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On the interpolation of sublinear operators

by

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Abstract. A wide class of interpolation theorems for linear operators extend to sublinear ones.

Let T be a mapping defined on a vector space X, taking values that are measurable functions on a measure space (Ω, μ). We say that T is *sublinear* if

$$|T(x_1+x_2)| \leqslant |T(x_1)|+|T(x_2)| \text{ a.e. } (x_1, x_2 \in X)$$

and

$$|T(kx)| \leqslant |k|\,|T(x)| \text{ a.e. } (x \in X,\ k \text{ scalar}).$$

The theory of interpolation of operators treats mainly linear operators. One exception is the Marcinkiewicz interpolation theorem [7], which holds for sublinear operators, and in fact for quasilinear ones ($|T(x_1+x_2)| \leqslant C(|T(x_1)|+|T(x_2)|)$ a.e.). The same is true for its abstract counterpart; the real method of interpolation. Using the K-method it is easily seen that if (X_0, X_1) is a couple of quasi-Banach spaces, (Y_0, Y_1) is a couple of quasi-Banach lattices of functions on a measure space Ω, and T is a quasi-linear operator from X_0+X_1 to Y_0+Y_1 that is bounded from X_0 to Y_0 and from X_1 to Y_1, then T is bounded from $(X_0, X_1)_{\theta q}$ to $(Y_0, Y_1)_{\theta q}$, cf. [2].

For the complex method, Calderón and Zygmund [1] have extended the Riesz interpolation theorem to sublinear operators on L^p spaces, while Weiss [6] treated sublinear operators of H^p onto L^p. Kraynek [5] extended this to Orlicz spaces. We will prove a general theorem including these results.

The customary procedure to interpolate the various sublinear operators that arise in analysis (maximal functions, square functions etc.) is to construct linearizations of them and use the theory for linear operators. The proof below is based on this idea and consists essentially of showing that any sublinear operator may be linearized (in a nonconstructive way).

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We study sublinear operators defined on X_0+X_1 where the couple X_0 and X_1 may be completely arbitrary as long as the interpolation space is defined. (Thus they do not have to be Banach spaces. Cf. [6], treating H^p, $p>0$. For remarks on the definition of complex interpolation spaces for quasi-Banach spaces, see [4].)

For simplicity we assume that the operators map into L^p-spaces, but the proof also works for more general Banach lattices of functions, e.g. Orlicz spaces, Lorentz spaces and weighted L^p-spaces. (In the discrete case (l^p-spaces) we may take $0<p_0, p_1<\infty$, but the proof does not allow this in general.)

THEOREM. *Let X_0 and X_1 be a couple of quasi-Banach spaces and let $X_\theta = (X_0, X_1)_\theta$ $(0<\theta<1)$ be a complex interpolation space. Let T be a sublinear operator of X_0+X_1 into $L^{p_0}(\mu)+L^{p_1}(\mu)$, $1\leqslant p_0, p_1\leqslant\infty$, such that*

$$\|Tx\|_{L^{p_0}(\mu)} \leqslant A_0\|x\|_{X_0} \quad \text{for} \quad x\in X_0$$

and

$$\|Tx\|_{L^{p_1}(\mu)} \leqslant A_1\|x\|_{X_1} \quad \text{for} \quad x\in X_1.$$

Then

$$\|Tx\|_{L^{p_\theta}(\mu)} \leqslant A_0^{1-\theta}A_1^\theta\|x\|_{X_\theta} \quad \text{for} \quad x\in X_\theta,$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

Proof. Let $x_0\in X_\theta$. First we assume that the measure μ is discrete. Let $\omega\in\Omega$; by the definition of sublinearity $x\to|Tx(\omega)|$ is a seminorm on X_0+X_1. By the Hahn-Banach theorem there exists a linear functional u_ω on X_0+X_1 such that $|u_\omega(x)|\leqslant|Tx(\omega)|$ and $u_\omega(x_0) = Tx_0(\omega)$. For $x\in X_0+X_1$ we define the function Ux by $Ux(\omega) = u_\omega(x)$. Then U is a linear operator of X_0+X_1 into the functions on Ω, $|Ux|\leqslant|Tx|$ and $Ux_0 = Tx_0$.

Consequently $\|Ux\|_{l^{p_i}}\leqslant\|Tx\|_{l^{p_i}}\leqslant A_i\|x\|_{X_i}$ $(i=0,1,\ x\in X_i)$. By interpolation

$$\|Tx_0\|_{l^{p_\theta}} = \|Ux_0\|_{l^{p_\theta}} \leqslant A_0^{1-\theta}A_1^\theta\|x_0\|_{X_\theta}.$$

In general this proof has to be modified to make Ux measurable. Therefore, let $E_1,\ldots,E_N$ be disjoint subsets of Ω with finite measures. $\int_{E_j}|Tx|\,d\mu$ is a seminorm and thus there exists a linear functional u_j such that $|u_j(x)|\leqslant\int_{E_j}|Tx|\,d\mu$ and $u_j(x_0) = \int|Tx_0|\,d\mu$. We define U by

$$Ux(\omega) = \sum\big(\mu(E_j)\big)^{-1}u_j(x)\,\chi_{E_j}(\omega).$$

Thus U is a linear operator, Ux_0 equals the conditional expectation $E(|Tx_0|\,|\mathscr{F}\{E_j\})$ on $\bigcup E_j$ and is zero elsewhere, and $|Ux|\leqslant E(|Tx|\,|\mathscr{F}\{E_j\})$, whence $\|Ux\|_{L^{p_i}}\leqslant\|Tx\|_{L^{p_i}}\leqslant A_i\|x\|_{X_i}$ for every x. Hence

$$\|Ux_0\|_{L^{p_\theta}}\leqslant A_0^{1-\theta}A_1^\theta\|x_0\|_{X_\theta}.$$

We can make $\|Ux_0\|_{L^{p_\theta}}$ arbitrarily close to $\|Tx_0\|_{L^{p_\theta}}$ by a proper choice of $\{E_j\}_1^N$, e.g.

$$E_j = \{\omega\colon j\varepsilon\leqslant|Tx_0(\omega)|<(j+1)\varepsilon\}$$

for suitable ε and N. Thus

$$\|Tx_0\|_{L^{p_\theta}}\leqslant A_0^{1-\theta}A_1\|x_0\|_{X_\theta},$$

and the proof is completed.

Remark. We have formulated this theorem for the complex method, but it is clear that it may be replaced by any method of interpolation. For example, the method in [3] yields interpolation theorems for sublinear operator between Orlicz spaces different from those in [5].

Note added in proof. The theorem does not extend to quasi-linear operators, see M. Cwikel, *A counterexample in nonlinear interpolation*, Proc. Amer. Math. Soc. 62 (1977), 62–66.

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