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On the extension of continuous linear maps in function spaces and the splitness of Dolbeaut complexes of holomorphic Banach bundles

by

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Abstract. The paper investigates the extension of continuous linear maps with values in the spaces of sections of coherent analytic sheaves over analytic spaces. It is shown that the space $H^0(X, \mathscr{S})$, where $\mathscr{S}$ is a coherent analytic sheaf over a paracompact analytic space X has the extension property with respect to the class of s-nuclear spaces if and only if it is isomorphic to C^A for some set A. We also investigate the existence of continuous linear projections of the space $C_\xi^\infty(R(X))$ onto $\mathcal{O}_\xi(X)$, where $R(X)$ is the regular part of X and ξ is a holomorphic Banach bundle over X. The splitness of Dolbeaut complexes of holomorphic Banach bundles over complex manifolds is considered. We prove that on complex manifolds which are increasing unions of open Stein sets these complexes split only at positive dimensions.

Introduction. In the present paper we consider extensions of continuous linear maps with values in some function spaces of complex analysis and the splitness of Dolbeaut complexes of holomorphic Banach bundles over complex manifolds. These problems have been investigated by several authors ([6], [8]). The paper contains three sections.

In § 1 we prove that the space $H^0(X, \mathscr{S})$ has the extension property with respect to the class of s-nuclear spaces if and only if it is isomorphic to C^A for some set A.

Section § 2 is devoted to the study of the existence of continuous linear projections of $C_\xi^\infty(R(X))$ onto $\mathcal{O}_\xi(X)$. It is shown that when X is Stein such a projection exists if and only if X is discrete.

In § 3 we investigate the splitness of Dolbeaut complexes of holomorphic Banach bundles over complex manifolds. We prove that on complex manifolds which are increasing unions of open Stein sets these complexes split only at positive dimensions. Let us note that the splitness of Dolbeaut complexes of holomorphic vector bundles over Stein manifolds has been established by Palamodov ([8]).

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§ 1. The extension of continuous linear maps with values in $H^0(X, \mathscr{S})$. Let E be a locally convex space. By $\mathscr{U}(E)$ we denote the set of all balanced convex neighbourhoods of zero in E. For each $U \in \mathscr{U}(E)$ let ϱ_U denote the Minkowski functional on E of U and let $E(U)$ be the completion of $E/\varrho_U = E/\varrho_U^{-1}(0)$ equipped with the norm ϱ_U. The canonical map from E into $E(U)$ is denoted by $\pi(U)$ and from $E(V)$, where $V \in \mathscr{U}(E)$, $V \subset U$, into $E(U)$ by $\omega(V, U)$. Finally by E' we denote the strongly dual space of E.

The space E is called *s-nuclear* ([9]) if for each $U \in \mathscr{U}(E)$ there exists a $V \in \mathscr{U}(E)$ such that $V \subset U$ and the map $\omega(V, U)$ is s-nuclear, i.e. $\omega(V, U)$ can be represented in the form

$$\omega(V, U)u = \sum_{j=1}^{\infty} \lambda_j u_j'(u) v_j, \tag{1.1}$$

where

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0,$$

$$\sum_{j=1}^{\infty} \lambda_j^p < \infty \quad \text{for} \quad p > 0, \tag{S}$$

$$u_j' \in E(V)', \quad v_j \in E(U), \quad \sup\{\|u_j'\|, \|v_j\|\} < \infty.$$

Let X be a paracompact analytic space and let $\mathscr{S}$ be a coherent analytic sheaf over X. It is known ([4]) that $H^0(W, \mathscr{S})$ has a unique locally convex space structure, where W is an open set in X, such that the restiction maps are continuous. In this section we prove the following

THEOREM 1.1. *Let X be a paracompact analytic space and let $\mathscr{S}$ be a coherent analytic sheaf over X. Then the following conditions are equivalent:*

(i) *$H^0(X, \mathscr{S}) \cong C^A$ for some set A.*

(ii) *Every continuous linear map from every subspace of every locally convex space into $H^0(X, \mathscr{S})$ can be extended to a continuous linear map.*

(iii) *The statement* (ii) *holds for all subspaces of all s-nuclear spaces.*

Proof. (i) ⇒ (ii) follows from the Hahn–Banach theorem.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (ii) is an immediate consequence of Propositions 1.2 and 1.3.

PROPOSITION 1.2 ([6]). *Let X be a paracompact analytic space and let $\mathscr{S}$ be a coherent analytic sheaf over X. Then $H^0(X, \mathscr{S})$ is s-nuclear.*

PROPOSITION 1.3. *Let E be an s-nuclear subspace of a locally convex space F. Then there exist an s-nuclear space $\tilde{E}$ and continuous linear maps $\tilde{\imath}$: $E \to \tilde{E}$ and h: $F \to \tilde{E}$ such that*

$$hi = \tilde{\imath} \quad \text{and} \quad \tilde{\imath} \text{ is an embedding.}$$

Here i denotes the canonical embedding of E into F.

We need the following

LEMMA 1.4 ([6]). *Every s-nuclear map from a subspace of a Banach space into a Banach space can be extended to an s-nuclear map.*

LEMMA 1.5 ([6]). *For every s-nuclear map T from a Banach space A into a Banach space B there exist s-nuclear maps $P: A \to C$ and $Q: C \to B$ such that $T = QP$.*

LEMMA 1.6 ([6]). *Let α: $A \to B$ and β: $B \to C$ be continuous linear maps between Banach spaces such that $i\alpha$ is s-nuclear for some embedding i and β is quasi-nuclear. Then $\beta\alpha$ is s-nuclear.*

Proof of Proposition 1.3. Let $U, V \in \mathscr{U}(E)$. We write $U < V$ if $U \supset V = \tilde{V} \cap E$, where $\tilde{V} \in \mathscr{U}(F)$ and $\omega(V, U)$ is s-nuclear. Let $U, V \in \mathscr{U}(E)$ and $U < V$. By Lemmas 1.4 and 1.5 there exist sequences $\{P_j(V, U)\}$ and $\{Q_j(V, U)\}$ of s-nuclear maps such that

$$\begin{aligned} &\tilde{\omega}(V, U) = Q_1(V, U)P_1(V, U), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &P_j(V, U) = Q_{j+1}(V, U)P_{j+1}(V, U), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{1.2}$$

where $\tilde{\omega}(V, U)$: $F(\tilde{V}) \to E(U)$ is an s-nuclear extension of $\omega(V, U)$. Fix $\tilde{W} \in \mathscr{U}(F)$. Let $\mathscr{F}(\tilde{W})$ denote the set of all finite sequences

$$a = \{(\tilde{V}_1, \tilde{U}_1), \ldots, (\tilde{V}_n, \tilde{U}_n); k_1, \ldots, k_n\},$$

where

$$\tilde{V}_j, \tilde{U}_j \in \mathscr{U}(F), \quad \tilde{W} \subset \tilde{V}_j \subset \tilde{U}_j, \quad U_j = \tilde{U}_j \cap E < V_j = \tilde{V}_j \cap E$$
$$\text{and} \quad k_j \in \{1, 2, \ldots\}, \ j = 1, \ldots, n.$$

For $a_j = \{(\tilde{V}_1^j, \tilde{U}_1^j), \ldots, (\tilde{V}_{n_j}^j, \tilde{U}_{n_j}^j), k_1^j, \ldots, k_{n_j}^j\} \in \mathscr{F}(\tilde{W})$, $j = 1, 2$ we write $a_1 < a_2$ if

$$\{(\tilde{V}_1^1, \tilde{U}_1^1), \ldots, (\tilde{V}_{n_1}^1, \tilde{U}_{n_1}^1)\} \subset \{(\tilde{V}_1^2, \tilde{U}_1^2), \ldots, (\tilde{V}_{n_2}^2, \tilde{U}_{n_2}^2)\}$$
$$\text{and} \quad \max k_j^1 < \min k_j^2.$$

For each $a \in \mathscr{F}(\tilde{W})$ by $\tilde{E}(\tilde{W}, a)$ we denote the space $F(\tilde{W})/\varrho_a$ equipped with the norm

$$\varrho_a(u) = \sum_{j=1}^{n} \|P_{k_j}(V_j, U_j)\,\omega(\tilde{W}, \tilde{V}_j)u\|. \tag{1.3}$$

Obviously, if $a_1 < a_2$, then the indentity map id: $F(\tilde{W}) \to F(\tilde{W})$ induces a continuous linear map $\tilde{\omega}(\tilde{W}, a_2, a_1)$: $\tilde{E}(\tilde{W}; a_2) \to \tilde{E}(\tilde{W}; a_1)$ and

$\tilde{\omega}(\tilde{W}, a_3, a_1) = \tilde{\omega}(\tilde{W}, a_3, a_2)\tilde{\omega}(\tilde{W}, a_2, a_1)$ for $a_3 > a_2$. Thus we can put

$$\tilde{E}(\tilde{W}) = \varprojlim \{\tilde{E}(\tilde{W}, a), \tilde{\omega}(\tilde{W}, a, b)\}.$$

From the construction of $\tilde{E}(\tilde{W})$ it follows that if $\tilde{W}_1, \tilde{W}_2 \in \mathscr{U}(F)$, $\tilde{W}_2 \subset \tilde{W}_1$ then the map $\omega(\tilde{W}_2, \tilde{W}_1)$ induces a continuous linear map $\tilde{\omega}(\tilde{W}_2, \tilde{W}_1)$: $\tilde{E}(\tilde{W}_2) \to \tilde{E}(\tilde{W}_1)$ such that

$$h(\tilde{W}_1)\omega(\tilde{W}_2, \tilde{W}_1) = \tilde{\omega}(\tilde{W}_2, \tilde{W}_1)h(\tilde{W}_2),$$

where $h(\tilde{W}_j)$: $F(\tilde{W}_j) \to \tilde{E}(\tilde{W}_j)$ denotes the canonical map. Thus putting

$$\tilde{E} = \varprojlim \{\tilde{E}(\tilde{W}), \tilde{\omega}(\tilde{W}, \tilde{W}')\}, \quad h = \varprojlim h(\tilde{W}) \text{ and } \tilde{\iota} = h\iota,$$

we get a locally convex space $\tilde{E}$ and continuous linear maps $\tilde{\iota}$: $E \to \tilde{E}$ and h: $F \to \tilde{E}$ such that $h\iota = \tilde{\iota}$. Hence to finish the proof it remains to establish the following:

(a) *$\tilde{\iota}$ is an embedding.*

(b) *$\tilde{E}$ is s-nuclear.*

Proof of (a). Let $\tilde{\iota}(u_\alpha) \to 0$ and let $U, V \in \mathscr{U}(E)$ be such that $U < V$. Applying (1.3) to $a = (V, U, 1)$, we have

$$\begin{aligned} \lim \varrho_U(u_\alpha) &= \lim \|\tilde{\omega}(V, U)\pi(\tilde{V})u_\alpha\| \\ &= \lim \|Q_1(V, U)P_1(V, U)\pi(\tilde{V})u_\alpha\| = 0. \end{aligned}$$

Hence $\mathscr{U}_\alpha \to 0$ and therefore $\tilde{\iota}$ is an embedding.

Proof of (b). It suffices to show that $\tilde{E}(\tilde{W})$ is s-nuclear for $\tilde{W} \in \mathscr{U}(F)$. Let $\tilde{W} \in \mathscr{U}(F)$ and let $a_1, a_2 \in \mathscr{F}(\tilde{W})$, $a_1 < a_2$. Consider the maps

$$\theta_1\colon \tilde{E}(\tilde{W}, a_1) \to \mathrm{Im}P_{k_1^1}(V_1^1, U_1^1) \oplus \ldots \oplus \mathrm{Im}P_{k_{n_1}^1}(V_{n_1}^1, U_{n_1}^1),$$

$$\theta_2\colon \tilde{E}(\tilde{W}, a_2) \to \mathrm{Im}P_{k_1^1}(V_1^1, U_1^1) \oplus \ldots \oplus \mathrm{Im}P_{k_{n_1}^1}(V_{n_1}^1, U_{n_1}^1)$$

given by the formulas

$$\theta_1(\bar{u}) = \left(P_{k_1^1}(V_1^1, U_1^1)\omega(\tilde{W}, \tilde{V}_1^1)u, \ldots, P_{k_{n_1}^1}(V_{n_1}^1, U_{n_1}^1)\omega(\tilde{W}, \tilde{V}_{n_1}^1)u\right),$$

where $u \in F(W)$, $\bar{u} = u \bmod \varrho_{a_1}^{-1}(0)$,

$$\begin{aligned} \theta_2(\bar{u}) = \big(&Q_{k_1^1+1}(V_1^1, U_1^1)P_{k_1^1+1}(V_1^1, U_1^1)\omega(\tilde{W}, \tilde{V}_1^1)u, \ldots \\ &\ldots, Q_{k_{n_1}^1+1}(V_{n_1}^1, U_{n_1}^1)P_{k_{n_1}^1+1}(V_{n_1}^1, U_{n_1}^1)\omega(\tilde{W}, \tilde{V}_{n_1}^1)u\big), \end{aligned}$$

where $u \in F(W)$, $\bar{u} = u \bmod \varrho_{a_2}^{-1}(0)$.

By (1.2) we have $\theta_2 = \theta_1\tilde{\omega}(\tilde{W}, a_2, a_1)$ and by (1.3) θ_1 is an embedding. Since $Q_j(V, U)$ is s-nuclear for $U < V$ and for $j \geqslant 1$, it follows that θ_2 is s-nuclear. Hence, by Lemma 1.6, $\tilde{\omega}(\tilde{W}, a_3, a_1)$ is s-nuclear for $a_3 > a_2$. This completes the proof of (b).

(ii) $\Rightarrow$ (i) results from the following

LEMMA 1.7. *Let Q be a Montel space and let θ denote the canonical embedding of Q into $J = \prod\{Q(U)\colon U \in \mathscr{U}(Q)\}$. Assume that there exists a continuous linear projection T of J onto $\mathrm{Im}\theta$. Then $Q \cong C^\Lambda$ for some set Λ.*

Proof. Since $J = \mathrm{Im}\,T \oplus \mathrm{Im}(\mathrm{id} - T)$, it follows that T is open. For each finite set $a \subset \mathscr{U}(Q)$ put

$$G(a) = \prod_{U \in a} S(U) \times \prod_{U \notin a} Q(U),$$

where $S(U)$ denotes the unit open ball in $Q(U)$. Since T is open, T induces a continuous linear open map

$$T(a)\colon J/\varrho_{G(a)} = \prod_{U \in a} Q(U) \to \mathrm{Im}\theta/\varrho_{TG(a)}.$$

Since Q is Montel, we infer that $T(a)$ is compact. Hence $\dim \mathrm{Im}\theta/\varrho_{TG(a)} < \infty$. On the other hand, since $\{TG(a)/m\}$ forms a basis of neighbourhoods of zero in $\mathrm{Im}\theta$ and $\mathrm{Im}\theta \cong Q$, we have $\dim Q(U) < \infty$ for $U \in \mathscr{U}(Q)$.

Let $\{u'_\alpha\colon \alpha \in \Lambda\}$ be a vector basis of Q'. Define a continuous linear map γ: $Q \to C^\Lambda$ by $\gamma u = \left(u'_\alpha(u)\right)$. Since $\varrho_U(u) = \sup\{|u'(u)|\colon u' \in U^\circ\}$, where U° is the polar of U and since U° is contained in a finite dimensional subspace of Q', it follows that γ is an embedding. Combining this with the relation $\overline{\mathrm{Im}\,\gamma} = C^\Lambda$, we infer that γ is an isomorphism.

The lemma is proved.

§ 2. The existence of continuous linear projections of $C_\xi^\infty(R(X))$ onto $\mathcal{O}_\xi(X)$. Let X be a paracompact analytic space and let ξ be a holomorphic Banach bundle over X. By $\mathcal{O}_\xi(X)$ we denote the space of holomorphic sections of ξ on X equipped with the compact-open topology and by $C_\xi^\infty(R(X))$ the space of C^∞-sections of ξ on $R(X)$ equipped with the topology of uniform convergence of all derivatives on compact sets in X. Since the restriction map $\mathcal{O}(X) \to \mathcal{O}(R(X))$ is an embedding ([5]), it follows that $\mathcal{O}_\xi(X)$ is contained in $C_\xi^\infty(R(X))$ as a subspace.

We prove the following

THEOREM 2.1. *Let X be a locally irreducible Stein space and let ξ be a holomorphic vector bundle over X. Let $\mathscr{S}$ be a coherent analytic subsheaf of the sheaf $\mathcal{O}_\xi$ of germs of holomorphic sections of ξ on X. Then $H^0(X, \mathscr{S})$ is complemented in $C_\xi^\infty(R(X))$ if and only if X is discrete.*

Since X is Stein and $\mathscr{S}$ is coherent by ([1], Lemma 3.4, p. 38), it suffices to prove the following

LEMMA 2.2. *Let X be a locally irreducible analytic space and let $\mathscr{S}$ and ξ be as in Theorem 2.1. Then $H^0(X, \mathscr{S})$ is complemented in $C_\xi^\infty(R(X))$ if and only if $H^0(X, \mathscr{S}) \cong C^A$ for some set A.*

Proof. Let $P\colon C_\xi^\infty(R(X)) \to H^0(X, \mathscr{S})$ be a continuous linear projection.

(a) Let X_a; $a \in A$ be components of X. Since X is locally connected, X_a are closed-open in X and hence $R(X_a)$ are closed-open in $R(X)$ and $R(X) = \bigcup_{a\in A} R(X_a)$. Thus we have

$$H^0(X, \mathscr{S}) = \prod_a H^0(X_a, \mathscr{S}) \quad \text{and} \quad C_\xi^\infty(R(X)) = \prod_a C_\xi^\infty(R(X_a)).$$

For each a let i_a denote the canonical embedding of $C_\xi^\infty(R(X_a))$ into $\prod_\beta C_\xi^\infty(R(X_\beta))$. Then the formula

$$P_a\sigma = (Pi_a\sigma)|X_a \quad \text{for} \quad \sigma \in C_\xi^\infty(R(X_a))$$

defines a continuous linear projection of $C_\xi^\infty(R(X_a))$ onto $H^0(X_a, \mathscr{S})$.

(b) By (a) we can assume X is connected. Hence by the local irreducibility of X it is easy to see that $H^0(X, \mathscr{S})$ has a continuous norm ϱ. Consider the map $\pi_\varrho P\colon C_\xi^\infty(R(X)) \to H^0(X, \mathscr{S})/\varrho$, where $\pi_\varrho\colon H^0(X, \mathscr{S}) \to H^0(X, \mathscr{S})/\varrho$ is the canonical map. Since $C_\xi^\infty(R(X)) = \lim\{C_\xi^\infty(G)\colon G$ is open in $R(X)\}$, it follows that there exists a relatively compact open set G in $R(X)$ and a linear map $Q\colon \operatorname{Im} R(X, G) \to H^0(X, \mathscr{S})/\varrho$ such that

$$\pi_\varrho P = QR(X, G), \tag{2.1}$$

where $R(X, G)\colon C_\xi^\infty(R(X)) \to C_\xi^\infty(G)$ is the restriction map. Let $\tilde{G}$ be a relatively compact neighbourhood of $\bar{G}$ in $R(X)$ and let $\varphi \in C^\infty(R(X))$, $\varphi|G = 1$, $\operatorname{supp}\varphi \subset \tilde{G}$. Define a continuous linear map $\hat{\varphi}\colon C_\xi^\infty(\tilde{G}) \to C_\xi^\infty(R(X))$ by multiplication by φ. Note that

$$R(X, G)\hat{\varphi} = R(\tilde{G}, G). \tag{2.2}$$

Let $\sigma \in H^0(X, \mathscr{S})$. Then by (2.1) and (2.2) we have

$$\begin{aligned}\pi_\varrho P\hat{\varphi}R(X, \tilde{G})\sigma &= QR(X, G)\hat{\varphi}R(X, \tilde{G})\sigma = QR(\tilde{G}, G)R(X, \tilde{G})\sigma \\ &= QR(X, G)\sigma = \pi_\varrho P\sigma = \pi_\varrho\sigma.\end{aligned}$$

Since π_ϱ is injective, we have

$$P\hat{\varphi}R(X, G)\sigma = \sigma \quad \text{for} \quad \sigma \in H^0(X, \mathscr{S}). \tag{2.3}$$

Let W be a relatively compact neighbourhood of $\bar{\tilde{G}}$ in $R(X)$ and let $\varrho_{\overline{W}}$ be a continuous norm on $H^0(X, \mathscr{S})$ defined by $\overline{W}$. Then the canonical map $q\colon H^0(X, \mathscr{S})/\varrho_{\overline{W}} \to C_\xi^\infty(\tilde{G})$ is continuous. By (2.3) we have

$$P\hat{\varphi}q\pi_{\varrho_{\overline{W}}} = \mathrm{id}. \tag{2.4}$$

Hence the identity map of $H^0(X, \mathscr{S})$ is factorized through a Banach space. Thus by the nuclearity of $H^0(X, \mathscr{S})$ we infer that $\dim H^0(X, \mathscr{S}) < \infty$.

THEOREM 2.3. *Let X be a Stein space and let ξ be a holomorphic Banach bundle over X. Then $\mathcal{O}_\xi(X)$ is complemented in $C_\xi^\infty(R(X))$ if and only if X is discrete.*

Proof. Let $P\colon C_\xi^\infty(R(X)) \to \mathcal{O}_\xi(X)$ be a continuous linear projection and let W be an irreducible branch of X. It suffices to show that $W = \{z_W\}$. Since W is irreducible, $\mathcal{O}_\xi(W)$ has a continuous norm. Applying the proof of Lemma 2.2 to the map $RP\colon C_\xi^\infty(R(X)) \to \mathcal{O}_\xi(X)$, we get a relatively compact open set G in $R(X)$ and a continuous linear map $Q\colon C_\xi^\infty(G) \to \mathcal{O}_\xi(W)$ such that

$$QR(X, G)\sigma = R\sigma \quad \text{for all } \sigma \in \mathcal{O}_\xi(X),$$

where $R\colon \mathcal{O}_\xi(X) \to \mathcal{O}_\xi(W)$ is the restriction map. Note that R is surjective ([13], Theorem 3.9). Let $\tilde{G}$ be a relatively compact neighbourhood of $\bar{G}$ in $R(X)$ and let $q\colon \widehat{\mathcal{O}_\xi(X)/\varrho_{\bar{G}}} \to C_\xi^\infty(G)$ be the restriction map. Then

$$Qq\pi_{\varrho_{\bar{G}}}\sigma = R\sigma \quad \text{for} \quad \sigma \in \mathcal{O}_\xi(X). \tag{2.5}$$

Whence, by the surjectivity of R it follows that $Qq\colon \widehat{\mathcal{O}_\xi(X)/\varrho_{\bar{G}}} \to \mathcal{O}_\xi(W)$ is surjective. Hence by the open mapping theorem it follows that $W = \{z_W\}$ if ξ is finite-dimensional.

Assume now that ξ is infinite-dimensional. By ([13], Theorem 3.13) there exists a $\sigma \in \mathcal{O}_\xi(X)$ such that $\sigma(z) \neq 0$ for every $z \in X$. Considering a 1-dimensional subbundle η of ξ generated by $\sigma(X)$. By ([13], Theorem 3.11) there exists a projection π of ξ onto η. It is easy to see that the formula

$$(\tilde{P}\sigma)z = \pi(P\sigma(z)) \quad \text{for} \quad \sigma \in C_\eta^\infty(R(X))$$

defines a continuous linear projection of $C_\eta^\infty(R(X))$ onto $\mathcal{O}_\eta(X)$. This implies that X is discrete.

The theorem is proved.

Remark 2.4. When X is an open Stein set in C^n and ξ is trivial, Theorem 2.3 has been proved by Poly [10].

§ 3. The splitness of Dolbeaut complexes of holomorphic Banach bundles. Let X be a paracompact complex manifold and let ξ be a holomorphic Banach bundle over X. For each $q \geq 0$ by Ω_ξ^q we denote the sheaf of germs of C^∞-forms of bidegree $(0, q)$ on X with values in ξ. We write $\Omega^q = \Omega_C^q$ where C is a trivial bundle over X with fibre C. By the Dolbeaut lemma and by the nuclearity of spaces $\Omega^q(U)$, where U are open sets in X, it follows that the sequence

$$0 \to \mathcal{O}_\xi \to \Omega_\xi^0 \xrightarrow{\bar{\partial}_\xi^0} \Omega_\xi^1 \xrightarrow{\bar{\partial}_\xi^1} \ldots \tag{3.1}$$

is exact. The complex of global sections of (3.1)

$$D(\xi)\colon 0\to \mathcal{O}_\xi(X)\to \Omega^0_\xi(X)\xrightarrow{\hat{\hat{\partial}}^0_\xi} \Omega^1_\xi(X)\xrightarrow{\hat{\hat{\partial}}^1_\xi}\dots \tag{3.2}$$

is called the *Dolbeaut complex of* ξ *on* X.

We say that the complex $D(\xi)$ *splits at* p if there exists a continuous linear map $\gamma_p\colon \operatorname{Im}\hat{\hat{\partial}}^p_\xi\to \Omega^p_\xi(X)$ such that $\hat{\hat{\partial}}^p_\xi\gamma_p = \mathrm{id}$.

We prove the following

THEOREM 3.1. *Let* ξ *be a holomorphic Banach bundle over a complex manifold* X *which is an increasing union of open Stein sets. Then* $D(\xi)$ *splits at* p *if and only if* $p > 0$.

The proof of Theorem 3.1 is based on the following

LEMMA 3.2 (Corollary 5.1 [7]). *Let*

$$0\to \{G_n, \beta^m_n\}\xrightarrow{\{f_n\}}\{F_n, \omega^m_n\}\xrightarrow{\{g_n\}}\{E_n, \alpha^m_n\}\to 0$$

be a complex of projective systems of Fréchet spaces and let $\operatorname{Ker} f_n = 0$, $\operatorname{Im} f_n = \operatorname{Ker} g_n$, $\operatorname{Im} g_n = E_n$ *and* $\overline{\operatorname{Im}\omega^n_{n+1}} = F_n$ *for* $n\geqslant 1$. *Then the following conditions are equivalent:*

(i) $\varprojlim g_n$ *is surjective.*

(ii) *For each* n_0 *there exists an* $n(n_0)\geqslant n_0$ *such that*

$$\operatorname{Im}\beta^{n_0}_n \text{ is dense in } \operatorname{Im}\beta^{n_0}_{n(n_0)} \quad \text{for} \quad n\geqslant n(n_0). \tag{3.3}$$

(iii) *For each* n_0 *there exists an* $n(n_0)\geqslant n_0$ *such that*

(3.4) *The canonical map* $\beta_{n_0}\colon \varprojlim\{G_n, \beta^m_n\}\to G_{n_0}$ *has a dense image in* $\operatorname{Im}\beta^{n_0}_n$ *for every* $n\geqslant n(n_0)$.

We need the following.

Let L be a quasi-complete locally convex space. Consider the sequence

$$0\to \mathcal{O}_\xi\varepsilon L\to \Omega^0_\xi\varepsilon L\to \Omega^1_\xi\varepsilon L\to\dots, \tag{3.5}$$

which is obtained by ϵ-tensoring the sequence (3.1) with L. By the association of ϵ-product and by ([6], Lemma 1.7) the sequence (3.5) is exact. Since the sheaf $\Omega^q_\xi\varepsilon L$ is fine for every $q\geqslant 0$, we get

$$H^q(X, \mathcal{O}_\xi\varepsilon L) = \operatorname{Ker}\hat{\hat{\partial}}^q_\xi\,\varepsilon L/\operatorname{Im}(\hat{\hat{\partial}}^{q-1}_\xi\varepsilon\,\mathrm{id}) \tag{3.6}$$

for every $q\geqslant 1$.

LEMMA 3.3. *Let* X *and* ξ *be as in Theorem* 3.1 *and let* L *be a quasi-complete locally convex space. Then*

$$H^q(X, \mathcal{O}_\xi\varepsilon L) = 0 \quad \text{for every } q\geqslant 2. \tag{3.7}$$

Proof. Let $\{X_n\}$ be an increasing exhaustion sequence of X consisting of open Stein sets in X. We write $X = \bigcup_{n=1}^{\infty} W_n$, where W_n is a relatively compact open set in X_n such that

$$W_n\subset \hat{W}_n = \mathcal{O}(X_n)\text{-hull}(W_n) \quad \text{for every } n\geqslant 1.$$

Since every holomorphic Banach bundle over a Stein space is complemented in some trivial Banach bundle and since

$$H^q(\hat{W}_n, \mathcal{O}\varepsilon L) = 0 \quad \text{for every } q\geqslant 2 \text{ and for every } n\geqslant 1$$

([2], Theorem B) similarly to [6] we get relation (3.7).

Proof of Theorem 3.1. We can assume that X is connected.

(i) Let $p>0$. Applying Lemma 3.3 and the relation (3.6) to $L = [\operatorname{Im}\hat{\hat{\partial}}^p_\xi]'_c = [\operatorname{Ker}\hat{\hat{\partial}}^{p+1}_\xi]'_c$, we have

$$\operatorname{Im}(\hat{\hat{\partial}}^p_\xi\varepsilon\,\mathrm{id}) = \operatorname{Im}\hat{\hat{\partial}}^p_\xi\varepsilon L. \tag{3.8}$$

Since $\operatorname{Im}\hat{\hat{\partial}}^p_\xi$ is Fréchet, it follows that $[(\operatorname{Im}\hat{\hat{\partial}}^p_\xi)'_c]'_c = \operatorname{Im}\hat{\hat{\partial}}^p_\xi$ and hence the indentity map $\mathrm{id}\colon \operatorname{Im}\hat{\hat{\partial}}^p_\xi\to\operatorname{Im}\hat{\hat{\partial}}^p_\xi$ belongs to $\operatorname{Im}\hat{\hat{\partial}}^p_\xi\varepsilon L$. Hence by (3.8) we infer that $D(\xi)$ splits at p.

(ii) Assume now that $D(\xi)$ splits at 0. Then $\mathcal{O}_\xi(X)$ is complemented in $\Omega^0_\xi(X)$ and $\operatorname{Im}\hat{\hat{\partial}}^0_\xi$ is closed. Note that $\hat{\hat{\partial}}^0_\xi = \varprojlim\hat{\hat{\partial}}^0_{\xi_n}$, where $\xi_n = \xi|X_n$. Since the map $\hat{\hat{\partial}}^0_{\xi_n}\colon \Omega^0_\xi(X_n)\to\operatorname{Ker}\hat{\hat{\partial}}^1_{\xi_n}$ is surjective, we infer that $\operatorname{Im}\hat{\hat{\partial}}^0_\xi$ is dense in $\varprojlim\operatorname{Ker}\hat{\hat{\partial}}^1_{\xi_n}$. Hence the map $\varprojlim\hat{\hat{\partial}}^0_{\xi_n}\colon \varprojlim\Omega^0_\xi(X_n)\to\varprojlim\operatorname{Ker}\hat{\hat{\partial}}^1_{\xi_n}$ is surjective. Thus by Lemma 3.2 it follows that the restriction maps $R_q\colon \mathcal{O}_\xi(X)\to\mathcal{O}_\xi(X_q)$ and $R^p_q\colon \mathcal{O}_\xi(X_q)\to\mathcal{O}_\xi(X_p)$ satisfy condition (3.4). Let $P\colon \Omega^0_\xi(X)\to\mathcal{O}_\xi(X)$ be a continuous linear projection. Since $\mathcal{O}_\xi(X)$ has a continuous norm, as in the proof of Lemma 2.2 the projection P is written in the form

$$P = QR(X, X_n) \quad \text{for some } n, \tag{3.9}$$

where $Q\colon \Omega^0_\xi(X_n)\to\mathcal{O}_\xi(X)$ is a continuous linear map. Let $q\geqslant n$ be such that

$$\overline{\operatorname{Im}R^n_q} = \overline{\operatorname{Im}R_n}. \tag{3.10}$$

Put

$$\tilde{X} = \bigcup\{W;\ W \text{ is a component of } X_q,\ W\cap X_n\neq\emptyset\}.$$

Note that $\tilde{X}$ is a closed-open Stein subset of X_q and

$$\operatorname{Ker}\big(R^n_q|\mathcal{O}(\tilde{X})\big) = 0. \tag{3.11}$$

Consider the continuous linear map $R(X, \tilde{X})QR(\tilde{X}, X_n)\colon \Omega_\xi^0(\tilde{X}) \to \mathcal{O}_\xi(\tilde{X})$. Let $\sigma \in \mathcal{O}_\xi(\tilde{X})$. By (3.10) and since $\tilde{X}$ is closed-open in X_q, it follows that there exists a sequence $\{\sigma_k\} \subset \mathcal{O}_\xi(X)$ such that $\lim_k R_n \sigma_k = R(\tilde{X}, X_n)\sigma$. Hence by (3.9) we have

$$\begin{aligned}\big(R(X, \tilde{X})QR(\tilde{X}, X_n)\sigma\big)|X_n &= \big(\lim_k R(X, \tilde{X})QR_n\sigma_k\big)|X_n \\ &= \big(\lim_k R(X, \tilde{X})P\sigma_k\big)|X_n \\ &= \big(\lim_k R(X, \tilde{X})\sigma_k\big)|X_n = \sigma|X_n.\end{aligned}$$

Hence by (3.11) we have $R(X, \tilde{X})QR(\tilde{X}, X_n)\sigma = \sigma$ for $\sigma \in \mathcal{O}_\xi(\tilde{X})$. Thus $\mathcal{O}_\xi(\tilde{X})$ is complemented in $\Omega_\xi^0(\tilde{X})$. This contradics Theorem 2.3. Hence $D(\xi)$ does not split at 0.

The theorem is proved.

Remark 3.4. When X is Stein and ξ is finite-dimensional, Theorem 3.1 has been established by Palamodov (Proposition 5.1 [8]).

Let X be a paracompact analytic space and let ξ be a holomorphic Banach bundle over X. The groups $H^q(X, \mathcal{O}_\xi) = \varinjlim H^q(\mathcal{U}, \mathcal{O}_\xi)$ are endowed with the inductive topology (where $\mathcal{U}$ is an open cover of X and $H^q(\mathcal{U}, \mathcal{O}_\xi) \underset{\text{top}}{=} \operatorname{Ker}\delta_\xi^q/\operatorname{Im}\delta_\xi^{q-1}$ if $\delta_\xi^q = \delta_\xi^q(\mathcal{U})\colon C^q(\mathcal{U}, \mathcal{O}_\xi) \to C^{q+1}(\mathcal{U}, \mathcal{O}_\xi)$ are the coboundary maps and the spaces $C^q(\mathcal{U}, \mathcal{O}_\xi)$ are equipped with the compact-open topology. In ([11]) Silva has proved that if the space X is an increasing union of open Stein sets and if $H^1(X, \mathcal{O})$ is Hausdorff, then X is Stein. The following theorem is an extension of this result:

THEOREM 3.5. *Let X be an analytic space which is an increasing union of open Stein sets. Let $H^1(X, \mathcal{O}_\xi)$ be Hausdorff for some non-zero holomorphic Banach bundle ξ over X. Then X is Stein.*

Proof. By the result of Silva we have to show that $H^1(X, \mathcal{O}) = 0$. Let $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}$ is an increasing sequence of open Stein sets. Consider the cover $\mathcal{U} = \{X_n\}_{n=1}^{\infty}$. Since $H^p(X_n, \mathcal{O}) = 0$ for $p, n \geqslant 1$, we have

$$H^1(X, \mathcal{O}) = \operatorname{Ker}\delta^1/\operatorname{Im}\delta^0 \tag{3.12}$$

and

$$\operatorname{Ker}\delta^1(\mathcal{U}_n) = \operatorname{Im}\delta^0(\mathcal{U}_n) \quad \text{for } n \geqslant 1, \tag{3.13}$$

where $\mathcal{U}_n = \{X_1, \ldots, X_n\}$. Note that $\delta^0(\mathcal{U}) = \varprojlim \delta^0(\mathcal{U}_n)$. Thus by (3.11) and (3.12) and by Lemma 3.2 it suffices to check that the restriction maps $R_q^p\colon \mathcal{O}(X_q) \to \mathcal{O}(X_p)$ satisfy condition (3.3).

Since $H^1(X, \mathcal{O}_\xi)$ is Hausdorff and since $\mathcal{U}$ is a Leray cover for $\mathcal{O}_\xi$, it follows that the canonical map $H^1(\mathcal{U}, \mathcal{O}_\xi) \to H^1(X, \mathcal{O}_\xi)$ is bijective. Hence $H^1(\mathcal{U}, \mathcal{O}_\xi)$ is Hausdorff. Thus $\operatorname{Im}\delta_\xi^0(\mathcal{U})$ is closed. Combining this with the surjectivity of the maps $\delta_{\xi_n}^0\colon C^0(\mathcal{U}_n, \mathcal{O}_\xi) \to \operatorname{Ker}\delta_{\xi_n}^1$, where $\xi_n = \xi|X_n$, we infer that the map $\varprojlim \delta_{\xi_n}^0\colon \varprojlim C^0(\mathcal{U}_n, \mathcal{O}_\xi) \to \varprojlim \operatorname{Ker}\delta_{\xi_n}^1$ is surjective. Hence the restriction maps $R_{\xi q}^p\colon \mathcal{O}_\xi(X_q) \to \mathcal{O}_\xi(X_p)$ satify condition (3.3).

(a) We first assume that ξ is infinite-dimensional. Given n_0. Take $q(n_0) \geqslant n_0$ such that (3.3) holds for $\beta_q^p = R_{\xi q}^p$. Let $q \geqslant q(n_0)$. By [13] we find an $\sigma \in \mathcal{O}_\xi(X_q)$ such that $\sigma(z) \neq 0$ for $z \in X_q$. Let η be a subbundle of $\xi|X_q$ spanned by $\sigma(X_q)$ and let $\theta\colon C \to \eta$ be a canonical isomorphism

$$\theta(z, \lambda) = \lambda\sigma(z) \quad \text{for} \quad (z, \lambda) \in X_q \times C.$$

By (3.3) and since η is complemented in $\xi|_{X_q}$ it follows that for each $\alpha \in \mathcal{O}(X_{q(n_0)})$ there exists a sequence $\{\sigma_n\} \subset \mathcal{O}_\eta(X_q)$ such that $\sigma_n|X_{n_0} \to \theta\alpha|X_{n_0}$. Hence $\theta^{-1}\sigma_n|X_{n_0} \to \alpha|X_{n_0}$ and thus the case where ξ is infinite-dimensional is proved.

(b) Assume now that ξ is finite-dimensional. Consider the infinite-dimensional holomorphic Banach bundle $\xi\mathscr{E}B$, where B is some infinite-dimensional Banach space. Note that

$$\operatorname{Ker}\big(\delta_\xi^1 \mathscr{E}\operatorname{id}\colon C^0(\mathcal{U}, \mathcal{O}_{\xi\mathscr{E}B}) \to C^1(\mathcal{U}, \mathcal{O}_{\xi\mathscr{E}B})\big) = \operatorname{Ker}\delta_\xi^1\mathscr{E}B.$$

Hence by the nuclearity of the space $C^0(\mathcal{U}, \mathcal{O}_\xi)$ it follows that $H^1(\mathcal{U}, \mathcal{O}_{\xi\mathscr{E}B}) = 0$. Combining this with (a), we infer that X is Stein.

The theorem is proved.

Remark 3.6. Fornaess has constructed a complex manifold which is an increasing union of open Stein sets such that $\dim\mathcal{O}(F) = 1$ ([3]). From Theorems 3.1 and 3.5 we obtain an extra property of F.

PROPOSITION 3.7. *Let ξ be a non-zero holomorphic Banach bundle over F. Then*

(i) *$D(\xi)$ splits only at $p > 0$.*

(ii) *$H^1(X, \mathcal{O}_\xi)$ is uncountable-dimensional and is not Hausdorff.*

Proof. (i) follows from Theorem 3.1. Theorem 3.6 implies that $H^1(X, \mathcal{O}_\xi)$ is not Hausdorff. The non-countability follows from (Theorem 1.4, [12]).

Remark 3.8. The necessity of the condition of Theorem 3.1 follows also from Theorems 2.3 and 3.5.

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On functions with bounded mixed variance

by

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Abstract. The concept of bounded mixed variance (BMV), which extends the notion of bounded mean oscillation (BMO), is discussed. The main result is that spaces of functions with bounded mixed variance are duals of certain "atomic H^1 spaces".

Introduction. In this note, we discuss a measure-theoretic concept which extends the notion of bounded mean oscillation described by John and Nirenberg in [11]. This concept, which we call *bounded mixed variance* generalizes the $\mathscr{L}^{p,\lambda}$ spaces of G. Stampacchia (see [1] and [13]), the BMO(ϱ) space of [10], and the duals of weighted Hardy spaces H^1 in [14], [15], [9], [12] and [8]. To emphasize the wide applicability of our remarks, we shall place them in the setting of a space X of homogeneous type, [3] and [4]. However, our main motivation and emphasis comes from the case $X = \partial D$, where D is a simply-connected bounded domain in $\boldsymbol{R}^n$ which is a Lipschitz or C^1 domain (see Remark 1.7 below).

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§ 1. Atomic spaces and preliminaries. Let $X = (X, d)$ be a space of homogeneous type (cf. [4], § 2) equipped with a pair of regular Borel measures μ and ν, mutually absolutely continuous and satisfying the doubling condition. That is, if $B = B_r(x) = \{y \in X\colon d(x, y) < r\}$ and $B^* = B_{2r}(x)$, then $\mu(B^*) \leqslant A\mu(B)$ and $\nu(B^*) \leqslant A\nu(B)$ for some constant A independent of the ball B.

Definition 1.0. Let $1 < q \leqslant \infty$. A function $a(x)$ is a $(1, q)$ *atom* of type (μ, ν) if its support is contained in some ball B and

$$\text{(i)} \qquad \Big\{\mu(B)^{-1} \int_B |a|^q \, d\mu\Big\}^{1/q} \leqslant \mu(B)^{-1},$$

$$\text{(ii)} \qquad \int a \, d\nu = 0.$$

Naturally, (i) is intended for the smallest ball B containing the support of $a(x)$ and the left-side of (i) equals the norm of a in $L^\infty(X, d\mu)$ if $q = \infty$. When $\mu(X) < \infty$, the constant $\nu(X)^{-1}$ is also considered to be an atom.