal operators on L_1+L_∞ which is the class of all functions of the form $f = g+h$ with $g\in L_1$ and $h \in L_\infty$. The method of proving the Fava inequality, as his proof shows, calls for the positivity of maximal operators. Recently the author [8] has generalized his inequality to the case of quasi-linear operators without assumption

* Some partial results in this paper were presented at the Meeting of Mathematical Society of Japan on October 4, 1979.