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# Spline approximation and Besov spaces on compact manifolds

by

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**Abstract.** We construct, for each $d$ and $m = 1, 2, \ldots$, spline systems on the cube $I^d$ which are Schauder bases in the (properly defined) Sobolev and Besov spaces $W_p^k(I^d)$ and $B_{p,q}^s(I^d)$, where $|k| \leqslant m$, $|s| < m$, $1 \leqslant p, q \leqslant \infty$. These bases satisfy the analogues of Bernstein's and Jackson's inequalities and so do their biorthogonal systems. This allows us, e.g. to obtain the interpolation formulae $B_{p,q}^s = (W_p^k, W_p^l)_{\vartheta,q}$, $\vartheta = (s-k)/(l-k)$, for the function spaces on a compact $d$-dimensional $C^\infty$ manifold $M$ for all $-\infty < k < s < l < \infty$ and $1 \leqslant p, q \leqslant \infty$.

**1. Introduction.** This paper consists of two parts. In Sections 1–3 we give a rather complete description of duality and (real) interpolation between Sobolev spaces $W_p^k(A)$ and $\mathring{W}_p^k(A)$, where $-\infty < k < \infty$, $1 \leqslant p \leqslant \infty$ and $A$ is either the cube $I^d = \langle 0, 1\rangle^d$ or a $d$-dimensional compact $C^\infty$ manifold (with or without boundary). This leads to equivalence of various interpolation definitions of Besov spaces $B_{p,q}^s(A)$, $\mathring{B}_{p,q}^s(A)$, where $-\infty < s < \infty$, $1 \leqslant p, q \leqslant \infty$.

We use real variables methods only. The most important tool is a result on approximation in Sobolev spaces on cubes (Theorem 5.16) which relies on the spline functions techniques developed in the second part (Sections 4 and 5). In Section 2 we use only a special case of Theorem 5.16 quoted as Lemma 2.1. The full contents of that result (and also a generalization thereof) are needed in [9] where we construct Schauder bases in $W_p^k(M)$ ($M$ being a compact $C^\infty$ manifold) which satisfy Bernstein's and Jackson's inequalities and so do their biorthogonal systems.

Let us describe our notation. By $A$ we shall denote a compact subset of a $d$-dimensional $C^\infty$ manifold (without boundary) $M$. (Other conditions on $A$ will be specified later.) We put

$$C^\infty(A) = \{f|_A \colon f \in C^\infty(M)\},$$

$$\mathring{C}^\infty(A) = \{g \in C^\infty(A) \colon \operatorname{supp} g \subseteq \operatorname{Int} A\}.$$

Letters $k, l, m, n, r, N$ will always denote integers and $p, q$ w satisfy $1 \leqslant p, q \leqslant \infty$. We set $p' = p/(p-1)$, $q' = q/(q-1)$.

We consider several "Sobolev spaces" on $A$ (the $\mathscr{C}$-spaces concide with the $\mathscr{M}$-spaces if and only if $1 < p < \infty$). The scheme of the definition is as follows.

Having defined a suitable norm $\|\cdot\|_{k,p}$ on $C^\infty(A)$, we let $\mathscr{C}W_p^k(A)$ denote the completion of $(C^\infty(A), \|\cdot\|_{k,p})$. Analogously, $\mathscr{C}\mathring{W}_p^k(A)$ will be the completion of $(\mathring{C}^\infty(A), \|\cdot\|^\circ_{k,p})$. Finally, we set

$$\mathscr{M}W_p^k(A) = \big(\mathscr{C}\mathring{W}_{p'}^{-k}(A)\big)^*, \qquad \mathscr{M}\mathring{W}_p^k(A) = \big(\mathscr{C}W_{p'}^{-k}(A)\big)^*.$$

It will follow that $\mathscr{M}W_p^k(A)$ is a linear subspace of $\mathscr{D}'(\operatorname{Int}A)$, the space of distributions on the interior of $A$, while

$$\mathscr{M}\mathring{W}_p^k(A) \subseteq \mathscr{D}'(A) := \{T \in \mathscr{D}'(M) \colon \operatorname{supp} T \subseteq A\}.$$

We shall fix a finite measure $\mu$ on $A$. (It will be the Lebesgue measure if $M = \mathbf{R}^d$.) One will have for $f \in C^\infty(A)$, $g \in \mathring{C}^\infty(A)$

(1.1) $$\Big|\int_A fg\,d\mu\Big| \leqslant c\,\|f\|_{k,p}\,\|g\|^\circ_{-k,p'},$$

where $c = c(k, \mu) < \infty$ does not depend on $f$ and $g$. It follows that the maps $J$, $\mathring{J}$ defined by

$$Jf = f d\mu \in \mathscr{D}'(\operatorname{Int}A), \qquad \mathring{J}g = g d\mu \in \mathscr{D}'(A)$$

can be extended to continuous linear operators

(1.2) $$J\colon \mathscr{C}W_p^k(A) \to \mathscr{M}W_p^k(A), \qquad \mathring{J}\colon \mathscr{C}\mathring{W}_p^k(A) \to \mathscr{M}\mathring{W}_p^k(A).$$

For suitable $A$ and $\mu$, the operators $J$, $\mathring{J}$ will be isomorphic embeddings for each $k, p$.

One has $J\big(\mathscr{C}W_p^k(A)\big) = \mathscr{M}W_p^k(A)$, $\mathring{J}\big(\mathscr{C}\mathring{W}_p^k(A)\big) = \mathscr{M}\mathring{W}_p^k(A)$ if and only if $1 < p < \infty$. (If $p \in \{1, \infty\}$, then the $\mathscr{M}$-spaces are non-separable.) If $k \geqslant 0$ and $M = \mathbf{R}^d$, then, under mild conditions on $A$, $\mathscr{C}W_p^k(A)$ can be identified with the classical space $W_p^k(A)$ if $1 \leqslant p < \infty$ and $\mathscr{M}W_p^k(A)$ can be identified with $W_p^k(A)$ if $1 < p \leqslant \infty$.

For the definition and properties of the real interpolation spaces $(\cdot,\cdot)_{\theta,q}$ we refer to [3]; some definitions are recalled in Section 2. The Besov spaces on $A$ will be defined as follows. If $s$ is a real number, we pick $l < s < r$, write $s = (1-\theta)l + \theta r$ and put

(1.3) $$B_{p,q}^s(A) = \big(\mathscr{M}W_p^l(A), \mathscr{M}W_p^r(A)\big)_{\theta,q} = \big(\mathscr{C}W_p^l(A), \mathscr{C}W_p^r(A)\big)_{\theta,q},$$

(1.4) $$\mathring{B}_{p,q}^s(A) = \big(\mathscr{M}\mathring{W}_p^l(A), \mathscr{M}\mathring{W}_p^r(A)\big)_{\theta,q} = \big(\mathscr{C}\mathring{W}_p^l(A), \mathscr{C}\mathring{W}_p^r(A)\big)_{\theta,q}.$$



The equality of the $\mathscr{C}$-and $\mathscr{M}$-interpolation spaces in (1.3) and (1.4) is not difficult. The main point is that the spaces $B_{p,q}^s(A)$ and $\mathring{B}_{p,q}^s(A)$ are the same (the respective norms being equivalent) for all choices of $l < s < r$. This is essentially equivalent (cf. [3], Section 3.5) to the following inequality of Ehrling–Nirenberg–Gagliardo type

(1.5) $$\|f\|_{\mathscr{M}W_p^k(A)} \leqslant C\,\|f\|^{1-\theta}_{\mathscr{M}W_p^l(A)}\,\|f\|^{\theta}_{\mathscr{M}W_p^r(A)}$$

for $f \in \mathscr{M}W_p^r(A)$, where $l < k < r$, $k = (1-\theta)l + \theta r$ and $C < \infty$ does not depend on $f$ (and the corresponding fact for the $\mathscr{M}\mathring{W}_p^k$ spaces).

The inequality (1.5) was known in the case $1 < p < \infty$, cf. [3] (and also if $l \geqslant 0$, cf. [1]) but the proof using multiplier theorems on $\mathbf{R}^d$ yielded constants that approached $\infty$ if $p \to 1$ or $p \to \infty$.

In Section 2 we shall comment on the relation of (1.3) and (1.4) with the intrinsic definition of $B_{p,q}^s(A)$ for $s > 0$.

Let us mention that $\mathring{B}_{p,q}^s(A)$ may not be closed in $B_{p,q}^s(A)$ for some $s > 0$. We let $\mathscr{C}B_{p,q}^s(A)$ denote the closure of $J\big(C^\infty(A)\big)$ in $B_{p,q}^s(A)$ and $\mathscr{C}\mathring{B}_{p,q}^s(A)$ denote the closure of $\mathring{J}\big(\mathring{C}^\infty(A)\big)$ in $\mathring{B}_{p,q}^s(A)$. We shall prove that

$$\big(\mathscr{C}\mathring{B}_{p,q}^s(A)\big)^* = B_{p',q'}^{-s}(A), \qquad \big(\mathscr{C}B_{p,q}^s(A)\big)^* = \mathring{B}_{p',q'}^{-s}(A),$$

and $\mathscr{C}B_{p,q}^s(A) = B_{p,q}^s(A)$, $\mathscr{C}\mathring{B}_{p,q}^s(A) = \mathring{B}_{p,q}^s(A)$, unless $q = \infty$.

In the rest of this section we set $M = \mathbf{R}^d$ and let $\mu$ be the Lebesgue measure. We assume that $\mu(A) = \mu(\operatorname{Int}A) > 0$.

Let $\|h\|_p = \big(\int_A |h(x)|^p\,d\mu(x)\big)^{1/p}$ be the $L_p$ norm of a function $h$ on $A$. Let $f \in C^\infty(A)$ and $g \in \mathring{C}^\infty(A)$. For $k \geqslant 0$ we set, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ being multi-indices,

(1.6) $$\|f\|_{k,p} = \Big(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|_p^p\Big)^{1/p}$$

and $\|g\|^\circ_{k,p} = \|g\|_{k,p}$. If $k < 0$, then

(1.7) $$\|f\|_{k,p} = \inf\Big(\sum_{|\alpha| \leqslant -k} \|f_\alpha\|_p^p\Big)^{1/p},$$

where the inf is extended over all the decompositions $f = \sum_{|\alpha| \leqslant -k} D^\alpha f_\alpha$ with $f_\alpha \in C^\infty(A)$. The definition of $\|g\|^\circ_{k,p}$ is similar, with $f_\alpha$ replaced by $g_\alpha \in \mathring{C}^\infty(A)$.

It is easy to check that (1.1) holds with $c = 1$. A converse estimate holds if, for instance, $A$ is star-shaped in the following sense. There is $x_0 \in \operatorname{Int}A$ such that every ray emanating from $x_0$ meets $\partial A$ at exactly one point.

LEMMA 1.8. *Assume that $A \subset \boldsymbol{R}^d$ is star-shaped. Then $J$, $\mathring{J}$ in* (1.2) *are isometric embeddings, i.e. for $f \in C^\infty(A)$, $g \in \mathring{C}^\infty(A)$ one has*

$$\|f\|_{k,p} = \sup\{\int_A f\tilde{g}\,d\mu:\ \tilde{g} \in \mathring{C}^\infty(A),\ \|\tilde{g}\|^{\circ}_{-k,p'} \leqslant 1\}, \tag{1.9}$$

$$\|g\|^{\circ}_{k,p} = \sup\{\int_A g\tilde{f}\,d\mu:\ \tilde{f} \in C^\infty(A),\ \|\tilde{f}\|_{-k,p'} \leqslant 1\}. \tag{1.10}$$

This lemma is proved by regularization. We shall sketch the argument at the end of this section.

Observe that, if $k \geqslant 0$ and $1 \leqslant p \leqslant \infty$, the space $\mathscr{M}W^k_p(A)$ may be regarded as the dual space of the quotient

$$\Big(\Big(\sum_{|\alpha|\leqslant k} \oplus\, \mathring{C}^\infty(A)\Big)/X,\ \|\cdot\|^{\circ}_{-k,p'}\Big),$$

where $X = \{(g_\alpha)_{|\alpha|\leqslant k}:\ \sum_{|\alpha|\leqslant k} D^\alpha g_\alpha = 0\}$. Hence $\mathscr{M}W^k_p(A)$ can be identified with $X^\perp \subset \big(\sum_{|\alpha|\leqslant k} \oplus \mathscr{M}W^0_p(A)\big)_p$. It is easy to check that $f = (f_\alpha)_{|\alpha|\leqslant k} \in X^\perp$ if and only if the distributional derivatives of $f_0$ satisfy $f_\alpha = (-1)^{|\alpha|} D^\alpha f_0$. If $1 < p \leqslant \infty$, then $\mathscr{M}W^0_p(A) = L_p(A)$ and hence $X^\perp$ can be identified with the classical Sobolev space $W^k_p(A)$. If $p = 1$, then $W^k_1(A)$ is a subset of $\mathscr{M}W^k_1(A)$ (equal to $J(\mathscr{C}W^k_1(A))$).

It is easy to see that for $k \geqslant 0$, $\mathscr{M}\mathring{W}^k_p(A)$ can be identified with a (closed) subspace of $\mathscr{M}W^k_p(A)$ consisting of elements which vanish in a suitable sense on $\partial A$.

LEMMA 1.11. *Let $U$: $C^\infty(A) \to \mathscr{C}W^k_p(A)$ be a linear operator. Then $U$ extends to a continuous map of $\mathscr{C}W^l_q(A)$ into $\mathscr{C}W^k_p(A)$ if and only if the formal adjoint $U^*$ maps $\mathscr{C}\mathring{W}^{-k}_{p'}(A)$ into $\mathscr{M}\mathring{W}^{-l}_{q'}(A)$.*

Proof. This follows immediately from Lemma 1.8.

By our previous remarks, if $k, l \leqslant 0$, then $\mathscr{C}\mathring{W}^{-k}_{p'}(A)$ and also $\mathscr{M}\mathring{W}^{-l}_{q'}(A)$ are subspaces of classical Sobolev spaces on $A$ (the case where $q = \infty$ may require a separate treatment), so the boundedness of $U^*$ is easier to check than that of $U$. For instance, let $k \leqslant l \leqslant 0$, $p = q$ and let $Uf = hf$, where $h \in C^\infty(A)$.

Clearly, a similar lemma can be stated for operators from $\mathscr{C}\mathring{W}^l_q(A)$ into $\mathscr{C}\mathring{W}^k_p(A)$.

It is easy to characterize the spaces $\mathscr{M}W^k_p(A)$ and $\mathscr{M}\mathring{W}^k_p(A)$ for $k < 0$ as certain quotient spaces. For instance, identifying $(\mathring{C}^\infty(A), \|\cdot\|^{\circ}_{-k,p'})$ with the subspace

$$\{((-1)^{|\alpha|} D^\alpha f)_{|\alpha|\leqslant -k}:\ f \in \mathring{C}^\infty(A)\} \subset \Big(\sum_{|\alpha|\leqslant -k} \oplus\, \mathscr{C}\mathring{W}^0_{p'}(A)\Big)_{p'},$$



we see that $T \in \mathscr{D}'(\operatorname{Int} A)$ satisfies $\|T\|_{\mathscr{M}W^k_p(A)} < 1$ if and only if one can write

$$T = \sum_{|\alpha|\leqslant -k} D^\alpha T_\alpha,$$

where $\sum \|T_\alpha\|^p_{\mathscr{M}W^0_p(A)} < 1$. An analogous result can be stated for elements of $\mathscr{M}\mathring{W}^k_p(A)$, $k < 0$.

Now we can sketch the proof of Lemma 1.8. To prove (1.9) we use the following fact.

There is a positive function $\varrho(\eta)$ so that for each $\eta > 1$ and $0 < \delta < \varrho(\eta)$ there is an operator $R = R_{\eta,\delta}$: $\mathscr{D}'(\operatorname{Int} A) \to C^\infty(A)$ such that

$$R^*(\mathscr{D}'(A)) \subseteq \mathring{C}^\infty(A), \tag{1.12}$$

$$\|R^*:\ \mathscr{M}\mathring{W}^l_q(A) \to \mathscr{C}\mathring{W}^l_q(A)\| \leqslant a(\eta, l, d), \tag{1.13}$$

where $a(\eta, l, d) \to 1$ as $\eta \to 1_+$, $l$, $d$ being fixed; given $f \in C^\infty(A)$, $\varepsilon > 0$, $\eta_0 > 1$, $k$ and $p$, one can find $1 < \eta \leqslant \eta_0$ and $0 < \delta < \varrho(\eta)$ so that

$$\|f - R_{\eta,\delta} f\|_{k,p} < \varepsilon. \tag{1.14}$$

Using these properties we find easily that, if $f \in C^\infty(A)$ and $\varepsilon > 0$, then choosing suitable $\eta_0$, $\eta$, $\delta$ we have $a(\eta, -k, d) < 1 + \varepsilon$ and for $T \in \mathscr{M}\mathring{W}^{-k}_{p'}(A)$ with $\|T\| \leqslant 1$ we obtain, using (1.14),

$$\langle T, f\rangle = \langle T, Rf\rangle + \langle T, f - Rf\rangle < \langle R^*T, f\rangle + \varepsilon.$$

Since $T$ can be chosen so that $\langle T, f\rangle = \|f\|_{k,p'}$ and by (1.13), $g = R^*T/(1+\varepsilon)$ satisfies $\|g\|^{\circ}_{-k,p'} \leqslant 1$, (1.9) follows immediately.

The proof of (1.10) is similar, the roles of $R$ and $R^*$ being reversed, so we shall only explain how (1.12), (1.13), (1.14) are obtained.

Without loss of generality we may assume that $x_0 = 0$. Let $B = \{x \in \boldsymbol{R}^d:\ \|x\| \leqslant 1\}$. Since $A$ is star-shaped and $x_0 = 0$, for any $\eta > 1$ there is $\varrho(\eta) > 0$ such that $A + \varrho(\eta)B \subseteq \eta A$. Thus, for $0 < \delta < \varrho(\eta)$,

$$(1/\eta)A + (\delta/\eta)B \subset \operatorname{Int} A. \tag{1.15}$$

Now fix a non-negative even function $\varphi \in C^\infty(\boldsymbol{R}^d)$ such that $\operatorname{supp}\varphi \subseteq B$ and $\int \varphi\,d\mu = 1$. Write $\varphi_\delta(x) = \delta^{-d}\varphi(x/\delta)$ for $x \in \boldsymbol{R}^d$, $\delta > 0$. Let

$$(V_\eta f)(x) = f(x/\eta)$$

for $f \in C^\infty(A)$, $x \in \eta A$, $\eta > 0$, and, if $T \in \mathscr{D}'(A)$, $\eta > 1$, let $T_{1/\eta} \in \mathscr{D}'(A)$ be defined by

$$\langle T_{1/\eta}, f\rangle = \langle T, V_\eta f\rangle.$$

If $\eta > 1$, $0 < \delta < \varrho(\eta)$, we put for $f \in C^\infty(A)$

$$R_{\eta,\delta} f = ((V_\eta f) * \varphi_\delta)|_A \in C^\infty(A).$$

For $T \in \mathscr{D}'(A)$ we find that $(R_{\eta,\delta})^* T = T_{1/\eta} * \varphi_{\delta/\eta}$, so that (1.12) follows easily from (1.15). To obtain (1.14), note that $\|\cdot\|_{k,p}$ is dominated by $\|\cdot\|_{m,\infty}$, where $m = \max\{0, k\}$, $\|V_\eta f - f\|_{m,\infty} < \frac{1}{2}\varepsilon$ if $\eta$ is close to 1 and $\|V_\eta f - (V_\eta f) * \varphi_\delta\|_{m,\infty} < \frac{1}{2}\varepsilon$ if $\delta > 0$ is small enough. Finally, (1.13) is obtained by straightforward computation. (One applies $R^*$ to a representation of $T \in \mathscr{M}\mathring{W}^l_q(A)$ as an element of a quotient space, if $l < 0$, or of a subspace of $(\sum_{|\alpha| \leqslant l} \oplus \mathscr{M}\mathring{W}^0_q)_q$, if $l \geqslant 0$.)

In Section 5 we shall use some nonisotropic Sobolev spaces on the cube $I^d$. If $m = (m_1, \ldots, m_d)$ is a multi-index and $1 \leqslant p \leqslant \infty$, we let $W^m_p(I^d)$ denote the completion of $C^\infty(I^d)$ in the norm

$$\|f\|_p^{(m)} = \sum_{0 \leqslant \alpha \leqslant m} \|D^\alpha f\|_p$$

and let $\mathring{W}^m_p(I^d)$ denote the closure of $\mathring{C}^\infty(I^d)$ in $W^m_p(I^d)$.

In Section 4 and 5 we do not need $\mathscr{M}W^k_p$ spaces and we write $W^k_p$ nstead of $\mathscr{C}W^k_p$.

**2. Sobolev and Besov spaces on the cube $I^d$.** In this section $d$ is a fixed integer $\geqslant 1$, $M = \mathbf{R}^d$ and $A$ is the standard cube $I^d = \langle 0, 1\rangle^d$ equipped with the Lebesgue measure. Hence we shall often use shorter notation, e.g. $\mathscr{C}W^k_p$ will mean $\mathscr{C}W^k_p(I^d)$ (cf. also Theorem 3.5 below).

The following lemma is a weaker version cf Corollary 5.19 below.

LEMMA 2.1. *Given $m \geqslant 0$, there exists a constant $C = C(m, d) < \infty$ and a sequence of linear projections $(P_i)_{i=1}^\infty$ in the space $L_2(I^d)$ with the following properties.*

(i) *Each $P_i$ is of the form $P_i f = \sum_{j=1}^{n_i} (f, g_j) f_j$, where $g_j \in \mathscr{C}\mathring{W}^m_\infty = \mathring{C}^m(I^d)$ and $f_j \in \mathscr{C}W^m_\infty = C^m(I^d)$ for $j = 1, \ldots, n_i$.*

(ii) *One has, for each $i = 1, 2, \ldots, 1 \leqslant p \leqslant \infty$ and $0 \leqslant k \leqslant k+l \leqslant m$*

$$\|P_i\colon \mathscr{C}W^k_p \to \mathscr{C}W^{k+l}_p\| \leqslant C 2^{il}, \tag{2.2}$$

$$\|E - P_i\colon \mathscr{C}W^{k+l}_p \to \mathscr{C}W^k_p\| \leqslant C 2^{-il}, \tag{2.3}$$

$$\|\mathring{P}_i\colon \mathscr{C}\mathring{W}^k_p \to \mathscr{C}\mathring{W}^{k+l}_p\| \leqslant C 2^{il}, \tag{2.4}$$

$$\|\mathring{E} - \mathring{P}_i\colon \mathscr{C}\mathring{W}^{k+l}_p \to \mathscr{C}\mathring{W}^k_p\| \leqslant C 2^{-il}. \tag{2.5}$$

*Here $\mathring{P}_i g = \sum_{j=1}^{n_i} (g, f_j) g_j$ is the Hilbert space adjoint of $P_i$ and $E\colon \mathscr{C}W^{k+l}_p \to \mathscr{C}W^k_p$, $\mathring{E}\colon \mathscr{C}\mathring{W}^{k+l}_p \to \mathscr{C}\mathring{W}^k_p$ are the natural inclusion maps.*

Proof. This follows from Corollary 5.19, if one sets $P_i = Q^{(m)}_{2^i}$ (and $\mathring{P}_i = \mathring{Q}^{(m)}_{2^i}$), and recalls the definition of the $Q^{(m)}_n$'s.



It is clear from (i) that the formulae for $P_i$ and $\mathring{P}_i$ define corresponding operators (which we denote by the same letter)

$$P_i\colon \mathscr{M}W^k_p \to \mathscr{M}W^r_p, \quad \mathring{P}_i\colon \mathscr{M}\mathring{W}^k_p \to \mathscr{M}\mathring{W}^r_p,$$

for all $-m \leqslant k, r \leqslant m$, $1 \leqslant p \leqslant \infty$. Also the embeddings $E\colon \mathscr{M}W^{k+l}_p \to \mathscr{M}W^k_p$, $\mathring{E}\colon \mathscr{M}\mathring{W}^{k+l}_p \to \mathscr{M}\mathring{W}^k_p$ are well defined if $l \geqslant 0$.

COROLLARY 2.6. *The operators $(P_i)$, $(\mathring{P}_i)$ of Lemma 2.1 satisfy estimates (2.2)–(2.5) (with a larger $C < \infty$) for all $-m \leqslant k \leqslant k+l \leqslant m$. This statement remains true if one replaces $\mathscr{C}$ by $\mathscr{M}$ in all those formulae.*

Proof. This is a standard exercise so we mention only few steps.

Looking at the norms of the operators adjoint to those in (2.2)–(2.5), we see that the statement is true if $\mathscr{C}$ is replaced by $\mathscr{M}$ and $-m \leqslant k \leqslant k+l \leqslant 0$. Restricting those adjoint operators to the closure of smooth functions in their domains we obtain (2.2)–(2.5) for $-m \leqslant k \leqslant k+l \leqslant 0$ (here we use Lemma 1.11 and the form of the $P_i$'s and $\mathring{P}_i$'s described in Lemma 2.1). Taking the adjoints of the latter operators we are again in the case $0 \leqslant k \leqslant k+l \leqslant m$ but now with $\mathscr{M}$ replacing $\mathscr{C}$. Finally, the case where $-m \leqslant k < 0 < k+l \leqslant m$ is an easy consequence. For instance, since $P_i \circ P_i = P_i$, we obtain easily

$$\|P_i\colon \mathscr{C}W^k_p \to \mathscr{C}W^{k+l}_p\| \leqslant \|P_i\colon \mathscr{C}W^k_p \to \mathscr{C}W^0_p\| \, \|P_i\colon \mathscr{C}W^0_p \to \mathscr{C}W^{k+l}_p\|$$
$$\leqslant C 2^{-ik} \cdot C 2^{i(k+l)} = C^2 2^{il}.$$

Remark 2.7. Comparing pairs of formulae (e.g. {(2.2), (2.4)}, {(2.3), (2.5)}) and reading subsequent proofs, one can guess that each result of this section has a counterpart in which $W$ is replaced by $\mathring{W}$, $\mathring{W}$ by $W$ and so on. This is the case and hence we shall state explicitly only one result or formula of such a pair, often without even mentioning the other one. Let us agree that (2.n°) will refer to the formula obtained from (2.n) by this procedure. E.g., (2.5) = (2.3°) and (2.3) = (2.5°). This convention will also be used in Section 3.

COROLLARY 2.8. *If $l < k$, $1 \leqslant p \leqslant \infty$, then the adjoint $E\colon \mathscr{M}W^k_p \to \mathscr{M}W^l_p$ to the embedding $\mathring{E}\colon \mathscr{C}\mathring{W}^{-l}_{p'} \to \mathscr{C}\mathring{W}^{-k}_{p'}$ is a compact one-to-one operator and*

$$E(\mathscr{M}W^k_p) \subseteq \mathscr{C}W^l_p \quad (= J(\mathscr{C}W^l_p)).$$

Proof. It is obvious that $E$ is one-to-one because $\mathring{E}$ has a dense range.

In order to prove the inclusion we use Corollary 2.6 with $m \geqslant \max\{|k|, |l|\}$. Note that, if $f \in \mathscr{M}W^k_p$, then $P_i f \in C^m \subseteq \mathscr{C}W^l_p$ and, by Corollary 2.6, formula (2.3) shows that $P_i f \to Ef$ in the norm of $\mathscr{M}W^l_p$.

Hence $Ef \in J(\mathscr{C}W_p^l)$, by Lemma 1.8. Also $E$ is compact, because $\|E-P_i\| \to 0$ and the $P_i$'s have finite rank.

Given two normed spaces $A_0$, $A_1$ with $A_1 \subseteq A_0$ (continuous embedding), set for $f \in A_0$, $t > 0$

$$K(t, f;\ A_0, A_1) = \inf\{\|f-f_1\|_{A_0} + t\|f_1\|_{A_1} : f_1 \in A_1\}.$$

If $A_2$ is another subspace of $A_0$, we say that the $K$-functionals of the pairs $(A_0, A_1)$, $(A_0, A_2)$ are *equivalent* if there is $c > 0$ so that, if $f \in A_0$, $f \neq 0$, then for $t > 0$

$$c \leqslant K(t, f;\ A_0, A_1)/K(t, f;\ A_0, A_2) \leqslant 1/c.$$

COROLLARY 2.9. *If $l < k$, $1 \leqslant p \leqslant \infty$, then the $K$-functionals of the pairs $(\mathscr{C}W_p^l, \mathscr{M}W_p^k)$ and $(\mathscr{C}W_p^l, \mathscr{C}W_p^k)$ are equivalent.*

Proof. Since the embedding $J\colon \mathscr{C}W_p^k \to \mathscr{M}W_p^k$ is continuous, there is $C < \infty$ so that for $f \in \mathscr{C}W_p^l$

$$K(t, f;\ \mathscr{C}W_p^l, \mathscr{M}W_p^k) \leqslant CK(t, f;\ \mathscr{C}W_p^l, \mathscr{C}W_p^k).$$

To prove a converse estimate it suffices to check that each element $f_1 \in \mathscr{M}W_p^k$ is the limit in $\mathscr{C}W_p^l$ of a sequence $(g_i) \subset \mathscr{C}W_p^k$ such that $\|g_i\|_{\mathscr{C}W_p^k} \leqslant C\|f_1\|_{\mathscr{M}W_p^k}$. This we have done in the proof of Corollary 2.8.

COROLLARY 2.10. *There exists $C = C_{m,d} < \infty$ such that, if $-m \leqslant l < k < r \leqslant m$, $k = (1-\theta)l + \theta r$, $1 \leqslant p \leqslant \infty$, then for $f \in \mathscr{M}W_p^k$ one has*

$$K(t, f;\ \mathscr{M}W_p^l, \mathscr{M}W_p^r) \leqslant C\min\{t^\theta, 1\}\|f\|_{\mathscr{M}W_p^k}.$$

Proof. We use again Corollary 2.6. One has

$$K(t, f;\ \mathscr{M}W_p^l, \mathscr{M}W_p^r) \leqslant \inf_i\{\|f-P_if\|_{\mathscr{M}W_p^l} + t\|P_if\|_{\mathscr{M}W_p^r}\}$$

and the right-hand side can be estimated using (2.2) and (2.3).

In the language of interpolation spaces Corollaries 2.8, 2.9, 2.10 can be expressed by the following formulae which are explained below. (We assume again that $l < k < r$, $k = (1-\theta)l + \theta r$, $1 \leqslant p, q \leqslant \infty$.)

$$(2.11)\qquad (\mathscr{M}W_p^l, \mathscr{M}W_p^r)_{\theta,q} = (\mathscr{C}W_p^l, \mathscr{M}W_p^r)_{\theta,q} = (\mathscr{C}W_p^l, \mathscr{C}W_p^r)_{\theta,q},$$

$$(2.12)\qquad (\mathscr{M}W_p^l, \mathscr{M}W_p^r)_{\theta,q}^\circ = (\mathscr{C}W_p^l, \mathscr{M}W_p^r)_{\theta,q}^\circ = (\mathscr{C}W_p^l, \mathscr{C}W_p^r)_{\theta,q}^\circ,$$

$$(2.13)\qquad \mathscr{M}W_p^k \subseteq (\mathscr{M}W_p^l, \mathscr{M}W_p^r)_{\theta,\infty},$$

$$(2.14)\qquad \mathscr{C}W_p^k \subseteq (\mathscr{C}W_p^l, \mathscr{C}W_p^r)_{\theta,\infty}^\circ,$$

$$(2.15)\qquad \mathscr{C}W_p^k \supseteq (\mathscr{M}W_p^l, \mathscr{M}W_p^r)_{\theta,1}.$$

Recall that the interpolation space $(A_0, A_1)_{\theta,q}$, where $A_1 \subseteq A_0$, $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$ consists of those $f \in A_0$ such that

$$\|f\| = \Big(\int_0^\infty \big(t^{-\theta}K(t, f;\ A_0, A_1)\big)^q\ t^{-1}dt\Big)^{1/q} < \infty.$$



By $(A_0, A_1)_{\theta,q}^\circ$ we denote the closure of $A_1$ in the space $(A_0, A_1)_{\theta,q}$. (By Theorem 3.4.2 (b) in [3], if $q < \infty$, then $A_1$ is dense in $(A_0, A_1)_{\theta,q}$.)

Now, the first equalities in (2.11) and (2.12) follow from Corollary 2.8 and the easy Theorem 3.4.2 (d) in [3]. The other equalities in (2.11) and (2.12) follow from Corollary 2.9 and the definitions.

Corollary 2.10 yields (2.13) immediately. Since, by (2.13) and (2.11),

$$\mathscr{C}W_p^k \subseteq \mathscr{M}W_p^k \subseteq (\mathscr{C}W_p^l, \mathscr{C}W_p^r)_{\theta,\infty} =: E$$

and $\mathscr{C}W_p^r$ contains $C^\infty(I^d)$ which is dense in $\mathscr{C}W_p^k$, the closure of $\mathscr{C}W_p^r$ in $E$ must contain $\mathscr{C}W_p^k$. This proves (2.14).

In order to prove (2.15), note that (2.14°) yields

$$\mathscr{C}\mathring{W}_{p'}^{-k} \subseteq (\mathscr{C}\mathring{W}_{p'}^{-r}, \mathscr{C}\mathring{W}_{p'}^{-l})_{1-\theta,\infty}^\circ.$$

Passing to the dual spaces and using the remark after Theorem 3.7.1 in [3], we obtain

$$\mathscr{M}W_p^k \supseteq (\mathscr{M}W_p^l, \mathscr{M}W_p^r)_{\theta,1} =: F.$$

Since a dense subset of $F$ (namely $\mathscr{M}W_p^r$) is, by Corollary 2.8, contained in the closed subspace $\mathscr{C}W_p^k$ of $\mathscr{M}W_p^k$, so is $F$. This proves (2.15).

Now we can prove some properties of Besov spaces on the cube $I^d$ announced in Section 1.

THEOREM 2.16. *Let $s$ be a real number and $1 \leqslant p, q \leqslant \infty$. The spaces $B_{p,q}^s(I^d)$, $\mathring{B}_{p,q}^s(I^d)$, $\mathscr{C}B_{p,q}^s(I^d)$, $\mathscr{C}\mathring{B}_{p,q}^s(I^d)$ defined in Section 1 do not depend on the choice of $l < s$ and $r > s$ (different choices lead to equivalent norms). Moreover, one has*

$$(2.17)\qquad (\mathscr{C}\mathring{B}_{p',q'}^{-s})^* = B_{p,q}^s,$$

$$(2.18)\qquad \mathscr{C}B_{p,q}^s = (\mathscr{C}W_p^l, \mathscr{C}W_p^r)_{\theta,q}^\circ,$$

*where $s = (1-\theta)l + \theta r$, $0 < \theta < 1$. In particular, $\mathscr{C}B_{p,q}^s = B_{p,q}^s$ if $1 \leqslant q < \infty$.*

Proof. Observe that (2.11) and (2.11°) prove the claim made after (1.4). In order to prove the independence of $l$ and $r$ it suffices to check that, if $k < l$ and $n > r$ and $s = (1-\tau)k + \tau n$, then

$$(2.19)\qquad (\mathscr{M}W^k, \mathscr{M}W_p^n)_{\tau,q} = (\mathscr{M}W_p^l, \mathscr{M}W_p^r)_{\theta,q}.$$

(2.19) follows from the reiteration theorem (cf. Theorem 3.5.3 in [3]). The assumptions of that theorem, i.e.

$$(\mathscr{M}W_p^k, \mathscr{M}W_p^n)_{\sigma,1} \subseteq \mathscr{M}W_p^j \subseteq (\mathscr{M}W_p^k, \mathscr{M}W_p^n)_{\sigma,\infty},$$

where $j = (1-\sigma)k + \sigma n$ and $j = l, r$, are satisfied by formulae (2.15) and (2.13).

By (1.3) and the definition of $(\mathscr{C}W^l_p, \mathscr{C}W^r_p)^\circ_{\theta,q}$, the latter space is the closure of $\mathscr{C}W^r_p$ in $B^s_{p,q}$. Since $C^\infty(I^d)$ is dense in $\mathscr{C}W^r_p$, the closure coincides with $\mathscr{C}B^s_{p,q}$. This proves (2.18).

The last assertion is true because, by Theorem 3.4.2 (b) in [3], $\mathscr{C}W^r_p$ is dense in $(\mathscr{C}W^l_p, \mathscr{C}W^r_p)_{\theta,q}$ if $1 \leqslant q < \infty$.

Finally, using (2.18°) and Theorem 3.7.1 in [3] we get

$$(\mathscr{C}\mathring{B}^{-s}_{p',q'})^* = \big((\mathscr{C}\mathring{W}^{-r}_{p'}, \mathscr{C}\mathring{W}^{-l}_{p'})^\circ_{1-\theta,q'}\big)^* = (\mathscr{M}W^l_p, \mathscr{M}W^r_p)_{\theta,q} = B^s_{p,q}.$$

This completes the proof of Theorem 2.16.

Corollary 2.20. *For each $k$ and $p$ there are natural inclusion maps*

$$B^k_{p,1} \subseteq \mathscr{C}W^k_p \subseteq \mathscr{C}B^k_{p,\infty}, \qquad \mathscr{M}W^k_p \subseteq B^k_{p,\infty}.$$

*Moreover, these maps are one-to-one.*

Proof. The inclusions follow directly from (2.15), (2.14), (2.18) and (2.13). To see that they are one-to-one note that the embeddings $\mathring{E}\colon \mathscr{C}\mathring{W}^{-k}_{p'} \to \mathscr{C}\mathring{B}^{-k}_{p',\infty}$, $\mathring{E}\colon \mathscr{C}\mathring{B}^{-k}_{p',1} \to \mathscr{C}\mathring{W}^{-k}_{p'}$ have a dense range, and consider the adjoint operators.

Remark. The space $B^s_{p,q}(A)$, for $s > 0$ and $A \subset \mathbf{R}^d$, is often defined intrinsically, e.g. in terms of $\omega_r(t,f)$ (where $r > s$), the $r$th order modulus of smoothness in $L_p$ (cf. [4], § 21). The fact that the latter space coincides with $\big(\mathscr{C}W^0_p(A), \mathscr{C}W^r_p(A)\big)_{s/r,q}$ for suitable $A \subset \mathbf{R}^d$, follows easily from the result of H. Johnen and K. Scherer [12].

They prove that, if $A \subset \mathbf{R}^d$ is a compact set with Lipschitz boundary, then for $f \in \mathscr{C}W^0(A)$, $r \geqslant 1$ and $t > 0$ one has

$$c\,\omega_r(t,f) \leqslant \tilde{K}\big(t^r, f; \mathscr{C}W^0_p(A), \mathscr{C}W^r_p(A)\big) \leqslant c^{-1}\omega_r(t,f),$$

where $c > 0$ depends only on $r$ and $A$. Here $\tilde{K}(s,f;\cdot,\cdot)$ is a modified $K$-functional which yields the same interpolation spaces as $K(s,f;\cdot,\cdot)$.

**3. Sobolev and Besov spaces on a compact manifold.** In this section $M$ is a $d$-dimensional $C^\infty$ manifold without boundary. We are mostly interested in two cases: either $M$ is compact and $A = M$, or $A \subset M$ is a compact subset with smooth boundary, $\partial A$. If $C \subseteq A$, we shall distinguish $\operatorname{Int} C$ (the interior relative to $M$) and $\operatorname{Int}_A C$ (the interior relative to $A$).

A subset $Q \subseteq M$ is said to be a $d$-cube if there exists a diffeomorphism $\phi\colon U \to \mathbf{R}^d$, where $U = \operatorname{Int} U \supset Q$, such that $\phi(Q) = I^d$. The norms $\|\cdot\|_{k,p}$ on $C^\infty(Q)$ and $\|\cdot\|^\circ_{k,p}$ on $\mathring{C}^\infty(Q)$ (cf. Section 1) can be introduced as follows. We fix a $\phi\colon Q \to I^d$ as in the definition and let for $f \in C^\infty(Q)$, $g \in \mathring{C}^\infty(Q)$,

$$\|f\|_{k,p} = \|f \circ \phi^{-1}\|_{\mathscr{C}W^k_p(I^d)},$$

$$\|g\|^\circ_{k,p} = \|g \circ \phi^{-1}\|_{\mathscr{C}\mathring{W}^k_p(I^d)}.$$



These norms depend on the choice of $\phi$. Choosing another admissible diffeomorphism $\psi$ of $Q$ onto $I^d$, we would obtain norms equivalent to those we defined above. This is easy to check directly if $k \geqslant 0$; in the case $k < 0$ one can use Lemma 1.11 and the subsequent comment. Consequently, the spaces $\mathscr{C}W^k_p(Q)$, $\mathscr{C}\mathring{W}^k_p(Q)$, $\mathscr{M}W^k_p(Q)$, $\mathscr{M}\mathring{W}^k_p(Q)$ are defined unambiguously, their norms being determined up to equivalence.

For each $x \in A$ there is a $d$-cube, say $Q_x$, such that $Q_x \subseteq A$ and $x \in \operatorname{Int}_A Q_x$. Hence we can find a finite family $\mathscr{Q} = (Q_i)_{i=1}^N$ of $d$-cubes so that

$$A = \bigcup_{i=1}^N \operatorname{Int}_A Q_i.$$

It is easy to construct $\varphi_1, \ldots, \varphi_N \in C^\infty(M)$ so that $\sum_{i\leqslant N} \varphi_i \equiv 1$ and

$$A \cap \operatorname{supp}\varphi_i \subseteq \operatorname{Int}_A Q_i,$$

for $i = 1, \ldots, N$. We shall fix such a sequence $\varphi_1, \ldots, \varphi_N$.

Now we set for $f \in C^\infty(A)$, $g \in \mathring{C}^\infty(A)$,

$$\|f\|_{\mathscr{C}W^k_p(A)} = \|f\|_{k,p} = \Big(\sum_{i=1}^N \|f|_{Q_i}\|^p_{\mathscr{C}W^k_p(Q_i)}\Big)^{1/p},$$

$$\|g\|_{\mathscr{C}\mathring{W}^k_p(A)} = \|g\|^\circ_{k,p} = \inf\Big(\sum_{i=1}^N \|g_i|_{Q_i}\|^p_{\mathscr{C}\mathring{W}^k_p(Q_i)}\Big)^{1/p},$$

where the inf is extended over all the sequences $g_1, \ldots, g_N \in \mathring{C}^\infty(A)$ such that $\operatorname{supp} g_i \subset \operatorname{Int} Q_i$ and $g = \sum_{i\leqslant N} g_i$. (Such sequences exist, e.g. take $g_i = g\varphi_i$.)

This definition of the norms $\|\cdot\|_{k,p}$ and $\|\cdot\|^\circ_{k,p}$ on $A$ depends on the choice of $\mathscr{Q}$. At the end of this section we shall sketch a proof that, choosing another family $\mathscr{Q}'$ of $d$-cubes, one obtains equivalent norms, and hence the Sobolev spaces on $A$ are again defined unambiguously.

In order to define suitable embeddings $J$ and $\mathring{J}$ we shall fix a smooth measure $\mu$ on $A$. (This means that, if $\phi\colon U \to \mathbf{R}^d$ is any chart of $M$ and $\nu$ is the measure on $U \cap A$ transported from the Lebesgue measure on $\mathbf{R}^d$ by means of $\phi$, then $d\mu = b\,d\nu$ where $b$ is (locally) a positive $C^\infty$ function.)

Let us check that (1.1) holds. On the cube $Q_i$, $1 \leqslant i \leqslant N$, we have $d\mu = b_i\,dx$, where $b_i \in C^\infty(Q_i)$, $b_i > 0$, and $dx$ is transported from the Lebesgue measure on $I^d$ by means of a suitable $\phi = \phi_i$. Observe that, if $f \in C^\infty(A)$, $g \in \mathring{C}^\infty(A)$, $g = \sum_{i\leqslant N} g_i$ as above, then for any $k, p$ we have, by Lemma 1.8,

$$\int_A fg\,d\mu = \sum_{i=1}^{N}\int_A fg_i\,d\mu = \sum_{i=1}^{N}\int_{Q_i} fg_ib_i\,dx$$
$$\leqslant \sum_{i=1}^{N}\|b_if|_{Q_i}\|_{\mathscr{C}W_p^k(Q_i)}\,\|g_i|_{Q_i}\|_{\mathscr{C}\mathring{W}_{p'}^{-k}(Q_i)}$$
$$\leqslant C\Big(\sum_{i=1}^{N}\|f|_{Q_i}\|^p_{\mathscr{C}W_p^k(Q_i)}\Big)^{1/p}\Big(\sum_{i=1}^{N}\|g_i|_{Q_i}\|^{p'}_{\mathscr{C}\mathring{W}_{p'}^{-k}(Q_i)}\Big)^{1/p'},$$

where the constant $C$ depends on $k$ and $b_1,\ldots,b_N$. Inequality (1.1) follows easily from this estimate and our definitions.

The embeddings $J$ and $\mathring{J}$ are defined as in Section 1.

LEMMA 3.1. *$J$ is an isomorphic embedding of $\mathscr{C}W_p^k(A)$ into $\mathscr{M}W_p^k(A)$, whose range does not depend on the choice of the smooth measure $\mu$. The same can be said about the map $\mathring{J}$ in* (1.2).

Proof. Choosing another smooth measure on $A$, say $\nu$, we would have $d\nu = b\,d\mu$, where $b\in C^\infty(A)$, $b>0$, hence $J_\nu f = bJ_\mu f$ for $f\in W_p^k(A)$. It is now clear that $J_\mu$ and $J_\nu$ have the same range.

The fact that $J$ is an isomorphic embedding follows easily from (1.9), because multiplication by $1/b_i$ is a continuous linear operation in $\mathscr{C}\mathring{W}_{p'}^{-k}(Q_i)$ for $1\leqslant i\leqslant N$.

The assertions about $\mathring{J}$ are proved similarly. In order to estimate $\|g\|^{\circ}_{k,p}$ by $\|\mathring{J}g\|_{\mathscr{M}\mathring{W}_p^k}$ it suffices to find, for $i=1,\ldots,N$, functions $f_i\in C^\infty(Q_i)$ so that $\|f_i\|_{-k,p'}\leqslant 2$ and

$$\int_{Q_i} f_i\varphi_i g\,b_i^{-1}\,d\mu \geqslant \|\varphi_i g\|^{\circ}_{k,p},$$

and then let $f=\sum_{i=1}^{N}\varphi_ib_i^{-1}f_i\in C^\infty(A)$. Indeed,

$$\int_A fg\,d\mu = \sum_{i=1}^{N}\int_{Q_i} f_i\varphi_ib_i^{-1}g\,d\mu \geqslant \sum_{i=1}^{N}\|\varphi_i g\| \geqslant \|g\|^{\circ}_{k,p}$$

and it is not difficult to check that $\|f\|_{-k,p'}\leqslant C$, where $C$ does not depend on $g$.

Let us show how the questions about Sobolev and Besov spaces on $A$ can be reduced to similar problems for $d$-cubes.

First we define the maps

$$C^\infty(A)\xrightarrow[j]{}\sum_{i=1}^{N}\oplus C^\infty(Q_i)\xrightarrow[S]{}C^\infty(A), \tag{3.2}$$



$$\mathring{C}^\infty(A)\xrightarrow[S^\circ]{}\sum_{i=1}^{N}\oplus\mathring{C}^\infty(Q_i)\xrightarrow[j^\circ]{}\mathring{C}^\infty(A), \tag{3.3}$$

setting

$$jf = (f|_{Q_i})_{i=1}^N,\qquad S(f_i) = \sum_{i=1}^{N}\varphi_if_i,$$
$$S^\circ g = (\varphi_i g|_{Q_i})_{i=1}^N,\qquad j^\circ(g_i) = \sum_{i=1}^{N} g_i.$$

It is easy to check that, for each $k,p$, if the spaces in (3.2) are given the $\|\cdot\|_{k,p}$ norms (the direct sum being in the sense of $l_p$), then $j$ is an isometry into, $S$ is continuous and $Sj$ is the identity on $C^\infty(A)$, so that $jS$ is a projection onto the range of $j$. Similarly in (3.3), $j^\circ$ is a quotient map, $S^\circ$ is continuous and $j^\circ S^\circ$ is the identity on $\mathring{C}^\infty(A)$. These properties are preserved if we pass to the completions. They are replaced by dual ones when we pass to the dual spaces. (Note that the operators $j, j^\circ$ (resp. $S$, $S^\circ$) are formally adjoint to each other with respect to the measure $\mu$.)

We can summarize these remarks in the following proposition.

PROPOSITION 3.4. *For each $k$, $p$ the operator $j$ is an isomorphism of $\mathscr{C}W_p^k(A)$ onto a subspace of $\sum_{i\leqslant N}\oplus\mathscr{C}W_p^k(Q_i)$ which is the range of the (continuous) projection $jS$. The same is true, if $\mathscr{C}$ is replaced by $\mathscr{M}$. Moreover, these embeddings and projections commute with the embedding $J$ of* (1.2).

There is an analogous statement about the embedding $S^\circ$ of $\mathscr{C}\mathring{W}_p^k(A)$ into $\sum_{i\leqslant N}\oplus\mathscr{C}\mathring{W}_p^k(Q_i)$, the projection $S^\circ j^\circ$ and the map $\mathring{J}$.

THEOREM 3.5 *All the results of Section 2, after Corollary 2.6, are valid if $I^d$ is replaced by a compact $d$-dimensional $C^\infty$ manifold $A$ (with or without boundary).*

Proof. Recall first that, if the manifold $A$ has $\partial A\neq\emptyset$, then $A$ can be embedded as a submanifold of a $d$-dimensional $C^\infty$ manifold, $M$, without boundary (e.g., the double of $A$).

Theorem 3.5 can be deduced easily using Proposition 3.4 and the results of Section 2. Alternatively, the proofs in Section 2 can be used verbatim, if instead of Lemma 1.8 one applies Lemma 3.1 and also a substitute is found for Corollary 2.6.

The operators $T_1, T_2, \ldots$, which take the place of $P_1, P_2, \ldots$ can be defined by the formula

$$T_i = S\circ(P_i^{(1)}\oplus P_i^{(2)}\oplus\ldots\oplus P_i^{(N)})\circ j,$$

where $P_i^{(s)}$ denotes the operator on the cube $Q_s$ obtained from the operator $P_i$ on the cube $I^d$ by means of the diffeomorphism $\Phi_s$, i.e. for $f \in C^\infty(A)$ we have

$$T_i f = \sum_{s=1}^{N} \varphi_s \Big(\big(P_i^{(s)}\big((f|_{Q_s})\Phi_s^{-1}\big)\big)\Phi_s\Big).$$

The $T_i$'s are no longer projections but this property was used only in order to deduce Corollary 2.6 from Lemma 2.1. All necessary estimates for operators $T_i$ and $\mathring{T}_i$ follow from the corresponding facts for $P_i$ and $\mathring{P}_i$. This is sufficient to complete the proof of Theorem 3.5.

It remains to check that the Sobolev spaces on $A$ defined above do not depend on our choice of $\mathcal{Q}$. Suppose we are given another family of $d$-cubes, say $\mathcal{Q}' = (Q_j')_{j=1}^n$ such that $(\mathrm{Int}_A Q_j')_{j=1}^n$ is a covering of $A$. Let $\psi_1, \ldots, \psi_n$ be a subordinate partition of unity. Let $\|\cdot\|_{k,p}'$ and $\|\cdot\|_{k,p}^{\circ\prime}$ denote the norms on $C^\infty(A)$ and $\mathring{C}^\infty(A)$, respectively, defined using $\mathcal{Q}'$ instead of $\mathcal{Q}$.

Assume first that $k \geqslant 0$. Recall the well known fact that, for $h \in \mathring{C}^\infty(A)$ such that $h \neq 0$, $\operatorname{supp} h \subset Q_i \cap Q_j'$, one has

$$c \leqslant \|h\|_{\mathscr{C}W_p^k(Q_i)} / \|h\|_{\mathscr{C}W_p^k(Q_j')} \leqslant 1/c, \tag{3.6}$$

where $c > 0$ does not depend on $h$. Now, if $f \in C^\infty(A)$, then it follows from the triangle inequality and the continuity of the multiplication by a smooth function that

$$\|f\|_{k,p} \sim \sum_{i=1}^{N}\sum_{j=1}^{n} \|f\varphi_i\psi_j\|_{\mathscr{C}W_p^k(Q_i)},$$

$$\|f\|_{k,p}' \sim \sum_{i=1}^{N}\sum_{j=1}^{n} \|f\varphi_i\psi_j\|_{\mathscr{C}W_p^k(Q_j')}.$$

By (3.6), the right-hand sides are equivalent quantities, hence so are the left-hand sides. Since $k \geqslant 0$, the equivalence of $\|\cdot\|_{k,p}^{\circ}$ and $\|\cdot\|_{k,p}^{\circ\prime}$ is now a trivial consequence.

The case where $k < 0$ follows now easily with the aid of Lemma 3.1. Indeed, the norm $\|\cdot\|_{k,p}$ on $C^\infty(A)$ is equivalent to the norm dual to $\|\cdot\|_{-k,p'}^{\circ}$ and, also by Lemma 3.1, $\|\cdot\|_{k,p}'$ is equivalent to the norm dual to $\|\cdot\|_{-k,p'}^{\circ\prime}$.

This completes the proof that the Sobolev spaces on $A$ do not depend on the initial choice of $\mathcal{Q}$, their norms being determined up to equivalence.

**4. Spline bases on the interval.** The first aim of this section is to prove Jackson type inequalities for the spline systems discussed in detail in [6], [7], [8]. We start with some definitions. For each integer $n \geqslant 1$ the dyadic partition $\pi_n = \{s_{n,j};\ j = 0, \pm 1, \ldots\}$ is defined as follows. For $j \leqslant 0$ we put $s_{n,j} = 0$ and for $j \geqslant n$ let $s_{n,j} = 1$. Moreover, for $n = 2^\mu + k$, $\mu \geqslant 0$, $1 \leqslant k \leqslant 2^\mu$, $k, \mu$ being integers, let



$$s_{n,j} = \begin{cases} j/2^{\mu+1} & \text{for } j = 1, \ldots, 2k, \\ (j-k)/2^\mu & \text{for } j = 2k+1, \ldots, n. \end{cases}$$

The $B$-splines of order $r$ (i.e. of degree $r-1$), $r \geqslant 1$, corresponding to the partition $\pi_n$ are defined by the formula

$$N_{n,j}^{(r)}(t) = (s_{n,j+r} - s_{n,j})[s_{n,j}, \ldots, s_{n,j+r}; (s-t)_+^{r-1}],$$

where the square bracket is the divided difference of $(s-t)_+^{r-1}$ of order $r$ taken in the variable $s$ at the points $s_{n,j}, \ldots, s_{n,j+r}$. For later use let

$$M_{n,j}^{(r)} = \frac{r}{s_{n,j+r} - s_{n,j}} N_{n,j}^{(r)}.$$

For the properties of the $B$-splines we refer e.g. to [5]. We mention here only some of them. Namely, $N_{n,j}^{(r)} \geqslant 0$ and $\operatorname{supp} N_{n,j}^{(r)} = \langle s_{n,j}, s_{n,j+r}\rangle$. The non-trivial $N_{n,j}^{(r)}$ functions are linearly independent over any fixed interval and

$$\sum_j N_{n,j}^{(r)}(t) = 1, \qquad \int_{-\infty}^{\infty} M_{n,j}^{(r)}(t)\,dt = 1.$$

The space of splines of order $r$ on the interval $I = \langle 0, 1\rangle$ corresponding to the partition $\pi_n$ is denoted as $S_n^r$ and we know that

$$S_n^r = \operatorname{span}[N_{n,j}^{(r)};\ j = -r+1, \ldots, n-1].$$

Moreover, it is convenient to put

$$S_n^r = \mathscr{P}_{n+r-1} \quad \text{for} \quad n = 2-r, \ldots, 0,$$

where $\mathscr{P}_k$ is the space of real polynomials of order $k$ (of degree $k-1$). It then follows that $\dim S_n^r = n+r-1$ for $n \geqslant 2-r$, and $S_n^r \subset S_{n+1}^r$. Using the scalar product $(f, g)$ of $L_2 = L_2(I)$ we define the spline orthonormal system $(f_j^{(r)}, j \geqslant 2-r)$ as follows; $f_{2-r}^{(r)} = 1$, $f_{j+1}^{(r)} \in S_{j+1}^r$, $f_{j+1}^r$ is orthogonal to $S_j^r$ and $\|f_{j+1}^{(r)}\|_2 = 1$, $f_{j+1}^{(r)}(s_{j+1,2k-1}) > 0$ whenever $j+1 = 2^\mu + k$, $1 \leqslant k \leqslant 2^\mu$. Notice that the functions $f_{2-r}^{(r)}, \ldots, f_1^{(r)}$ are simply the first $r$ orthonormal Legendre polynomials on $I$. In connection with the investigation of these spline systems in the spaces of differentiable functions it is necessary to consider along with $(f_j^{(r)})$ the associated biorthogonal systems. To define them let

$$Hf(t) = \int_t^1 f, \quad Gf(t) = \int_0^t f, \quad Df(t) = \frac{d}{dt} f(t).$$

Now define for $j \geqslant 2-r+|k|$, $|k|<r$, $k$ being an integer, $f_j^{(r,0)} = f_j^{(r)}$ and

$$f_j^{(r,k)} = \begin{cases} D^k f_j^{(r)} & \text{for} \quad 0<k<r, \\ H^{-k} f_j^{(r)} & \text{for} \quad -r<k<0. \end{cases}$$

It then follows that for $|k|<r$ we have

$$(f_i^{(r,k)}, f_j^{(r,-k)}) = \delta_{ij}, \qquad i, j \geqslant 2-r+|k|. \tag{4.1}$$

Thus the operator, defined if $|k|<r$,

$$P_n^{(r,k)} f = \sum_{j=2-r+|k|}^{n} (f, f_j^{(r,-k)}) f_j^{(r,k)} \tag{4.2}$$

is a projection in $L_2$ and $P_n^{(r)} = P_n^{(r,0)}$ is an orthogonal projection in $L_2$. Defining, for $|k|<r$,

$$S_n^{r,k} = \operatorname{span}[f_j^{(r,k)};\ j = 2-r+|k|, \ldots, n]$$

we find that for $0 \leqslant k < r$ $S_n^{r,k} = S_n^{r-k}$ and for $-r<k<0$

$$S_n^{r,k} = \{f \in S_n^{r-k}\colon\ D^j f(0) = D^j f(1) = 0,\ j = 0, \ldots, -k-1\}.$$

Using the $B$-splines we find in general, $|k|<r$,

$$S_n^{r,k} = \operatorname{span}[N_{n,j}^{r-k};\ j = -r+1+k_+, \ldots, n-1-k_-], \tag{4.3}$$

where $k_\pm = \max(\pm k, 0)$. It is important that the operators $P_n^{(r,k)}$: $L_p(I) \to L_p(I)$ are uniformly bounded in $n$ and $p$ (cf. [6]), i.e. for some constant $C_r$ we have

$$\|P_n^{(r,k)}\|_p \leqslant C_r, \qquad |k|<r,\ n \geqslant 2-r+|k|,\ 1 \leqslant p \leqslant \infty. \tag{4.4}$$

Lemma 4.5. *Let $-r<k<r-1$, $n \geqslant 2-r+|k+1|$. Then*

$$DP_n^{(r,k)} f = P_n^{(r,k+1)} Df \tag{4.6}$$

*holds for $f \in W_1^1(I)$ if $k \geqslant 0$ and for $f \in \mathring{W}_1^1(I)$ if $k<0$.*

Lemma 4.7. *Let $N \geqslant k \geqslant 0$ and $\sum_{j=0}^{N} a_j = 0$. Then*

$$\sum_{j=0}^{k-1} a_j b_j + \Big(\sum_{i=k}^{N} a_i\Big) b_k = \sum_{j=0}^{k-1} \Big(\sum_{i=0}^{j} a_i\Big)(b_j - b_{j+1}).$$

The easy proofs of these lemmas are omitted.

Lemma 4.8 (Jackson type inequality). *Let $-r<k<r-1$, $1 \leqslant p \leqslant \infty$. Then there is a constant $C_r$ such that*

$$\|f - P_n^{(r,k)} f\|_p \leqslant C_r \frac{1}{n} \|D(f - P_n^{(r,k)} f)\|_p, \qquad n \geqslant 1, \tag{4.9}$$

*holds for $f \in W_1^1(I)$ if $k \geqslant 0$ and for $f \in \mathring{W}_1^1(I)$ if $k<0$.*



Proof. In case $k=0$ (4.9) is proved in [7] (cf. Lemma 4.1). Let now $k>0$. Since $P_n^{(r,k)}$ is a projection onto $S_n^{r-k}$, it follows by (4.4) that uniformly in $p$ and $n$, $1 \leqslant p \leqslant \infty$, $n \geqslant 1$

$$\|f - P_n^{(r,k)} f\|_p \sim \|f - P_n^{(r-k)} f\|_p.$$

Moreover, according to Lemma 4.5 we have

$$\begin{aligned} \|D(f - P_n^{(r,k)} f)\|_p &= \|Df - P_n^{(r,k+1)} Df\|_p \\ &\sim \|Df - P_n^{(r-k,1)} Df\|_p \sim \|D(f - P_n^{(r-k)} f)\|_p. \end{aligned}$$

Thus, by the quoted result we obtain (4.9) for $0 \leqslant k < r-1$. In the case $k<0$ the proof is based on the same idea of Freud and Popov as in the case of $k=0$ (cf. [11], [7]), and it is presented below.

The order of approximation of $\|f - P_n^{(r,k)} f\|_p$ is equivalent to the best approximation of $f$ by elements from $S_n^{r,k}$, whence with some constant $C_r$

$$\|f - P_n^{(r,k)} f\|_p \leqslant C_r \|f - h\|_p \tag{4.10}$$

holds for $h \in S_n^{r,k}$. Our aim is to choose a proper $h$ to get the right-hand side of (4.9). To this end let

$$g = D(f - P_n^{(r,k)} f), \qquad a_j = \int_{I_{n,j}} g, \qquad I_{n,j} = \langle s_{n,j}, s_{n,j+1} \rangle, \qquad b_j = M_{n,j}^{(r-k-1)},$$

and let

$$h = P_n^{(r,k)} f + G\Big[\sum_{j=0}^{n+k-1} a_j b_j + \Big(\sum_{i=n+k}^{n-1} a_i\Big) b_{n+k}\Big]. \tag{4.11}$$

Since $k<0$ and $f \in \mathring{W}_1^1(I)$ it follows that

$$\int_I g = \sum_{j=0}^{n-1} a_j = 0.$$

Thus Lemma 4.7 can be applied to get the formula

$$h = P_n^{(r,k)} f + G\Big[\sum_{j=0}^{n+k-1} \Big(\sum_{i=0}^{j} a_i\Big)(b_j - b_{j+1})\Big].$$

However, for $j = 0, \ldots, n+k-1$,

$$G(b_j - b_{j+1}) = G(M_{n,j}^{(r-k-1)} - M_{n,j+1}^{(r-k-1)}) = GDN_{n,j}^{(r-k)} = N_{n,j}^{(r-k)}$$

and therefore

$$h = P_n^{(r,k)} f + \sum_{j=0}^{n+k-1} \Big(\sum_{i=0}^{j} a_i\Big) N_{n,j}^{(r-k)},$$

whence by (4.3) we infer that $h \in \mathring{S}_n^{r,k}$. Thus we know that (4.10) holds. Since by assumptions $f - P_n^{(r,k)} f \in \mathring{W}_1^1(I)$, it follows that $Gg = f - P_n^{(r,k)} f$, and by definition (4.11) we obtain

$$f(t) - h(t) = Gg(t) - \sum_{j=0}^{n+k-1} a_j Gb_j(t) - \Big(\sum_{i=n+k}^{n-1} a_i\Big) Gb_{n+k}(t) = \int_I g(I_t - H_t), \tag{4.12}$$

where $I_t$ is the characteristic function of $\langle 0, t\rangle$ and

$$H_t(s) = \begin{cases} Gb_j(t) & \text{for} \quad s \in I_{n,j},\ j = 0, \ldots, n+k-1, \\ Gb_{n+k}(t) & \text{for} \quad s \in (s_{n,n+k}, 1\rangle. \end{cases}$$

Now, $0 \leqslant Gb_j(t) \leqslant 1$ and therefore $|f(t) - h(t)| \leqslant \|g\|_1 \leqslant \|g\|_p$, whence by (4.10) we infer inequality (4.9) for $1 \leqslant n \leqslant r - 2k$. Since now on it is assumed that $n > r - 2k$. Let

$$\int_I |f - h|^p = J_1 + J_2 + J_3$$

be the decomposition corresponding to

$$I = \langle 0, s_{n,r-k}\rangle \cup \langle s_{n,r-k}, s_{n,n+k}\rangle \cup \langle s_{n,n+k}, 1\rangle.$$

Using (4.12) we obtain for the first integral

$$J_1 = \sum_{i=0}^{r-k-1} \int_{I_{n,i}} \Big| \sum_{j=0}^{i-1} a_j + \int_{s_{n,i}}^{t} g(s)\,ds - \sum_{j=0}^{i} a_j Gb_j(t) \Big|^p dt$$
$$\leqslant \sum_{i=0}^{r-k-1} \int_{I_{n,i}} \Big| \sum_{j=0}^{i-1} a_j \big(1 - Gb_j(t)\big) + \int_{I_{n,i}} g(s) \big(I_{\langle t, s_{n,i}\rangle}(s) - Gb_j(t)\big)\,ds \Big|^p dt$$
$$\leqslant \sum_{i=0}^{r-k-1} \int_{I_{n,i}} dt \Big( \int_0^{s_{n,i+1}} |g(s)|\,ds \Big)^p \leqslant \Big( 2\,\frac{r-k}{n} \|g\|_p \Big)^p.$$

Similarly, for the second integral we have

$$J_2 = \sum_{i=r-k}^{n+k+1} \int_{I_{n,i}} dt \Big| \sum_{j=0}^{i-1} a_j + \int_{s_{n,i}}^{t} g(s)\,ds - \sum_{j=0}^{i-r+k+1} a_j - \sum_{j=i-r+k+2}^{i} a_j Gb_j(t) \Big|^p dt$$
$$\leqslant \frac{2}{n} \sum_{i=r-k}^{n+k-1} \Big( \int_{s_{n,i-r+k+2}}^{s_{n,i+1}} |g(s)|\,ds \Big)^p \leqslant \Big( \frac{C_r}{n} \|g\|_p \Big)^p.$$



Finally, for the third integral we have

$$J_3 = \sum_{i=n+k}^{n-1} \int_{I_{n,i}} \Big| \sum_{j=i-r+k}^{i-1} a_j \big(1 - Gb_j(t)\big) + \int_{I_{n,i}} g(s) \big(I_{\langle s_{n,i}, t\rangle}(s) - Gb_j(t)\big)\,ds -$$
$$- \Big( \int_{s_{n,n+k}}^{1} g(s)\,ds \Big) Gb_{n+k}(t) \Big|^p dt \leqslant \frac{2(-k)}{n} \sum_{i=n+k}^{n} \Big( 2 \int_{s_{n,n+2k-r}}^{1} |g| \Big)^p \leqslant \Big( \frac{C_r}{n} \|g\|_p \Big)^p,$$

and this completes the proof.

Now we are ready to define suitable for our purpose spline systems.

Definition 4.13. For given integer $m \geqslant 0$ we define for $n \geqslant -m$

$$F_n^{(m)} = f_n^{(2r,r)}, \qquad \mathring{F}_n^{(m)} = f_n^{(2r,-r)},$$

where $m = r - 2$. Notice that $F_n^{(m)} \in C^m(I)$ and $\mathring{F}_n^{(m)} \in \mathring{C}^{(m)}(I)$. According to (4.1) we have

$$(F_i^{(m)}, \mathring{F}_j^{(m)}) = \delta_{i,j}, \qquad i, j \geqslant -m. \tag{4.14}$$

Defining

$$Q_n^{(m)} f = \sum_{j=-m}^{n} (f, \mathring{F}_j^{(m)}) F_j^{(m)},$$
$$\mathring{Q}_n^{(m)} f = \sum_{j=-m}^{n} (f, F_j^{(m)}) \mathring{F}_j^{(m)},$$

we find by Lemma 4.5 the following formulae, $0 \leqslant k \leqslant m$, $n \geqslant k - m$,

$$\begin{aligned} D^k Q_n^{(m)} f &= P_n^{(2r, r+k)} D^k f, \qquad f \in W_1^k(I), \\ D^k \mathring{Q}_n^{(m)} f &= P_n^{(2r, -r+k)} D^k f, \qquad f \in \mathring{W}_1^k(I). \end{aligned} \tag{4.15}$$

It should be clear now (cf. (4.4)) that for each $k$, $0 \leqslant k \leqslant m$, the systems $(D^k F_j^{(m)}, j \geqslant k - m)$ and $(D^k \mathring{F}_j^{(m)}, j \geqslant k - m)$ are bases in $W_p^0$ and $\mathring{W}_p^0(I)$, $1 \leqslant p \leqslant \infty$, respectively.

Theorem 4.16. *Let $0 \leqslant l \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$. Then there is a constant $C_m$ such that Bernstein's and Jackson's inequalities, i.e.*

$$\|D^k Q_n^{(m)} f\|_p \leqslant C_m n^{k-l} \|D^l Q_n^{(m)} f\|_p, \tag{4.17}$$

$$\|D^l (f - Q_n^{(m)} f)\|_p \leqslant C_m n^{l-k} \|D^k (f - Q_n^{(m)} f)\|_p \tag{4.18}$$

*hold for $f \in W_p^k(I)$, $n \geqslant 1$. Both these inequalities hold true if we replace $Q$ by $\mathring{Q}$ and assume that $f \in \mathring{W}_p^k(I)$.*

Proof. For the proof of (4.17) see e.g. [6], [10]. The combination of (4.15) and of Lemma 4.8 gives (4.18).

**5. Bases and projections in function spaces on the cube.** Let us start with some notation: $d$ is a fixed positive integer, $\boldsymbol{t} = (t_1, \ldots, t_d)$,

$$(g_1 \otimes \ldots \otimes g_d)(\boldsymbol{t}) = g_1(t_1) \ldots g_d(t_d), \quad \boldsymbol{t} \in \boldsymbol{R}^d, \tag{5.1}$$

$$\boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} = P_{n_1}^{(r_1,k_1)} \otimes \ldots \otimes P_{n_d}^{(r_d,k_d)}, \quad |k_i| < r_i. \tag{5.2}$$

Definition (5.2) means that $\boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})}$ is linear and, if $g$ is as in (5.1), we have

$$\boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} g = (P_{n_1}^{(r_1,k_1)} g_1) \otimes \ldots \otimes (P_{n_d}^{(r_d,k_d)} g_d).$$

Moreover, define

$$S_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}} = S_{n_1}^{r_1,k_1} \otimes \ldots \otimes S_{n_d}^{r_d,k_d}, \tag{5.3}$$

i.e. $S^{\boldsymbol{r},\boldsymbol{k}}$ is the linear span of those $g$ given as in (5.1) with $g_i \in S^{r_i,k_i}$. It now follows that $\boldsymbol{P}_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}}$ is a projection onto $S_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}}$. Inequality (4.4) implies now that there is a constant $C_{\boldsymbol{r}}$ (cf. [2]) such that for the norm of $\boldsymbol{P}_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}}$: $L^p(I^d) \to L^p(I^d)$, $1 \leqslant p \leqslant \infty$, we have

$$\|\boldsymbol{P}_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}}\|_p \leqslant C_{\boldsymbol{r}}, \quad n_i \geqslant 2 - r_i + |k_i|, \quad i = 1, \ldots, d. \tag{5.4}$$

Actually $C_{\boldsymbol{r}} = C_{r_1} \ldots C_{r_d}$. For later convenience we denote by $\boldsymbol{P}_{n,i}^{(r,k)}$ the linear operator

$$\boldsymbol{P}_{n,i}^{(r,k)} = E_1 \otimes \ldots \otimes E_{i-1} \otimes P_n^{(r,k)} \otimes E_{i+1} \otimes \ldots \otimes E_d, \tag{5.5}$$

where the $E_j$'s are copies of the identity operator. Now,

$$\boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} = \boldsymbol{P}_{n_1,1}^{(r_1,k_1)} \circ \ldots \circ \boldsymbol{P}_{n_d,d}^{(r_d,k_d)}, \tag{5.6}$$

and clearly

$$\|\boldsymbol{P}_{n,i}^{(r,k)}\|_p \leqslant C_r, \quad |k| < r. \tag{5.7}$$

Denote by $S_{n,i}^{r,k}$ the range of the projection $\boldsymbol{P}_{n,i}^{(r,k)}$ in $L_p(I^d)$. It then follows by (5.6) that

$$S_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}} = \bigcap_{i=1}^{d} S_{n_i,i}^{r_i,k_i}. \tag{5.8}$$

LEMMA 5.9. *Let $1 \leqslant p \leqslant \infty$, $|k_j| < r_j$, $j = 1, \ldots, d$. Then there is a constant $C_{\boldsymbol{r}}$ such that for $f \in L_p(I^d)$ we have*

$$C_{\boldsymbol{r}}^{-1} \|f - \boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} f\|_p \leqslant \sum_{j=1}^{d} \|f - \boldsymbol{P}_{n_j,j}^{(r_j,k_j)} f\|_p \leqslant C_{\boldsymbol{r}} \|f - \boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} f\|_p,$$

*for $n_j \geqslant 2 - r_j + |k_j|$, $j = 1, \ldots, d$.*

Proof. Since $\boldsymbol{P}_{n,j}^{(r,k)}$ and $\boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})}$ are projections, we have for $f \in L_p(I^d)$ uniformly in $p$, $1 \leqslant p \leqslant \infty$,

$$\|f - \boldsymbol{P}_{n,j}^{(r,k)} f\|_p \sim \inf\{\|f - g\|_p \colon g \in S_{n,j}^{(r,k)}\},$$
$$\|f - \boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} f\|_p \sim \inf\{\|f - g\|_p \colon g \in S_{\boldsymbol{n}}^{\boldsymbol{r},\boldsymbol{k}}\}.$$



Thus the right-hand side inequality in Lemma 5.9 follows by (5.8). The left-hand side inequality we obtain using (5.6) by a telescoping argument. Indeed,

$$\boldsymbol{E} - \boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} = \sum_{j=1}^{d} \boldsymbol{P}_{n_1,1}^{(r_1,k_1)} \circ \ldots \circ \boldsymbol{P}_{n_{j-1},j-1}^{(r_{j-1},k_{j-1})} (\boldsymbol{E} - \boldsymbol{P}_{n_j,j}^{(r_j,k_j)}),$$

and therefore an application of (5.7) completes the proof.

In analogy to Lemma 4.5 we have the following formulae. If $\boldsymbol{0} \leqslant \boldsymbol{k} + \boldsymbol{a} < \boldsymbol{r}$, $\boldsymbol{a}$ being a multi-index and $f \in W_p^{\boldsymbol{a}}(I^d)$, then

$$D^{\boldsymbol{a}} \boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k})} f = \boldsymbol{P}_{\boldsymbol{n}}^{(\boldsymbol{r},\boldsymbol{k}+\boldsymbol{a})} D^{\boldsymbol{a}} f. \tag{5.10}$$

Moreover, for each $i = 1, \ldots, d$, we have

$$D^{\boldsymbol{a}} \boldsymbol{P}_{n_i,j}^{(r_i,k_i)} f = \boldsymbol{P}_{n_i,j}^{(r_i,k_i+a_i)} D^{\boldsymbol{a}} f. \tag{5.11}$$

Formulae (5.10) and (5.11) hold true if $-\boldsymbol{r} < \boldsymbol{k} + \boldsymbol{a} < \boldsymbol{0}$ and $f \in \mathring{W}_p^{\boldsymbol{a}}(I^d)$. Let us now define

$$\boldsymbol{Q}_n^{(m)} = Q_n^{(m)} \otimes \ldots \otimes Q_n^{(m)},$$
$$\mathring{\boldsymbol{Q}}_n^{(m)} = \mathring{Q}_n^{(m)} \otimes \ldots \otimes \mathring{Q}_n^{(m)},$$

where $n \geqslant -m$, $m \geqslant 0$, and the number of factors is $d$.

LEMMA 5.12. *Let $0 \leqslant l \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$. Then there is a constant $C_{r,d}$ such that for $n \geqslant 1$ and $f \in W_p^k(I^d)$*

$$\sum_{|\alpha|=l} \|D^\alpha (f - \boldsymbol{Q}_n^{(m)} f)\|_p \leqslant C_{r,d} n^{l-k} \sum_{|\beta|=k} \|D^\beta (f - \boldsymbol{Q}_n^{(m)} f)\|_p. \tag{5.13}$$

*Inequality* (5.13) *remains true after replacing $\boldsymbol{Q}$ by $\mathring{\boldsymbol{Q}}$ and assuming that $f \in \mathring{W}_p^k(I^d)$.*

Proof. Let $\boldsymbol{r} = (r, \ldots, r)$, $\boldsymbol{n} = (n, \ldots, n)$, $|\boldsymbol{a}| = 1$, $m = r - 2$. According to (5.10) and Lemma 5.9 we have

$$\|D^\alpha (f - \boldsymbol{Q}_n^{(m)} f)\|_p = \|D^\alpha f - \boldsymbol{P}_{\boldsymbol{n}}^{(2\boldsymbol{r},\boldsymbol{r}+\boldsymbol{a})} D^\alpha f\|_p \leqslant C_{r,d} \sum_{i=1}^{d} \|D^\alpha f - \boldsymbol{P}_{n,i}^{(2r,r+a_i)} D^\alpha f\|_p.$$

Now, by Lemma 4.8, (5.11) and Fubini's theorem

$$\|D^\alpha f - \boldsymbol{P}_{n,i}^{(2r,r+a_i)} D^\alpha f\|_p \leqslant C_r n^{l-k} \|D^{\alpha+(k-l)e_i} f - \boldsymbol{P}_{n,i}^{(2r,r+a_i+k-l)} D^{\alpha+(k-l)e_i} f\|_p$$
$$\leqslant C_r n^{l-k} \sum_{|\beta|=k} \|D^\beta f - \boldsymbol{P}_{n,i}^{(2r,r+\beta_i)} D^\beta f\|_p,$$

and once more by Lemma 5.9 and by (5.10) for fixed $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = k$ we get

$$\sum_{i=1}^{d} \|D^\beta f - \boldsymbol{P}_{n,i}^{(2r,r+\beta_i)} D^\beta f\|_p \leqslant C_r \|D^\beta f - \boldsymbol{P}_{\boldsymbol{n}}^{(2\boldsymbol{r},\boldsymbol{r}+\boldsymbol{\beta})} D^\beta f\|_p$$
$$= C_r \|D^\beta (f - \boldsymbol{P}_{\boldsymbol{n}}^{(2\boldsymbol{r},\boldsymbol{r})} f)\|_p = C_r \|D^\beta (f - \boldsymbol{Q}_n^{(m)} f)\|_p.$$

The combination of all these inequalities gives (5.13). For $f \in \mathring{W}_p^k(I^d)$ the proof of (5.13) with $Q$ replaced by $\mathring{Q}$ is similar and it is omitted.

Using Fubini's theorem and (4.17) we establish easily the Bernstein's inequality

(5.14) $$\|D^k Q_n^{(m)} f\|_p \leqslant C_{r,d} n^{|k|-|l|} \|D^l Q_n^{(m)} f\|_p,$$

which holds for $n \geqslant 1$, $0 \leqslant l \leqslant k \leqslant (m, \ldots, m)$, $1 \leqslant p \leqslant \infty$. Clearly $Q$ can be replaced by $\mathring{Q}$ in (5.14). Before we state the main result let us introduce, for $\boldsymbol{n} = (n_1, \ldots, n_d)$, $n_i \geqslant -m$, the functions

$$\boldsymbol{F}_{\boldsymbol{n}}^{(m)} = F_{n_1}^{(m)} \otimes \ldots \otimes F_{n_d}^{(m)}, \qquad \mathring{\boldsymbol{F}}_{\boldsymbol{n}}^{(m)} = \mathring{F}_{n_1}^{(m)} \otimes \ldots \otimes \mathring{F}_{n_d}^{(m)}.$$

Now, if $n \geqslant -m$, $\boldsymbol{n} = (n, \ldots, n)$, $\boldsymbol{m} = (m, \ldots, m)$, we let

$$Q_n^{(m)} f = \sum_{-\boldsymbol{m} \leqslant \boldsymbol{j} \leqslant \boldsymbol{n}} (f, \mathring{\boldsymbol{F}}_{\boldsymbol{j}}^{(m)}) \boldsymbol{F}_{\boldsymbol{j}}^{(m)},$$

$$\mathring{Q}^{(m)} f = \sum_{-\boldsymbol{m} \leqslant \boldsymbol{j} \leqslant \boldsymbol{n}} (f, \boldsymbol{F}_{\boldsymbol{j}}^{(m)}) \mathring{\boldsymbol{F}}_{\boldsymbol{j}}^{(m)}.$$

Clearly $Q_n^{(m)}$ and $\mathring{Q}_n^{(m)}$ are projections in $L_2(I^d)$ and are adjoint to each other.

As a consequence of (5.7), (5.10), (5.11) and Lemma 5.9 we get

PROPOSITION 5.15. *There exists $C = C_{m,d}$ such that, for $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$ and $n \geqslant -m$,*

$$\|Q_n^{(m)}\colon W_p^k(I^d) \to W_p^k(I^d)\| \leqslant C$$

*and, for $f \in W_p^k(I^d)$, $\|f - Q_n^{(m)} f\|_{k,p} \to 0$. In fact the system $(\boldsymbol{F}_{\boldsymbol{n}}^{(m)})$, suitably ordered (e.g., as in [6]), is a Schauder basis in $W_p^k(I^d)$ for $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$.*

*Analogous facts are true for $(\mathring{\boldsymbol{F}}_n^{(m)})$, $(\mathring{Q}_n^{(m)})$ and $\mathring{W}_p^k(I^d)$.*

THEOREM 5.16. *Let $0 \leqslant k \leqslant k+l \leqslant m$, $1 \leqslant p \leqslant \infty$. Then there is $C = C_{m,d}$ so that Bernstein's and Jackson's inequalities*

(5.17) $$\|Q_n^{(m)} f\|_{k+l,p} \leqslant C n^l \|Q_n^{(m)} f\|_{k,p},$$

(5.18) $$\|f - Q_n^{(m)} f\|_{k,p} \leqslant C n^{-l} \|f - Q_n^{(m)} f\|_{k+l,p},$$

*hold for $n \geqslant 1$ and $f \in W_p^k(I^d)$ (resp. $f \in W_p^{k+l}(I^d)$).*

*Inequalities (5.17) and (5.18) remain valid after replacing $Q_n^{(m)}$ by $\mathring{Q}_n^{(m)}$, provided that $f \in \mathring{W}_p^k(I^d)$ (resp. $f \in \mathring{W}_n^{k+l}(I^d)$).*

Proof. Inequality (5.17) follows easily from (5.14)

$$\|Q_n^{(m)} f\|_{k+l,p} \leqslant \Big( \sum_{|\alpha| \leqslant l} + \sum_{l < |\alpha| \leqslant k+l} \Big) \|D^\alpha Q_n^{(m)} f\|_p$$

$$\leqslant C n^l \|Q_n^{(m)} f\|_p + n^l \sum_{0 < |\beta| \leqslant k} \|D^\beta Q_n^{(m)} f\|_p \leqslant C n^l \|Q_n^{(m)} f\|_{k,p}.$$



The proof of inequality (5.18) is similar. Using Lemma 5.12 we obtain

$$\|f - Q_n^{(m)} f\|_{k,p} \leqslant \sum_{i=0}^{k} \sum_{|\alpha| = i} \|D^\alpha (f - Q_n^{(m)} f)\|_p$$

$$\leqslant C n^{-l} \sum_{i=0}^{k} \sum_{|\beta| = i+l} \|D^\beta (f - Q_n^{(m)} f)\|_p \leqslant C n^{-l} \|f - Q_n^{(m)} f\|_{k+l,p}.$$

The proof of (5.17) and (5.18) for $\mathring{Q}_n^{(m)}$ is analogous.

COROLLARY 5.19. *Let $0 \leqslant k \leqslant k+l \leqslant m$, $1 \leqslant p \leqslant \infty$. Then, for some $C = C_{m,d}$, we have for $n \geqslant 1$*

$$\|Q_n^{(m)}\colon W_p^k(I^d) \to W_p^{k+l}(I^d)\| \leqslant C n^l,$$

$$\|E - Q_n^{(m)}\colon W_p^{k+l}(I^d) \to W_p^k(I^d)\| \leqslant C n^{-l},$$

*where we set $\boldsymbol{E} f = f \in W_p^k(I^d)$ for $f \in W_p^{k+l}(I^d)$.*

*Moreover, $\|f - Q_n^{(m)} f\|_{k,p} = o(n^{-l})$ for $f \in W_p^{k+l}(I^d)$.*

*Analogous estimates hold true for the operators $\mathring{Q}_n^{(m)}$, $n \geqslant 1$ and $f \in \mathring{W}_p^{k+l}(I^d)$.*

Proof. Since the $Q_n^{(m)}$'s are idempotents, the corollary follows readily from Theorem 5.16 and Proposition 5.15.

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# A criterion for subharmonicity of a function of the spectrum

by

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**Abstract.** The following is a special case of the general result proven in the paper.
Let $\chi\colon F_c(\boldsymbol{C})\to[-\infty, +\infty)$, where $F_c(\boldsymbol{C})$ denotes the collection of all non-empty compact subsets of the complex plane $\boldsymbol{C}$. Assume that $\chi(K)\leqslant\chi(L)$, if $K\subset L$, and $\chi(\bigcap K_n) = \lim\chi(K_n)$, whenever $K_{n+1}\subset K_n$ for $n = 1, 2, \ldots$. Then conditions (a) and (b) are equivalent: (a) for every analytic function $a$ from $G\subset\boldsymbol{C}$ into a Banach algebra $A$ the function $\lambda\to\chi\big(\sigma(a(\lambda))\big)$ is subharmonic; (b) the same for $A$ commutative.
An application to uniform algebras is given.

**1. Introduction.** Consider a typical situation: we are given a Banach algebra $A$ (the case $A$ = the algebra of all bounded operators on a Banach space $X$ being the most interesting) and an analytic function $a\colon G\to A$, where $G\subset\boldsymbol{C}$ is open; suppose that we are interested in studying the behaviour of the set-valued function $K(\lambda) = \sigma\big(a(\lambda)\big)$ (= the spectrum). One way of doing it is to consider some characteristics $\chi$ of compact sets, and to analyse the functions $\lambda\to\chi\big(K(\lambda)\big)$. In many instances $\chi\big(K(\lambda)\big)$ was found to be subharmonic (e.g. for $\chi(K) = \log\max\{|z|\colon z\in K\}$, cf. Vesentini [14], and $\chi(K) = \log\operatorname{diam}(K)$, cf. Aupetit [1], and $\chi(K) = n$th diameter of $K$ or the logarithmic capacity of $K$, cf. Słodkowski [10]).

In the realm of uniform algebras J. Wermer [16] began to study the multifunction $K(\lambda) = \hat{g}\big(\hat{f}^{-1}(\lambda)\big)$, where $f, g\in A$, a uniform algebra on a compact space $X$, and $\lambda\in\sigma(f)\setminus f(X)$. Here, too, $\chi\big(K(\lambda)\big)$ is subharmonic for the same characteristics $\chi$ as above, cf. [3], [5], [8], [10], [17], [18].

Since this approach has resulted already in many interesting applications to uniform algebras and operator theory (see [1], [2], [3], [8], [12], [16], [18]), it seems worthwhile to find out some general and easily applicable conditions on $\chi$, that would guarantee subharmonicity of $\chi\big(K(\lambda)\big)$ for $K(\lambda) = \sigma\big(a(\lambda)\big)$ or $g\big(f^{-1}(\lambda)\big)$. (Cf. [1], Ch. 3, § 1, Remarque.)

Incidentally, each concrete $\chi$ mentioned above fulfils trivially the following condition. (This observation was made by B. Aupetit.)

(*) If $a\colon G\to A$ is analytic, and $A$ is a commutative Banach algebra then $\lambda\to\chi\big(\sigma(a(\lambda))\big)$ is subharmonic.