[5] R. Narasimhan, *Introduction to the Theory of Analytic Spaces*, Lecture Notes in Math. 25, Springer-Verlag, Berlin–Heidelberg–New York 1966.
[6] Nguyen van Khue, *On the cohomology groups of $\mathcal{S}\in L$*, Studia Math. 72 (1982), 183–197.
[7] P. V. Palamodov, *Homological methods in theory of locally convex spaces*, Uspekhi Mat. Nauk 1 (1971), 3–64.
[8] — *On Stein manifolds the Dolbeaut complexes split at positive dimensions*, Mat. Sb. 88 (1972), 287–315.
[9] A. Pietsch, *Nuclear Locally Convex Spaces*, Akademie Verlag, Berlin 1972.
[10] J. B. Poly, *Sur opérateurs differentiels et les morphismes directes*, C. R. Acad. Sci. Paris 270 (10) (1970), 647–649.
[11] A. Silva, *Rungescher satz and a condition for Steiness for the limit of an increasing sequence of Stein spaces*, Ann. Inst. Fourier 28 (1978), 187–200.
[12] Y. T. Siu, *Non-countable dimension of cohomology groups of analytic sheaves and domains of holomorphy*, Math. Z. 102 (1967), 17–29.
[13] M. G. Zajdenberg, S. G. Krejn, P. A. Kusment, A. A. Pankov, *Banach bundles and linear operators*, Uspekhi Mat. Nauk 5 (1975), 101–157.

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On functions with bounded mixed variance

by

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Abstract. The concept of bounded mixed variance (BMV), which extends the notion of bounded mean oscillation (BMO), is discussed. The main result is that spaces of functions with bounded mixed variance are duals of certain "atomic H^1 spaces".

Introduction. In this note, we discuss a measure-theoretic concept which extends the notion of bounded mean oscillation described by John and Nirenberg in [11]. This concept, which we call *bounded mixed variance* generalizes the $\mathcal{L}^{p,\lambda}$ spaces of G. Stampacchia (see [1] and [13]), the BMO(ϱ) space of [10], and the duals of weighted Hardy spaces H^1 in [14], [15], [9], [12] and [8]. To emphasize the wide applicability of our remarks, we shall place them in the setting of a space X of homogeneous type, [3] and [4]. However, our main motivation and emphasis comes from the case $X = \partial D$, where D is a simply-connected bounded domain in $\boldsymbol{R}^n$ which is a Lipschitz or C^1 domain (see Remark 1.7 below).

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§ 1. Atomic spaces and preliminaries. Let $X = (X, d)$ be a space of homogeneous type (cf. [4], § 2) equipped with a pair of regular Borel measures μ and ν, mutually absolutely continuous and satisfying the doubling condition. That is, if $B = B_r(x) = \{y \in X\colon d(x, y) < r\}$ and $B^* = B_{2r}(x)$, then $\mu(B^*) \leqslant A\mu(B)$ and $\nu(B^*) \leqslant A\nu(B)$ for some constant A independent of the ball B.

Definition 1.0. Let $1 < q \leqslant \infty$. A function $a(x)$ is a $(1, q)$ *atom* of type (μ, ν) if its support is contained in some ball B and

(i) $$\Big\{\mu(B)^{-1}\int_B |a|^q\,d\mu\Big\}^{1/q} \leqslant \mu(B)^{-1},$$

(ii) $$\int a\,d\nu = 0.$$

Naturally, (i) is intended for the smallest ball B containing the support of $a(x)$ and the left-side of (i) equals the norm of a in $L^\infty(X, d\mu)$ if $q = \infty$. When $\mu(X) < \infty$, the constant $\nu(X)^{-1}$ is also considered to be an atom.

More generally, if $0 < p < q$ and $1 \leqslant q \leqslant \infty$, replacing the right-side of (i) by $\mu(B)^{-1/p}$, we have the definition of (p, q) atoms of type (μ, ν). In the case $\nu = \mu$, we obtain the ordinary (p, q) atoms of Coifman–Weiss.

DEFINITION 1.1. If $1 < q \leqslant \infty$, we denote be $h^{1,q}_\mu(d\nu)$ the Banach space of bounded linear functionals f on $\mathrm{Lip}(\alpha)$ having an *atomic decomposition*

$$f = \sum_{j=0}^{\infty} \lambda_j a_j, \quad \text{with} \quad \sum |\lambda_j| < \infty, \tag{1.1}$$

where the a_j are $(1, q)$ atoms of type (μ, ν). Each f has norm

$$\|f\|_{1,q} = \inf\Big\{\sum |\lambda_j|\Big\}$$

with infimum taken over all representations of the form (1.1).

Since μ and ν are assumed to be mutually absolutely continuous, the density

$$d\nu/d\mu = \omega(x), \quad 0 < \omega(x) < \infty \ \text{(a.e.)}, \tag{1.2}$$

satisfies

$$\omega \in L^1(d\mu) \quad \text{and} \quad \omega^{-1} \in L^1(d\nu). \tag{1.2'}$$

In general, denoting by ω^s the Borel measures given by

$$d\omega^s = \omega(x)^s d\mu, \quad s \text{ real}, \tag{1.3}$$

we have $\omega^0 = \mu$ and $\omega^1 = \nu$. Recall that for any $1 < p < \infty$, a positive density h belongs to $A_p(d\mu)$—respectively, $h \in B_p(d\nu)$—if there exist positive constants A_p and B_p (both $\geqslant 1$) such that

$$\Big\{\mu(B)^{-1} \int_B h\, d\mu\Big\}\Big\{\mu(B)^{-1} \int_B (1/h)^{1/(p-1)} d\mu\Big\}^{p-1} \leqslant A_p \tag{A_p}$$

—respectively,

$$\Big\{\nu(B)^{-1} \int_B h^p d\nu\Big\}^{1/p} \leqslant B_p \nu(B)^{-1} \int_B h\, d\nu \tag{B_p}$$

—uniformly in B. As p increases, the classes $B_p(d\nu)$ decrease while the classes $A_p(d\mu)$ increase. With $\omega = d\nu/d\mu$, it is easy to see that

$$\omega \in A_p(d\mu) \quad \text{if and only if} \quad \omega^{-1} \in B_q(d\nu),\ 1/p + 1/q = 1. \tag{1.4}$$

For convenience, we shall use the abbreviations

$$h^r_{\mu(B)} = \Big\{\mu(B)^{-1} \int_B h^r d\mu\Big\}^{1/r}, \quad h_{\mu(B)} = h^1_{\mu(B)} \tag{1.5}$$

and the notations

$$\begin{aligned} A_\infty &= \bigcup_{p>1} A_p, \quad & A_* &= \bigcap_{p>1} A_p \\ B_\infty &= \bigcup_{q>1} B_q, \quad & B_* &= \bigcap_{q>1} B_q. \end{aligned} \tag{1.6}$$

It is known that $A_\infty = B_\infty$ (see [2], where a direct characterization of A_∞ is also given). A simple relationship between A_* and B_* is as follows.

LEMMA 1.2. *Let $h \in A_2(d\mu)$. Then,*

$$h^{-1} \in A_*(d\mu) \quad \text{if and only if} \quad h \in B_*(d\mu). \tag{1.7}$$

Proof. Note that, by (1.5), $h \in A_2(d\mu)$ if and only if

$$1 \leqslant h_{\mu(B)} h^{-1}_{\mu(B)} \leqslant A_2,$$

that is to say,

$$\{h_{\mu(B)}\}^{-1} \approx h^{-1}_{\mu(B)}, \quad \text{uniformly in } B.$$

Let $1 < p < 2$ and $r = 1/(p-1)$. Then, $h^{-1} \in A_p(d\mu)$ if and only if

$$h^{-1}_{\mu(B)} h^r_{\mu(B)} \leqslant A_p \Leftrightarrow h^r_{\mu(B)} \leqslant B_p h_{\mu(B)}$$

since $h \in A_2(d\mu)$; that is, if and only if $h \in B_r(d\mu)$.

COROLLARY 1.3. *For $\omega = d\nu/d\mu$, the following are equivalent;*

(i) $\omega \in A_*(d\mu) \cap B_*(d\mu)$,

(ii) $\omega^{-1} \in A_*(d\nu) \cap B_*(d\nu)$,

(iii) $\omega^{-1} \in A_*(d\mu) \cap B_*(d\mu)$.

The proof follows directly from (1.4) and (1.7). Another useful result is as follows.

LEMMA 1.4. *If $d\nu/d\mu = \omega \in A_*(d\mu) \cap B_*(d\mu)$ and $1 < p < \infty$, then*

$$h_{\nu(B)} \approx h_{\mu(B)} \quad \textit{for each } h \in A_p(d\mu), \tag{1.8}$$

$$A_p(d\nu) = A_p(d\mu), \tag{1.9}$$

and

$$B_p(d\nu) = B_p(d\mu). \tag{1.9'}$$

Proof. If $h \in A_p(d\mu)$, then $h \in B_r(d\mu)$ for some $1 < r < \infty$, since $A_\infty = B_\infty$. With $r' = r/(r-1)$, Hölder's inequality yields

$$(h\omega)_{\mu(B)} \leqslant h^r_{\mu(B)} \omega^{r'}_{\mu(B)} \leqslant C h_{\mu(B)} \omega_{\mu(B)}$$

since $h \in B_r(d\mu)$ and $\omega \in B_*(d\mu)$. Hence, dividing by $\omega_{\mu(B)} = \nu(B)/\mu(B)$, we see that

$$h_{\nu(B)} \leqslant C h_{\mu(B)}. \tag{i}$$

Applying the same argument to the function $g = h^{-1/(p-1)}$ which is in $A_{p'}(d\mu)$, we obtain the estimate $g_{\nu(B)} \leqslant C g_{\mu(B)}$. Raising it to the $(p-1)^{\text{st}}$ power and combining it with (i) it follows that $A_p(d\mu) \subset A_p(d\nu)$.

Conversely, if $h \in A_p(d\nu)$ then $h \in B_r(d\nu)$ for some $r \in (1, \infty)$. Therefore, using also the assumption that $\omega \in A_*(d\mu)$, we obtain

$$\begin{aligned} h_{\mu(B)} &= \mu(B)^{-1} \int_B h\, \omega^{1/r} \omega^{-1/r}\, d\mu \\ &\leqslant \Big\{\mu(B)^{-1} \int_B h^r d\nu\Big\}^{1/r} \Big\{\mu(B)^{-1} \int_B (1/\omega)^{r'/r} d\mu\Big\}^{1/r'} \\ &= [\nu(B)/\mu(B)]^{1/r}\, h^r_{\nu(B)}\, [(1/\omega)^{1/(r-1)}_{\mu(B)}]^{1/r} \\ &\leqslant C[\nu(B)/\mu(B)]^{1/r}\, h_{\nu(B)}\, [\omega_{\mu(B)}]^{-1/r} = C h_{\nu(B)}. \end{aligned}$$

In other words, $h_{\mu(B)} \leqslant C h_{\nu(B)}$ uniformly in B. Hence, by (i) above, formula (1.8) follows. The rest of the proof of (1.9) is also clear. Finally, (1.8) and (1.9) yield (1.9′) since

$$h \in B_p(d\mu) \Leftrightarrow h^p_{\mu(B)} \approx h_{\mu(B)} \text{ uniformly in } B.$$

COROLLARY 1.5. *Let* $\omega = d\nu/d\mu \in A_* \cap B_*$. *Then, for every positive density* h *and any real* s, $h_{\omega^s(B)} \approx h_{\omega^{s-1}(B)}$ *uniformly in* B; *hence*

$$(1.10) \qquad A_p(d\omega^s) = A_p(d\omega^{s-1}) \quad \text{and} \quad B_p(d\omega^s) = B_p(d\omega^{s-1})$$

for every $1 < p < \infty$.

The conclusion comes from iterating Lemma 1.4.

PROPOSITION 1.6. *If* $\omega = d\nu/d\mu \in A_* \cap B_*$ *then* ω *multiplies* A_2; *that is, for every real* s,

$$(1.11) \qquad h \in A_2(d\omega^s) \quad \text{if and only if} \quad (\omega h) \in A_2(d\omega^s).$$

Proof. By (1.10), it suffices to consider $s = 0$. Then, $h \in A_2(d\mu)$ implies $h \in B_r(d\mu)$ for some $r \in (1, \infty)$. Hence, for all balls B,

$$\begin{aligned} (\omega h)_{\nu(B)} (h^{-1}\omega^{-1})_{\nu(B)} &= \mu(B)^2 \nu(B)^{-2} \Big\{\frac{1}{\mu(B)} \int_B h\omega^2 d\mu\Big\} (h^{-1})_{\mu(B)} \\ &\leqslant \mu(B)^2 \nu(B)^{-2} h^r_{\mu(B)} [\omega^{2r'}_{\mu(B)}]^2 (h^{-1})_{\mu(B)} \\ &\leqslant C h_{\mu(B)} (h^{-1})_{\mu(B)}, \quad \text{since} \quad \omega \in B_*. \end{aligned}$$

Hence, $h \in A_2(d\mu)$ implies that $(\omega h) \in A_2(d\nu) = A_2(d\mu)$ by (1.9). By Corollary 1.3, $\omega^{-1} \in A_* \cap B_*$ and so the converse also holds.

Remark 1.7. Let $X = \partial D$, where $D \subset \boldsymbol{R}^n$ is a bounded Lipschitz domain (see [5], [6], etc.) starshaped about a fixed point P_0 say, and let d = Euclidean distance. If $d\nu = d\omega$ is the harmonic measure for D (evaluated at P_0), then $(X, \nu, d) = (\partial D, \omega, d)$ is a space of homogeneous type with $\nu(X) = \omega(\partial D) = 1$. If $d\sigma$ = surface measure on ∂D, then $\mu = \sigma$ and $\nu = \omega$ are mutually absolutely continuous. Moreover, the density

$$d\nu/d\mu = d\omega/d\sigma = \omega(Q) = K(P_0, Q), \qquad Q \in \partial D = X,$$

called the *Poisson kernel* for D (evaluated at P_0), satisfies $\omega \in B_2(d\sigma)$, see [5]. Thus equivalently,

$$(1.12) \qquad \omega \in A_2(d\omega) \quad \text{if} \quad D \text{ is a Lipschitz domain.}$$

In the particular case when D is a C^1-domain (e.g. [6], [7], etc.) the main result of § 1 in [8] shows that ω and ω^{-1} are in $A_*(d\sigma)$. Consequently,

$$(1.13) \qquad \omega, \omega^{-1} \in A_*(d\sigma) \cap B_*(d\sigma) \quad \text{if} \quad D \text{ is a } C^1\text{-domain.}$$

§ 2. Bounded mixed variance and duality. From now on, let us suppose that $\mu(X) < \infty$. (At any rate, the case $\mu(X) = +\infty$ is slightly easier.) Accordingly, the constant function $\nu(X)^{-1}$ is considered to be a $(1, q)$ atom of type (μ, ν), while the constant $\mu(X)^{-1}$ is considered to be a $(1, q)$ atom of type (ν, μ).

DEFINITION 2.1. A function $g \in L^1(X, d\nu)$ has *bounded mixed variance* of type (μ, ν) if

$$(2.0) \qquad \mathscr{V}_{\mu,\nu}(g) = \sup\Big\{\mu(B)^{-1} \int_B |g - g_{\nu(B)}|^2 \omega\, d\nu\Big\}^{1/2} < \infty$$

with supremum taken over all balls $B \subset X$, $\omega = d\nu/d\mu$.

Note that, adapting to our abstract setting a result in [14], condition (2.0) is equivalent to

$$(2.1) \qquad \|g\|_{*,\mu,\nu} = \sup\Big\{\mu(B)^{-1} \int_B |g - g_{\nu(B)}|\, d\nu\Big\} < \infty$$

whenever $\omega \in A_2(d\nu)$. The class of all such functions g will be denoted by $\mathrm{BMV}_\mu(d\nu) = \mathrm{BMV}_\mu(X, d\nu)$. Equipped with the norm

$$(2.2) \qquad \mathscr{N}_{\mu,\nu}(g) = \Big|\nu(X)^{-1} \int_X g\, d\nu\Big| + \mathscr{V}_{\mu,\nu}(g)$$

the space $\mathrm{BMV}_\mu(d\nu) = \{g \in L^1(X, d\nu) : \mathscr{V}_{\mu,\nu}(g) < \infty\}$ is complete.

Our next remarks, including Theorem 2.3 below, extend the corresponding "unweighted" results of Coifman–Weiss (see [4], pages 631–633).

LEMMA 2.2. *If* $\omega = d\nu/d\mu \in A_2(d\nu)$, *then* $|g| \in \mathrm{BMV}_\mu(d\nu)$ *for all* $g \in \mathrm{BMV}_\mu(d\nu)$. *Hence*, $\mathrm{BMV}_\mu(d\nu)$ *forms a lattice*.

Proof. As usual, the proof amounts to showing that, if for each B and $g \in L^1(d\nu)$ there exists a constant g_B for which

$$(2.0') \qquad \Big\{\mu(B)^{-1} \int_B |g - g_B|^2 \omega\, d\nu\Big\}^{1/2} \leqslant K$$

with constant K independent of B, then $g \in \mathrm{BMV}_\mu(d\nu)$ with variance $\mathscr{V}_{\mu,\nu}(g) \leqslant CK$. Since $\omega \in A_2(d\nu)$ is equivalent to $\omega \in B_2(d\mu)$, it is easy to see that $\mathscr{V}_{\mu,\nu}(g) \leqslant (1+B_2)K$.

THEOREM 2.3. *If $d\nu/d\mu = \omega \in A_2(d\nu)$, then $\mathrm{BMV}_\mu(d\nu)$ is the dual of the atomic space $h_\mu^{1,2}(d\nu)$ for some pairing $\langle f, g\rangle_\omega$ such that, for all $g \in \mathrm{BMV}_\mu(d\nu)$,*

$$\langle f, g\rangle_\omega = \int_X fg\,d\nu, \qquad \text{for all atoms } f \in h_\mu^{1,2}(d\nu). \tag{2.3}$$

Proof. By Lemma 2.2 and standard arguments (see [4]), in order to verify the inclusion $\mathrm{BMV}_\mu(d\nu) \subset [h_\mu^{1,2}(d\nu)]^*$, it suffices to consider the case when f is an atom. For the constant atom $f = \nu(X)^{-1}$, it is clear that $|\langle f, g\rangle_\omega| \leqslant \mathscr{N}_{\mu,\nu}(g)$. Otherwise, let $f = a$ be a (1,2) atom of type (μ, ν) supported by B. Since $\int a\,d\nu = 0$ then, for any $g \in \mathrm{BMV}_\mu(d\nu)$,

$$\begin{aligned}|\langle a, g\rangle_\omega| &= \Big|\int a[g-g_{\nu(B)}]\,d\nu\Big| = \Big|\int_B a\omega^{-1/2}[g-g_{\nu(B)}]\,\omega^{1/2}\,d\nu\Big| \\ &\leqslant \Big\{\int_B a^2 d\mu\Big\}^{1/2}\Big\{\int_B |g-g_{\nu(B)}|^2\omega\,d\nu\Big\}^{1/2} \\ &\leqslant \Big\{\mu(B)^{-1}\int_B |g-g_{\nu(B)}|^2\omega\,d\nu\Big\}^{1/2} \leqslant \mathscr{V}_{\mu,\nu}(g).\end{aligned}$$

Conversely, if $\Lambda \in [h_\mu^{1,2}(d\nu)]^*$ with norm $\|\Lambda\|$, we fix a ball $B \subset X$ and denote by χ_B its characteristic function. Consider the linear subspace

$$\Omega^2(B, d\nu) = \Big\{\gamma \in L^2(B, d\nu) \colon \int_B \gamma\omega^{1/2}\,d\nu = 0\Big\}$$

which is the orthogonal complement of $\omega^{1/2}$ in $L^2(B, d\nu)$. In fact, denoting by $\|\ \|_2$ the norm in $L^2(B, d\nu)$, we have

$$\|\omega^{1/2}\|_2 = \mu(B)^{1/2}\Big\{\frac{1}{\mu(B)}\int_B \omega^2\,d\mu\Big\}^{1/2} \leqslant B_2\nu(B)\mu(B)^{-1/2} < \infty$$

since $\omega \in B_2(d\mu)$. For each $\gamma \in \Omega^2(B, d\nu)$, we observe that the function

$$a = \gamma\chi_B\,\omega^{1/2}\|\gamma\|_2^{-1}\mu(B)^{-1/2}$$

is a (1,2) atom of type (μ, ν) supported in B. Thus, $\gamma\chi_B$ belongs to $h_\mu^{1,2}(d\nu)$ with norm $\leqslant \mu(B)^{1/2}\|\gamma\|_2$ and so $\Lambda(\gamma\chi_B)$ is defined and satisfies $|\Lambda(\gamma\chi_B)| \leqslant \|\Lambda\|\mu(B)^{1/2}\|\gamma\|_2$. Extending Λ by Hahn–Banach to all of $L^2(B, d\nu)$, we deduce the existence of some $g \in L^2(B, d\nu)$ such that

$$\Lambda(\gamma) = \int_B \gamma g\,d\nu, \qquad \text{for all } \gamma \in L^2(B, d\nu). \tag{2.4}$$

Moreover, a familiar argument (see [4], page 633) yields then a function $g \in L^2(X, d\nu)$ such that (2.4) holds for every ball $B \subset X$.

It remains to show that $g \in \mathrm{BMV}_\mu(d\nu)$ with norm dominated by $\|\Lambda\|$. Note first that

$$\Big|\nu(X)^{-1}\int_X g\,d\nu\Big| = |\Lambda(b)| \leqslant \|\Lambda\| \tag{i}$$

if $b = \nu(X)^{-1}$ is the constant atom. Secondly, we claim that

$$\mathscr{V}_{\mu,\nu}(g) \leqslant (1+B_2)\|\Lambda\|. \tag{ii}$$

By (2.0), observe that

$$\mathscr{V}_{\mu,\nu}(g) = \sup_\varphi\Big|\int_B \varphi[g-g_{\nu(B)}]\,\omega\,d\nu/\mu(B)\Big|$$

for all φ supported in balls B and satisfying

$$\int_B \varphi^2\omega\,d\nu/\mu(B) \leqslant 1. \tag{iii}$$

Thus, we must show that for all $f = \varphi\omega$, with φ as above,

$$\sup_f\Big|\int_B f[g-g_{\nu(B)}]\,d\nu/\mu(B)\Big| \leqslant (1+B_2)\|\Lambda\|. \tag{2.5}$$

But, for any such f, Schwarz's inequality and (iii) imply that

$$|f_{\nu(B)}| = \frac{\mu(B)}{\nu(B)}\Big|\frac{1}{\mu(B)}\int_B (\varphi\omega)\,\omega\,d\mu\Big| \leqslant B_2$$

since $\omega \in B_2(d\mu)$. Hence, using again (iii) we see that the function

$$b = (1+B_2)^{-1}[f-f_{\nu(B)}]\mu(B)^{-1}\chi_B$$

is a (1, 2) atom of type (μ, ν) supported in B, for every $f = \varphi\omega$ as above. Consequently, applying Λ to any such atom b, it follows from (2.4) that

$$(1+B_2)|\Lambda(b)| = \Big|\int_B [f-f_{\nu(B)}]g\,d\nu/\mu(B)\Big| = \Big|\int_B f[g-g_{\nu(B)}]\,d\nu/\mu(B)\Big|$$

since also $[g-g_{\nu(B)}]$ has $d\nu$-integral zero over B. This proves (2.5) and hence (ii) above. Therefore, $g \in \mathrm{BMV}_\mu(d\nu)$ with norm $\leqslant (2+B_2)\|\Lambda\|$.

Remark 2.4. Since $\omega \in B_2(d\mu)$ implies the estimate

$$\mathscr{V}_{\mu,\nu}(g) \leqslant \sup_B\Big[\Big\{\mu(B)^{-1}\int_B |g\omega|^2 d\mu\Big\}^{1/2} + B_2|g\omega|_{\mu(B)}\Big],$$

we have in this case that

$$(g\omega) \in L^\infty(d\mu) \quad \text{implies} \quad g \in \mathrm{BMV}_\mu(d\nu) \tag{2.6}$$

with $\mathscr{V}_{\mu,\nu}(g) \leqslant (1+B_2)\|g\omega\|_{\infty,\mu}$. In particular,

$$\omega \in B_2(d\mu) \quad \text{implies} \quad \omega^{-1} \in \mathrm{BMV}_\mu(d\nu) \tag{2.7}$$

with mixed variance $\mathscr{V}_{\mu,\nu}(d\mu/d\nu) \leqslant (1+B_2)$. In view of this and of Theorem 2.3, it seems more natural to include in the definition of $\mathrm{BMV}_\mu(d\nu)$ the condition that $\omega = d\nu/d\mu \in B_2(d\mu)$. Thus, we shall do so from now on.

Assuming merely that $\omega = d\nu/d\mu \in A_\infty(d\mu)$, it follows that $\mathrm{BMV}_\mu(d\mu) = \mathrm{BMV}_\nu(d\nu) = \mathrm{BMO}(X)$ is the space of all functions with bounded mean oscillation on X. By a special case of Theorem 2.3, we then have

$$[h_\nu^{1,2}(d\nu)]^* = \mathrm{BMO}(X) = [h_\mu^{1,2}(d\mu)]^*. \tag{2.8}$$

Moreover, by Theorem (2.25) of [4], we have the formula

$$f \in h_\nu^1(d\nu) \quad \text{if and only if} \quad (f\omega) \in h_\mu^1(d\mu) \tag{2.9}$$

where $h^1 = h^{1,\infty} = h^{1,q}$ for all $1 < q < \infty$ ([4], Theorem A).

Despite formula (2.8), we cannot conclude from these facts that any density $\omega \in A_\infty(d\mu)$ multiplies BMO. This would be false even for $\omega \in A_*(d\mu)$ since such densities can still be unbounded. (Compare with [17], Theorem 1.2.)

§ 3. Bounded p-variance and other remarks. Our earlier definition of bounded mixed variance referred to an *ordered pair* of measures (μ, ν) such that $\omega = d\nu/d\mu \in B_2(d\mu)$. But, in certain applications (see [8], [16]), one must also consider the pair (ν, μ). Since $A_\infty = B_\infty$, we only have that

$$\omega^{-1} = d\mu/d\nu \in B_p(d\nu), \tag{3.1}$$

for *some* $1 < p < \infty$ depending on ω. Therefore, the interchange of measures μ and ν in formula (2.1) gives rise to the following notion of bounded mixed p-variance.

DEFINITION 3.1. Let $1 < p < \infty$. A function $g \in L^1(X, d\mu)$ has *bounded mixed p-variance of type* (ν, μ) if

$$\mathscr{V}_{\nu,\mu}^p(g) = \sup \Big\{\nu(B)^{-1} \int_B |g - g_{\mu(B)}|^p \omega^{1-p} d\mu\Big\}^{1/p} < \infty, \tag{3.2}$$

where the sup is taken over all balls $B \subset X$ and $\omega^{-1} = d\mu/d\nu \in B_p(d\nu)$.

Note that, by (1.4) and Theorem 4 of [14], (3.2) is equivalent to

$$\|g\|_{*,\nu,\mu} = \sup_B \Big\{\nu(B)^{-1} \int_B |g - g_{\mu(B)}| d\mu\Big\} < \infty. \tag{3.2'}$$

Hence, the class of all such g will be denoted by $\mathrm{BMV}_\nu(d\mu) = \mathrm{BMV}_\nu(X, d\mu)$. Equipped with the norm

$$\mathscr{N}_{\nu,\mu}(g) = \Big|\mu(X)^{-1} \int_X g\, d\mu\Big| + \mathscr{V}_{\nu,\mu}^p(g) \tag{3.3}$$

the space $\mathrm{BMV}_\nu(d\mu) = \{g \in L^1(X, d\mu) \colon \mathscr{V}_{\nu,\mu}^p(g) < \infty\}$ is complete.

LEMMA 3.2. *If g is in $\mathrm{BMV}_\nu(d\mu)$, so is $|g|$. Hence, $\mathrm{BMV}_\nu(d\mu)$ is a lattice.*

Proof. As before, it suffices to show that the uniform bound

$$\Big\{\nu(B)^{-1} \int_B |g - g_B|^p \omega^{1-p} d\mu\Big\}^{1/p} \leqslant K$$

implies that $g \in \mathrm{BMV}_\nu(d\mu)$ or, equivalently, that $\|g\|_{*,\nu,\mu} < \infty$.

Let $q = p/(p-1)$.

THEOREM 3.3. *$\mathrm{BMV}_\nu(d\mu)$ is the dual of the atomic space $h_\nu^{1,q}(d\mu)$ for some pairing $\langle f, g\rangle$ such that, for all $g \in \mathrm{BMV}_\nu(d\mu)$,*

$$\langle f, g\rangle = \int_X fg\, d\mu, \quad \textit{for all atoms } f \in h_\nu^{1,q}(d\mu). \tag{3.4}$$

Proof. Since the argument is analogous to the proof of Theorem 2.3, we will be brief. For any atom $f = a \in h_\nu^{1,q}(d\mu)$ with support in B, since $p^{-1} + q^{-1} = 1$ and $d\mu = \omega^{-1} d\nu$, we have

$$|\langle a, g\rangle| = \Big|\int a[g - g_{\mu(B)}]\omega^{-1} d\nu\Big| \leqslant \Big\{\Big|\int_B |a|^q d\nu\Big\}^{1/q} \Big\{\int_B |\ldots|^p \omega^{-p} d\nu\Big\}^{1/p}$$
$$\leqslant \Big\{\nu(B)^{-1} \int_B |g - g_{\mu(B)}|^p \omega^{1-p} d\mu\Big\}^{1/p} \leqslant \mathscr{V}_{\nu,\mu}^p(g)$$

and the inclusion $\mathrm{BMV}_\nu(d\mu) \subset [h_\nu^{1,q}(d\mu)]^*$ follows as before.

Conversely, given $\Lambda \in [h_\nu^{1,q}(d\mu)]^*$ with norm $\|\Lambda\|$ and fixing some B, we let now $L_0^q(B, d\mu) = \{\gamma \in L^q(B, d\mu) \colon \int_B \gamma\, d\mu = 0\}$ with norm $\|\gamma\|_q$. For each $\gamma \in L_0^q(B, d\mu)$, the function

$$a = \gamma \chi_B \mu(B)^{1/q} \nu(B)^{-1} \|\gamma\|_q^{-1}$$

is a $(1, q)$ atom of type (ν, μ). Thus, $\Lambda(\gamma\chi_B)$ is defined and $|\Lambda(\gamma\chi_B)| \leqslant \|\Lambda\| \nu(B) \mu(B)^{-1/q} \|\gamma\|_q$.

By Hahn–Banach and the Riesz Representation Theorem, there exists some $g \in L^p(B, d\mu)$ such that

$$\Lambda(\gamma) = \int_B \gamma g\, d\mu, \quad \text{for all } \gamma \in L^q(B, d\mu). \tag{3.4'}$$

As before, it follows that there is a $g \in L^p(X, d\mu)$ such that (3.4′) holds for every ball B. Finally, we claim that

$$\mathscr{V}_{\nu,\mu}^p(g) \leqslant (1 + B_p)\|\Lambda\|. \tag{$*$}$$

From (3.2), it follows that

$$\mathscr{V}_{\nu,\mu}^p(g) = \sup_B \Big|\int_B f[g - g_{\mu(B)}] d\mu/\nu(B)\Big| \tag{3.5}$$

for all $f = \varphi\omega^{1-p}$ supported in balls B and with norm $\leqslant 1$ in $L^q(B, d\mu/\nu(B))$. For any such f, Hölder's inequality implies that

$$|f_{\mu(B)}| \leqslant \frac{\nu(B)}{\mu(B)}\Big\{\frac{1}{\nu(B)}\int_B \omega^{-p}\,d\nu\Big\}^{1/p} \leqslant B_p$$

since $\omega^{-1} \in B_p(d\nu)$, by hypothesis. Consequently, the function

$$b = (1+B_p)^{-1}[f-f_{\mu(B)}]\chi_B\,\omega(B)^{-1}$$

is a $(1, q)$ atom of type (ν, μ) supported in B and the rest of the proof of $(*)$ follows as before.

Remark 3.4. Since $\omega^{-1} \in B_p(d\nu)$ we have that

$$(3.6)\qquad (g/\omega) \in L^\infty(d\nu) \quad \text{implies} \quad g \in \mathrm{BMV}_\nu(d\mu)$$

with mixed variance $\mathscr{V}^p_{\nu,\mu}(g) \leqslant (1+B_p)\,\|g/\omega\|_{\infty,\nu}$. In particular, if $1 < p < \infty$ then

$$(3.7)\qquad \omega^{-1} \in B_p(d\nu) \quad \text{implies} \quad \omega \in \mathrm{BMV}_\nu(d\mu)$$

with mixed variance $\mathscr{V}^p_{\nu,\mu}(d\nu/d\mu) \leqslant (1+B_p)$.

Next, let us consider a weighted variant of Theorem (2.25) of [4].

THEOREM 3.5. *Let $d\omega^s = \omega(x)^s\,d\mu$ for any real s, and let $h^1 = h^{1,\infty}$. If $\omega \in A_*(d\mu)\cap B_*(d\mu)$, then*

$$(3.8)\qquad a \in h^1_{\omega^s}(d\omega^{s-1}) \quad \textit{if and only if} \quad (a\omega) \in h^1_{\omega^{s-1}}(d\omega^{s-2}),$$

$$(3.8')\qquad a \in h^1_{\omega^{s-1}}(d\omega^{s}) \quad \textit{if and only if} \quad (a\omega) \in h^1_{\omega^{s-2}}(d\omega^{s-1}).$$

Since $d\omega^1 = d\nu$ and $d\omega^0 = d\mu$, we are mainly interested in the cases $s = 1$ and $s = 2$ of this result. We begin with a lemma.

LEMMA 3.6. *Let $1 < q < \infty$ and $q' = q/(q-1)$.*

(3.9) *If $\omega^{-1} \in B_{q'}(d\mu)$, then $h^{1,q}_\nu(d\mu) = h^{1,\infty}_\nu(d\mu)$.*

(3.9′) *If $\omega \in B_{q'}(d\mu)$, then $h^{1,q}_\mu(d\nu) = h^{1,\infty}_\mu(d\nu)$.*

Proof. The two parts are similar, and are simple analogues of Theorem A in [4]. In (3.9′) for example, it suffices to show that any atom $a \in h^{1,q}_\mu(d\nu)$ may be decomposed in the form (1.1) with respect to $(1, \infty)$ atoms a_j of type (μ, ν). Following now the notation of [4], let a be supported in a sphere S_0 and let $b = a\mu(S_0)$. We consider Whitney-type Decompositions of bounded open subsets

$$U^\alpha = \{x \in X\colon [\mathscr{M}_{\mu,q}b](x) > \alpha\}, \quad \text{where} \quad \mathscr{M}_{\mu,q}b = \{\mathscr{M}_\mu|b|^q\}^{1/q}$$

and

$$(3.10)\qquad [\mathscr{M}_\mu f](x) = \sup\{\mu(S)^{-1}\int_S |f|\,d\mu\colon S \text{ is a sphere, } x \in S\}.$$

Let χ_j denote the characteristic functions of the spheres S_j which form an M-disjoint covering of U^α, and define

$$\eta_j(x) = \begin{cases} 0, & \text{if} \quad x \notin U^\alpha, \\ \chi_j(x)/\sum_j \chi_j(x), & \text{if} \quad x \in U^\alpha. \end{cases}$$

Since $\int b\,d\nu = \mu(S_0)\int a\,d\nu = 0$ by assumption, we choose

$$g_0(x) = \begin{cases} b(x) & \text{if} \quad x \notin U^\alpha, \\ \sum_j (\eta_j b)_{\nu(S_j)}\chi_j(x), & \text{if} \quad x \in U^\alpha, \end{cases}$$

$$h_j(x) = b(x)\eta_j(x) - (\eta_j b)_{\nu(S_j)}\chi_j(x) \quad \text{for all } x \in X,$$

so that

$$(3.11)\qquad b(x) = g_0(x) + \sum_j h_j(x).$$

Clearly, each h_j is supported in the sphere S_j and $\int h_j\,d\nu = 0$. In order to infer that also the bounded part g_0 satisfies $\int g_0\,d\nu = 0$, we need to check that the sum in (3.11) is convergent in $L^1(d\nu)$ norm.

But, by the M-disjointness of the spheres S_j, we see that

$$\sum_j \|h_j\|_{1,\nu} \leqslant 2\sum_j \int_{S_j} |b|\,d\nu \leqslant 2M\int_{U^\alpha} |b|\,d\nu$$
$$\leqslant 2M\mu(S_0)\Big\{\mu(S_0)^{-1}\int_{S_0} |b|\,\omega\,d\mu\Big\}.$$

Hence, using Hölder's inequality with exponent q, the desired conclusion follows whenever $\omega \in B_{q'}(d\mu)$. The rest of the proof is identical to the argument in [4].

Proof of Theorem 3.5. Let us consider (3.8) in the case $s = 1$, say. For any atom $a \in h^{1,\infty}_\nu(d\mu)$, it suffices to show that $b = (a\omega)$ belongs to $h^{1,2}_\mu(d\omega^{-1})$. In fact, with ω^{-1} in place of ω, (3.9′) shows that $h^{1,2}_\mu(d\omega^{-1}) = h^{1,\infty}_\mu(d\omega^{-1})$ provided $\omega^{-1} = (d\omega^{-1}/d\mu)$ is in $B_2(d\mu)$. The latter holds by our assumptions and Corollary 1.3. If the atom a is supported by B, we have

$$\Big\{\mu(B)^{-1}\int_B b^2\,d\mu\Big\}^{1/2} \leqslant \nu(B)^{-1}\Big\{\mu(B)^{-1}\int_B \omega^2\,d\mu\Big\}^{1/2} \leqslant B_2\mu(B)^{-1}$$

since $\omega \in B_2(d\mu)$. It follows at once that $(a\omega) \in h^{1,2}_\mu(d\omega^{-1})$.

Conversely, for any $b \in h^{1,\infty}_\mu(d\omega^{-1})$ supported by B, it suffices to check that $a = (b/\omega)$ belongs to $h^{1,2}_\nu(d\mu)$, by virtue of (3.9) with $q = 2$. We have $\int a\,d\mu = \int b\,\omega^{-1}\,d\mu = 0$ and

$$\Big\{\nu(B)^{-1}\int_B a^2\,d\nu\Big\}^{1/2} \leqslant \mu(B)^{-1}\Big\{\nu(B)^{-1}\int_B (1/\omega)^2\,d\nu\Big\}^{1/2} \leqslant B_2\nu(B)^{-1}$$

since $\omega \in A_2(d\mu)$ implies that $(1/\omega) \in B_2(d\nu)$.

The proofs of the other cases are all analogous.

Remark 3.7. Since $\omega \in B_2(d\mu)$ if and only if $\omega \in A_2(d\nu)$, combining (3.9′) with Theorem 2.3 yields the duality

$$\mathrm{BMV}_\mu(d\nu) = [h^1_\mu(d\nu)]^*, \quad \text{where} \quad h^1 = h^{1,\infty}. \tag{3.12}$$

On the other hand, if $\omega^{-1} \in B_p(d\nu) \cap B_p(d\mu)$ then, combining (3.9) with Theorem 3.3 we obtain

$$\mathrm{BMV}_\nu(d\mu) = [h^1_\nu(d\mu)]^*, \quad \text{where} \quad h^1 = h^{1,\infty}. \tag{3.13}$$

In particular, by Corollary 1.3, formula (3.13) is valid whenever $\omega \in A_*(d\mu) \cap \cap B_*(d\mu)$.

We close with a result about multipliers of $\mathrm{BMV}_\nu(d\mu)$ on $(X, \nu, \mu) = (\partial D, \omega, \sigma)$.

Proposition 3.8. *If $\omega = (d\nu/d\mu) \in A_*(d\mu)$, then, for all $0 < \alpha \leqslant 1$, $\mathrm{Lip}(\alpha)$ is contained in $\mathrm{BMV}_\nu(d\mu)$ and multiplies $\mathrm{BMV}_\nu(d\mu)$ continuously.*

Proof. It suffices to show that for all $f \in \mathrm{BMV}_\nu(d\mu)$ and $g \in \mathrm{Lip}(\alpha)$

$$\|fg\|_{*,\nu,\mu} \leqslant C\|g\|_\alpha \mathcal{N}_{\nu,\mu}(f), \tag{3.14}$$

where $\|g\|_\alpha = \|g\|_\infty + C_\alpha$ as usual.

Fixing an arbitrary ball $B = B_r(x_0)$ in ∂D, we have

$$\int_B |fg - f_{\mu(B)} g(x_0)|\, d\mu \leqslant \|g\|_\infty \int_B |f - f_{\mu(B)}|\, d\mu + |f_{\mu(B)}| \int_B |g - g(x_0)|\, d\mu.$$

Hence, with $C(B) = f_{\mu(B)} g(x_0)$,

$$\text{(i)} \qquad \frac{1}{\nu(B)} \int_B |fg - C(B)|\, d\mu \leqslant \|g\|_\alpha \Big\{ \mathcal{N}_{\nu,\mu}(f) + \Big| \int_B f\, d\mu \Big| r^\alpha/\nu(B) \Big\}.$$

Using $f = f\omega^{(1-p)/p}\omega^{(p-1)/p}$ and Hölder's inequality yields

$$\Big| \int_B f\, d\mu \Big| \leqslant \|f\|_{L^p(d\omega^{1-p})} \nu(B)^{1/p'} \leqslant C_p \mathcal{N}_{\nu,\mu}(f) \nu(B)^{1/p'}$$

where $p' = p/(p-1)$ and $1 < p < \infty$.

Finally, using the $A_*(d\mu)$ property, it follows that for any $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that

(ii) $\nu(B) \geqslant (1/c_\varepsilon)\mu(B)^{1+\varepsilon}$, uniformly in B. Now, $\mu(B) = \sigma(B) \approx r^{n-1}$. Thus, choosing $\varepsilon = 1$ and fixing any $p > 2(n-1)/\alpha$, we have

$$\text{(iii)} \qquad \Big| \int_B f\, d\mu \Big| r^\alpha/\nu(B) \leqslant C_p \mathcal{N}_{\nu,\mu}(f) r^{\alpha - 2(n-1)/p}$$

and the conclusion follows at once from (i) and (iii).

References

[1] S. Campanato, *Proprietà di hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa 17 (1963), 175–188.

[2] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[3] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Lect. Notes in Math. vol. 242, Springer-Verlag, Berlin and New York 1971.

[4] —, — *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[5] B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. 65 (1977), 275–288.

[6] — *On the Poisson integral for Lipschitz and C^1 domains*, Studia Math. 66 (1979), 13–24.

[7] E. Fabes, M. Jodeit and N. Rivière, *Potential techniques for boundary value problems on C^1-domains*, Acta Math. 141 (1978), 165–186.

[8] E. Fabes, C. Kenig and U. Neri, *Carleson measures, H^1 duality and weighted BMO in non-smooth domains*, Indiana Univ. Math. J. 30 (1981), 547–581.

[9] J. García-Cuerva, *Weighted H^p spaces*, Diss. Math. 162, Warszawa, 1979.

[10] S. Janson, *On functions with conditions on the mean oscillation*, Ark. Mat. 14 (1976), 189–196.

[11] F. John and L. Nirenberg *On functions of bounded mean oscillation*. Comm. Pure Appl. Math. 14 (1961), 415–426.

[12] C. E. Kenig, *Weighted H^p spaces on Lipschitz domains*, Amer. J. Math. 102 (1980), 129–163.

[13] N. G. Meyers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. 15 (1964), 717–721.

[14] B. Muckenhoupt and R. L. Wheeden, *Weighted bounded mean oscillation and the Hilbert transform*, Studia Math. 54 (1976), 221–237.

[15] —, — *On the dual of weighted H^1 of the half-space*, ibid. 63 (1978), 57–79.

[16] U. Neri, *Weighted H^1-BMO dualities*, Univ. of Md. TR81-15, submitted to Proc. Harmonic Analysis Conference in honor of A. Zygmund (Chicago, March 1981).

[17] D. A. Stegenga, *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation*, Amer. J. Math. 98 (1976), 573–589.

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