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A characterization of a two-weight norm inequality for maximal operators

by

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Abstract. It is proved that if $w(x)$ and $v(x)$ are non-negative functions on R^n and $1 < p \leqslant q \leqslant \infty$, $p < \infty$, then

$$\Big(\int [Mf(x)]^q w(x)\,dx\Big)^{1/q} \leqslant C\Big(\int |f(x)|^p v(x)\,dx\Big)^{1/p}$$

for all f in $L^p(v)$ if and only if

$$\Big(\int_Q [M(\chi_Q v^{1-p'})(x)]^q w(x)\,dx\Big)^{1/q} \leqslant C\Big(\int_Q v(x)^{1-p'}\,dx\Big)^{1/p} < \infty$$

for all cubes Q, where M denotes the maximal operator

$$Mf(x) = \sup_{x\in Q \text{ cube}} |Q|^{-1}\int_Q |f(x)|\,dx.$$

More generally, it is shown that the analogue of this result holds with M replaced by the weighted fractional maximal operator

$$M_{\mu,a}f(x) = \sup_{x\in Q} |Q|_\mu^{a/n-1}\int_Q |f|\,d\mu$$

provided $0 \leqslant a < n$ and μ is a positive Borel measure on R^n satisfying a doubling condition.

§ 1. Introduction. Throughout this paper, Q will denote a cube in R^n with sides parallel to the co-ordinate planes. For $r > 0$, rQ will denote the cube with centre that of Q and diameter r times that of Q. If μ is a positive Borel measure on R^n, we set $|Q|_\mu = \int_Q d\mu$. We use $L^p(\mu)$ to denote the usual Lebesgue space on $(R^n, d\mu)$ and we denote by M the maximal operator

$$Mf(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(x)|\,dx \quad \text{for suitable } f.$$

Finally, the letter C will be used to denote a positive constant not necessarily the same at each occurrence.

In [4] B. Muckenhoupt showed that for $1 < p < \infty$ and w, v non-negative functions on R^n, the weak-type inequality

$$\int_{\{Mf>\lambda\}} w(x)\,dx \leqslant C\lambda^{-p}\int |f(x)|^p v(x)\,dx \tag{1.1}$$

holds for all f in $L^p(v)$ if and only if

$$\Big(|Q|^{-1}\int_Q w(x)\,dx\Big)\Big(|Q|^{-1}\int_Q v(x)^{1-p'}\,dx\Big)^{p-1} \leqslant C \tag{1.2}$$

for all cubes Q. In addition it was shown that if $w = v$, then (1.2) in fact implies the strong-type inequality

$$\int [Mf(x)]^p w(x)\,dx \leqslant C\int |f(x)|^p v(x)\,dx \tag{1.3}$$

for all f in $L^p(v)$. However (1.2) is not in general sufficient for (1.3) ([4]; p. 218).

In [6] B. Muckenhoupt and R. L. Wheeden showed that (1.3) implies

$$\Big(|Q|^{-1}\int [M\chi_Q(x)]^p w(x)\,dx\Big)\Big(|Q|^{-1}\int_Q v(x)^{1-p'}\,dx\Big)^{p-1} \leqslant C \tag{1.4}$$

for all cubes Q, that in the presence of various additional assumptions on f or w and v, (1.4) implies (1.3) and conjectured that (1.4) is sufficient for (1.3) in general. In § 4 below an example is given to show that this conjecture is false.

One of the main results of this paper is that for $1 < p < \infty$, (1.3) is equivalent to the following condition on w and v.

$$\int_Q [M(\chi_Q v^{1-p'})(x)]^p w(x)\,dx \leqslant C\int_Q v(x)^{1-p'}\,dx < \infty \tag{1.5}$$

for all cubes Q. This is a special case of the result mentioned in the abstract which in turn is a special case of Theorem B in § 3 below.

In the next section we characterize the two-weight norm inequality for certain dyadic maximal operators and then use this in § 3 to obtain results on the usual non-dyadic maximal operators. The final section contains an example of weights w and v on R that satisfy (1.4) with $p = 2$ but not (1.3) with $p = 2$.

§ 2. Weighted norm inequalities for dyadic maximal operators. Let $R_+^n = [0, \infty)^n$ and $Z_+^n = \{0, 1, 2, \ldots\}^n$. Throughout this section, Q will denote a set contained in R_+^n of the form $\prod_{i=1}^n [x_i, x_i+2^k)$, $x \in 2^k Z_+^n$ for some k in Z. Such sets will be referred to as dyadic cubes. If μ is a positive Borel measure on R_+^n and $0 \leqslant \alpha < n$, we define the dyadic weighted fractional maximal operator $M_{\mu,\alpha}$ by

$$M_{\mu,\alpha}f(x) = \sup_{\substack{x\in Q \text{ dyadic cube}\\ |Q|_\mu > 0}} |Q|_\mu^{\alpha/n-1}\int_Q |f|\,d\mu.$$

THEOREM A. *Suppose $0 \leqslant \alpha < n$, $1 < p \leqslant q \leqslant \infty$, $p < \infty$. Let μ, ν, and ω be positive Borel measures on R_+^n with μ locally finite. Then*

$$\|M_{\mu,\alpha}f\|_{L^q(\omega)} \leqslant C\|f\|_{L^p(\nu)} \tag{2.1}$$

for all f in $L^p(\nu)$ if and only if $\mu \ll \nu$ and

$$\Big\|\chi_Q M_{\mu,\alpha}\Big(\chi_Q \frac{d\mu}{d\nu}^{p'-1}\Big)\Big\|_{L^q(\omega)} \leqslant C\Big\|\chi_Q\frac{d\mu}{d\nu}^{p'-1}\Big\|_{L^p(\nu)} < \infty \tag{2.2}$$

for all dyadic cubes $Q \subset R_+^n$.

Proof. Assume that (2.1) holds. Suppose, in order to derive a contradiction, that $E \subset R_+^n$ is a bounded Borel set satisfying $|E|_\mu > 0 = |E|_\nu$ and set $f = \chi_E$ in (2.1). Now $M_{\mu,\alpha}f > 0$ on R_+^n and thus the left side of (2.1) is positive while the right side is zero. This contradiction shows that $\mu \ll \nu$. Suppose now, in order to derive a contradiction, that

$$\int_Q \frac{d\mu}{d\nu}^{p'-1} d\mu = \int_Q \frac{d\mu}{d\nu}^{p'} d\nu = \infty.$$

Then there is f in $L^p(\nu)$ such that

$$\int_Q f\frac{d\mu}{d\nu}\,d\nu = \infty.$$

Thus $M_{\mu,\alpha}f \equiv \infty$ and the left side of (2.1) is infinite while the right side is finite. This contradiction shows that

$$\int_Q \frac{d\mu}{d\nu}^{p'-1} d\mu < \infty$$

for all dyadic cubes Q. Finally, if we let

$$f = \chi_Q \frac{d\mu}{d\nu}^{p'-1}$$

in (2.1) we obtain (2.2).

Conversely assume that $\mu \ll \nu$ and that (2.2) holds. We first establish (2.1) with $M_{\mu,\alpha}$ replaced by the smaller operator $M_{\mu,\alpha}^R$, $R > 0$, where

$$M_{\mu,\alpha}^R f(x) = \sup_{\substack{x\in Q \text{ dyadic cube}\\ |Q|_\mu > 0,\ \operatorname{diam} Q \leqslant R}} |Q|_\mu^{\alpha/n-1}\int_Q |f|\,d\mu.$$

We shall need the following elementary covering lemma.

DEFINITION. Let Ω be a collection of sets. A set Q in Ω is *maximal* (*relative to* Ω) if $Q \subset Q'$, $Q' \in \Omega$ implies $Q = Q'$.

LEMMA 1. *Let Ω be a collection of dyadic cubes satisfying $\sup\limits_{Q\in\Omega} \operatorname{diam} Q < \infty$. Then every cube in Ω is contained in some maximal cube and the maximal cubes are mutually disjoint.*

We now return to the proof of Theorem A. Let f be in $L^p(\nu)$ and for each k in Z, let $\{Q_j^k\}_{j\in J_k}$ be an enumeration of the cubes maximal relative to the collection

$$\Big\{Q \text{ dyadic};\ |Q|_\mu > 0,\ \operatorname{diam} Q \leqslant R,\ |Q|_\mu^{a/n-1} \int_Q |f|\, d\mu > 2^k\Big\}.$$

From Lemma 1 we obtain

$$\{M_{\mu,a}^R f > 2^k\} = \bigcup_{j\in J_k} Q_j^k \quad \text{for} \quad k \text{ in } Z, \tag{2.3}$$

$$Q_i^k \cap Q_j^k = \emptyset \quad \text{for} \quad i \neq j,\ k \text{ in } Z, \tag{2.4}$$

$$|Q_j^k|_\mu^{a/n-1} \int_{Q_j^k} |f|\, d\mu > 2^k \quad \text{for} \quad j \in J_k,\ k \text{ in } Z. \tag{2.5}$$

From (2.5) and Hölder's inequality we have

$$\begin{aligned} |Q_j^k|_\mu^{1-a/n} &\leqslant 2^{-k} \int_{Q_j^k} |f| \frac{d\mu}{d\nu}\, d\nu \\ &\leqslant 2^{-k}\Big(\int_{Q_j^k} |f|^p\, d\nu\Big)^{1/p}\Big(\int_{Q_j^k} \frac{d\mu}{d\nu}^{p'}\, d\nu\Big)^{1/p'} \\ &= 2^{-k}|Q_j^k|_\sigma^{1/p'}\Big(\int_{Q_j^k} |f|^p\, d\nu\Big)^{1/p}, \end{aligned} \tag{2.6}$$

where we have set

$$d\sigma = \frac{d\mu}{d\nu}^{p'}\, d\nu = \frac{d\mu}{d\nu}^{p'-1}\, d\mu.$$

Note that $|Q_j^k|_\sigma$ is finite by (2.2) and positive since $|Q_j^k|_\mu > 0$.

We now dispose of the case $1 < p < q = \infty$. Since

$$M_{\mu,a}\Big(\chi_{Q_j^k} \frac{d\mu}{d\nu}^{p'-1}\Big) \geqslant |Q_j^k|_\mu^{a/n-1}\, |Q_j^k|_\sigma \quad \text{on} \quad Q_j^k, \tag{2.7}$$

we obtain from (2.2) with $q = \infty$ that $|Q_j^k|_\mu^{a/n-1}|Q_j^k|_\sigma \leqslant C\,|Q_j^k|_\sigma^{1/p}$ whenever $|Q_j^k|_\omega \neq 0$. Using (2.6) we have

$$|Q_j^k|_\sigma^{1/p'} \leqslant C\,|Q_j^k|_\mu^{1-a/n} \leqslant C 2^{-k} |Q_j^k|_\sigma^{1/p'} \Big(\int_{Q_j^k} |f|^p\, d\nu\Big)^{1/p}.$$

Since $|Q_j^k|_\sigma$ is positive and finite, we obtain that $2^k \leqslant C\,\|f\|_{L^p(\nu)}$ whenever $|Q_j^k|_\omega \neq 0$ and in view of (2.3) this yields (2.1) for $q = \infty$.

We now suppose $q < \infty$. From (2.2) and (2.7) we obtain

$$|Q_j^k|_\omega \leqslant |Q_j^k|_\mu^{q-qa/n}|Q_j^k|_\sigma^{q/p-q} \leqslant 2^{-kq}\Big(\int_{Q_j^k} |f|^p\, d\nu\Big)^{q/p} \quad \text{by (2.6)} \tag{2.8}$$

and using (2.3) we have

$$|\{M_{\mu,a}^R f > 2^k\}|_\omega = \sum_{j\in J_k} |Q_j^k|_\omega \leqslant (2^{-k}\|f\|_{L^p(\nu)})^q$$

by (2.8) since $p \leqslant q$. In particular $|\{M_{\mu,a}^R f = \infty\}|_\omega = 0$ and if $\overset{*}{Q}{}_j^k = Q_j^k \setminus \{M_{\mu,a}^R f > 2^{k+1}\}$, then

$$\begin{aligned} \int [M_{\mu,a}^R f]^q\, d\omega &\leqslant 2^q \sum_{k\in Z} 2^{kq} |\{2^k < M_{\mu,a}^R f \leqslant 2^{k+1}\}|_\omega \\ &\leqslant 2^q \sum_{k,j\in J_k} |\overset{*}{Q}{}_j^k|_\omega \Big[|Q_j^k|_\mu^{a/n-1} \int_{Q_j^k} |f|\, d\mu\Big]^q \quad \text{by} \quad \text{(2.3) and (2.5)} \\ &= 2^q \sum_{k,j\in J_k} |\overset{*}{Q}{}_j^k|_\omega \Big[|Q_j^k|_\mu^{a/n-1} \int_{Q_j^k} \frac{d\mu}{d\nu}^{p'-1} d\mu\Big]^q \Big[|Q_j^k|_\sigma^{-1} \int_{Q_j^k} |f| \frac{d\mu}{d\nu}^{1-p'} d\sigma\Big]^q \\ &= 2^q \sum_{k,j\in J_k} \gamma_j^k \Big[|Q_j^k|_\sigma^{-1} \int_{Q_j^k} |f| \frac{d\mu}{d\nu}^{1-p'} d\sigma\Big]^q, \end{aligned} \tag{2.9}$$

where the non-negative numbers γ_j^k satisfy

$$\begin{aligned} \gamma_j^k &= |\overset{*}{Q}{}_j^k|_\omega \Big[|Q_j^k|_\mu^{a/n-1} \int_{Q_j^k} \frac{d\mu}{d\nu}^{p'-1} d\mu\Big]^q \\ &\leqslant \int_{\overset{*}{Q}{}_j^k} \Big[M_{\mu,a}\Big(\chi_{Q_j^k} \frac{d\mu}{d\nu}^{p'}\Big)\Big]^q d\omega. \end{aligned} \tag{2.10}$$

Now let $\Omega = \{(k,j);\ k \in Z,\ j \in J_k\}$ and let γ be the measure on Ω that assigns mass γ_j^k to (k,j). Define

$$T\colon\ (L^1 + L^\infty)(R^n, d\sigma) \to L^\infty(\Omega, d\gamma)$$

by

$$Tg = \Big\{|Q_j^k|_\sigma^{-1}\int_{Q_j^k}|g|\,d\sigma\Big\}_{(k,j)\in\Omega}, \qquad g\in(L^1+L^\infty)(\sigma).$$

Clearly T is sublinear and of strong-type (∞,∞). We claim that T is of weak-type $(1, q/p)$. Let $\lambda>0$. If $\{I_i\}_i$ denotes the cubes maximal relative to the collection

$$\Big\{Q_j^k;\ |Q_j^k|_\sigma^{-1}\int_{Q_j^k}|g|\,d\sigma>\lambda\Big\} \tag{2.11}$$

then using Lemma 1 we have

$$|\{Tg>\lambda\}|_\gamma = \sum\Big\{\gamma_j^k;\ |Q_j^k|_\sigma^{-1}\int_{Q_j^k}|g|\,d\sigma>\lambda\Big\} = \sum_i\sum_{Q_j^k\subset I_i}\gamma_j^k$$

$$\leqslant \sum_i\sum_{Q_j^k\subset I_i}\int_{\tilde{Q}_j^k}\Big[M_{\mu,a}\Big(\chi_{I_i}\frac{d\mu}{d\nu}^{p'-1}\Big)\Big]^q d\omega$$

by (2.10) and the fact that $Q_j^k\subset I_i$

$$\leqslant \sum_i\int_{I_i}\Big[M_{\mu,a}\Big(\chi_{I_i}\frac{d\mu}{d\nu}^{p'-1}\Big)\Big]^q d\omega \qquad \text{by (2.4)}$$

$$\leqslant C\sum_i |I_i|_\sigma^{q/p} \qquad \text{by (2.2)}$$

$$\leqslant C\Big(\sum_i |I_i|_\sigma\Big)^{q/p} \qquad \text{since } p\leqslant q$$

$$\leqslant C\Big(\sum_i\frac{1}{\lambda}\int_{I}|g|\,d\sigma\Big)^{q/p} \leqslant C\Big(\frac{1}{\lambda}\int|g|\,d\sigma\Big)^{q/p}$$

since the I_i are in the collection (2.11) and are mutually disjoint by Lemma 1. This establishes that T is of weak-type $(1, q/p)$.

The Marcinkiewicz interpolation theorem now implies that T is of strong-type (p, q) and thus using (2.9) we have

$$\int[M_{\mu,a}^R f]^q d\omega \leqslant 2^q\Big\|T\Big(f\frac{d\mu}{d\nu}^{1-p'}\Big)\Big\|_{L^q(\gamma)}^q$$

$$\leqslant C_{p,q}\Big\|f\frac{d\mu}{d\nu}^{1-p'}\Big\|_{L^p(\sigma)}^q \leqslant C_{p,q}\|f\|_{L^p(\nu)}^q$$

since

$$d\sigma = \frac{d\mu}{d\nu}^{p'}d\nu \quad\text{and}\quad p(1-p')+p'=0.$$

Thus (2.1) has been established for $M_{\mu,a}^R$ with a constant independent of $R>0$. Now use the fact that $M_{\mu,a}^R f\uparrow M_{\mu,a}f$ as $R\uparrow\infty$ together with the monotone convergence theorem to obtain (2.1) for $M_{\mu,a}$. This completes the proof of Theorem A.

Remark 1. If the measure σ in the above argument had satisfied the doubling condition $|2Q|_\sigma\leqslant C|Q|_\sigma$ for all cubes, we could have appealed to Hormander's version of Carleson's theorem ([3]; Theorems 2.3 and 2.4–see also P.L. Duren's extension to $p<q$ in [2]) to deduce (2.1) directly from (2.9) and the fact that the γ_j^k satisfy the following "Carleson measure" condition

$$\sum_{Q_j^k\subset 4Q_s^t}\gamma_j^k\leqslant C|Q_s^t|_\sigma^{q/p}, \qquad t\in Z,\ s\in J_t$$

which itself is an immediate consequence of (2.2) if σ satisfies a doubling condition. The argument used above to obtain (2.1) from (2.9) is an adaptation of the techniques used in [3] and [2].

§ 3. Weighted norm inequalities for maximal operators. Throughout this section Q will denote a set contained in R^n of the form $\prod_{i=1}^n[x_i, x_i+h]$ for some x in R^n, $h>0$. Such sets will be called cubes. Suppose μ is a positive Borel measure on R^n satisfying the following doubling condition.

$$|2Q|_\mu\leqslant C|Q|_\mu \quad\text{for all cubes } Q. \tag{D}$$

Remark 2. If μ is a positive Borel measure satisfying (D), then $0<|Q|_\mu<\infty$ and $|\partial Q|_\mu=0$ for all cubes Q where ∂Q denotes the boundary of Q.

Then if $0\leqslant a<n$, we define the weighted fractional maximal operator $M_{\mu,a}$ (not to be confused with the "dyadic" $M_{\mu,a}$ defined in § 2) by

$$M_{\mu,a}f(x) = \sup_{x\in Q}|Q|_\mu^{a/n-1}\int_Q|f|\,d\mu.$$

THEOREM B. *Suppose $0\leqslant a<n$, $1<p\leqslant q\leqslant\infty$, $p<\infty$. Let μ,ν, and ω be positive Borel measures on R^n and assume that μ satisfies the doubling condition* (D). *Then*

$$\|M_{\mu,a}f\|_{L^q(\omega)}\leqslant C\|f\|_{L^p(\nu)} \tag{3.1}$$

for all f in $L^p(\nu)$ if and only if $\mu\ll\nu$ and

$$\Big\|\chi_Q M_{\mu,a}\Big(\chi_Q\frac{d\mu}{d\nu}^{p'-1}\Big)\Big\|_{L^q(\omega)}\leqslant C\Big\|\chi_Q\frac{d\mu}{d\nu}^{p'-1}\Big\|_{L^p(\nu)}<\infty \tag{3.2}$$

for all cubes $Q\subset R^n$.

Proof. The proof that (3.1) implies both $\mu \ll \nu$ and (3.2) is virtually identical to the proof given in Theorem A that (2.1) implies $\mu \ll \nu$ and (2.2). We do not repeat the details.

Conversely assume that $\mu \ll \nu$ and (3.2) holds. We use an argument of C. Fefferman and E. M. Stein ([9], p. 112). We say that a cube contained in R^n is a closed dyadic cube if it is of the form $\prod_{i=1}^{n} [x_i, x_i+2^k]$, $x \in 2^k Z^n$ for some k in Z. For t in R^n define ${}^tM_{\mu,a}$ by

$$ {}^tM_{\mu,a}f(x) = \sup_{\substack{x\in Q \\ Q-t \text{ closed dyadic cube}}} |Q|_\mu^{a/n-1} \int_Q |f|\,d\mu . $$

Remark 2 shows that (3.2) implies (2.2) and it now follows easily from Theorem A (applied to translations and reflections of the cone R^n_+) that

$$ \|{}^tM_{\mu,a}f\|_{L^q(\omega)} \leqslant C\|f\|_{L^p(\nu)} $$

for all f in $L^p(\nu)$ with a constant C independent of t in R^n. For $R > 0$, set

$$ M^R_{\mu,a}f(x) = \sup_{x\in Q,\, l(Q)\leqslant R} |Q|_\mu^{a/n-1} \int_Q |f|\,d\mu , $$

where $l(Q)$ denotes the side length of the cube Q. The proof of Theorem B can be completed by the monotone convergence theorem if we can show that (3.1) holds with $M_{\mu,a}$ replaced by $M^R_{\mu,a}$ and with a constant C independent of R. That this is so is an immediate consequence of the following lemma.

LEMMA 2. *If μ satisfies* (D) *and $0 \leqslant a < n$, then there is a constant $C < \infty$ such that*

$$ M^{2^k}_{\mu,a}f(x) \leqslant C \int_{[-2^{k+2},\,2^{k+2}]^n} {}^tM_{\mu,a}f(x)\,\frac{dt}{2^{n(k+3)}} $$

for all x in R^n, k in Z, and locally μ-integrable f.

Proof of Lemma 2. Fix x in R^n and k in Z. Suppose I is a cube satisfying $x \in I$,

$$ |I|_\mu^{a/n-1} \int_I |f|\,d\mu > \tfrac{1}{2} M^{2^k}_{\mu,a}f(x) $$

where $2^{j-1} < l(I) \leqslant 2^j$, j in Z, $j \leqslant k$. Let Ω consist of the t in $[-2^{k+2}, 2^{k+2}]^n$ such that there is $Q_t \supset I$ with $Q_t - t$ a closed dyadic cube and $l(Q_t) = 2^{j+1}$.

For t in Ω, $Q_t \subset 7I$ and so $|Q_t|_\mu \leqslant C|I|_\mu$ by (D). Thus

$$ {}^tM_{\mu,a}f(x) \geqslant |Q_t|_\mu^{a/n-1} \int_{Q_t} |f|\,d\mu \geqslant C^{a/n-1}|I|_\mu^{a/n-1} \int_I |f|\,d\mu > \frac{C^{a/n-1}}{2} M^{2^k}_{\mu,a}f(x). $$

It is geometrically evident that the Lebesgue measure of Ω is at least $2^{n(k+1)}$ and so

$$ \int_{[-2^{k+2},\,2^{k+2}]^n} {}^tM_{\mu,a}f(x)\,\frac{dt}{2^{n(k+3)}} \geqslant \frac{C^{a/n-1}}{2^{2n+1}} M^{2^k}_{\mu,a}f(x). $$

This completes the proof of Lemma 2 and hence also the proof of Theorem B.

Remark 3. Condition (3.1) has been studied in connection with the following weakened form of (3.2).

$$ \text{(3.3)} \qquad \Big[|Q|_\mu^{-\beta} \int_Q d\omega\Big]^{1/q} \Big[|Q|_\mu^{-1} \int_Q \frac{d\mu}{d\nu}^{p'-1} d\mu\Big]^{1/p'} \leqslant C $$

for all cubes Q, $\beta/q = 1/p - a/n$. Here one assumes $0 < \beta \leqslant 1$, $0 \leqslant a < n$, and $1 < p \leqslant q < \infty$. Clearly, (3.1) implies (3.3). In [7] the author has shown that if

$$ d\mu \quad \text{and} \quad d\sigma = \frac{d\mu}{d\nu}^{p'-1} d\mu $$

are comparable measures in the sense of R. R. Coifman and C. Fefferman ([1]; p. 248) then (3.3) implies (3.1). In another direction, B. Muckenhoupt and R. L. Wheeden showed ([5]—see also Welland [8]) that if μ is Lebesgue measure on R^n, $d\omega = w\,d\mu$, $d\nu = w^{p/q}d\mu$ and $1/q = 1/p - a/n$, then (3.3) again implies (3.1). Of course (3.3) is not in general sufficient for (3.1) ([4]; p. 218).

§ 4. A counterexample.

Recall the following example due to B. Muckenhoupt ([4]; p. 218) of a pair of weights (w, v) satisfying (1.2) but not (1.3). Let

$$ f(x) = x^{-1}|\log x|^{-2}\chi_{(0,\frac12)}(x). $$

Then $Mf(x) \approx x^{-1}|\log x|^{-1}$ for x in $(0, \frac12)$ and if $w = (Mf)^{-1}\chi_{(0,\frac12)}$ and $v = f^{-1}$, then the pair (w, v) satisfies (1.2) with $p = 2$ while $\int (Mf)^2 w = \infty$ and $\int f^2 v < \infty$. Note however that (w, v) fails to satisfy (1.4).

Our example of a pair (w, v) satisfying (1.4) but not (1.3) is essentially obtained by rearranging the above function f and setting $w = (Mf)^{-1}$ and $v = f^{-1}$. We now give the details. Let

$$ I^0_1 = [0, 1], $$

$$ I^1_1 = [0, \tfrac14],\ I^1_2 = [\tfrac34, 1], $$

$$ I^2_1 = [0, \tfrac{1}{16}], \quad I^2_2 = [\tfrac{3}{16}, \tfrac14], \quad I^2_3 = [\tfrac34, \tfrac{13}{16}], \quad I^2_4 = [\tfrac{15}{16}, 1], $$

$$ I^3_1 = [0, \tfrac{1}{64}],\ \text{etc.} $$

Set

$$f = \chi_{I_1^0} + \sum_{k=1}^{\infty} \frac{2^k}{k^2} \sum_i \chi_{I_i^k}.$$

Then for $k \geqslant 1$

$$\int_{I_i^k} f = \Big(\sum_{j \leqslant k} \frac{2^j}{j^2}\Big)|I_i^k| + \sum_{j>k} \frac{2^j}{j^2} \sum_{I_s^j \subset I_i^k} |I_s^j| \tag{4.1}$$

$$= \Big(\sum_{j \leqslant k} \frac{2^j}{j^2}\Big)|I_i^k| + \sum_{j>k} \frac{2^j}{j^2} 2^{j-k} 4^{-j}$$

$$\approx \frac{2^{-k}}{k} = |I_i^k| \frac{2^k}{k}.$$

From (4.1) we easily obtain that

$$Mf \geqslant M\chi_{I_1^0} + \sum_{k=1}^{\infty} \frac{2^k}{k} \sum_i \chi_{I_i^k} \tag{4.2}$$

and setting $w = (Mf)^{-1} \chi_{I_1^0}$ and $v = f^{-1}$ we have

$$\int (Mf)^2 w = \int_0^1 Mf \geqslant 1 + \sum_{k=1}^{\infty} \frac{2^k}{k} 2^k 4^{-k} = \infty, \tag{4.3}$$

$$\int f^2 v = \int_0^1 f = 1 + \sum_{k=1}^{\infty} \frac{2^k}{k^2} 2^k 4^{-k} < \infty.$$

Finally we claim that (w, v) satisfies (1.4) with $p = 2$. We give details in the case $Q = I_1^k = [0, 4^{-k}]$, $k \geqslant 1$, and leave the general case to the interested reader. We first note that (4.2) implies

$$\int_{I_i^k} w(x)\,dx \leqslant k 2^{-k} |I_i^k| = k 8^{-k} \tag{4.4}$$

for $k \geqslant 1$ and thus

$$\int (M\chi_{I_1^k}(x))^2 w(x)\,dx \leqslant \int_{I_1^k} w(x)\,dx + \sum_{j=1}^{k-1} (4^{j-k+1})^2 \int_{I_1^j} w(x)\,dx + \Big(\int_{I_1^0} w(x)\,dx\Big)(4^{-k})^2 \tag{4.5}$$

$$\lesssim \sum_{j=1}^{k} (4^{j-k+1})^2 j 8^{-j} + 16^{-k} \quad \text{by (4.4)}$$

$$\approx k 8^{-k}.$$

From (4.1) we obtain $\int_{I_1^k} v(x)^{-1}\,dx = \int_{I_1^k} f \approx 2^{-k}/k$ and multiplying this by inequality (4.5) we obtain

$$\Big(\int (M\chi_{I_1^k})^2 w\Big)\Big(\int_{I_1^k} v^{-1}\Big) \leqslant 16^{-k} = |I_1^k|^2. \tag{4.6}$$

Inequality (4.6) shows that (1.4) holds with $p = 2$ for cubes $Q = I_1^k$, $k \geqslant 1$ and the verification of (1.4) for general cubes Q follows a similar line of reasoning. Inequality (4.3) on the other hand shows that (1.3) fails with $p = 2$ for the pair (w, v).

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