Classical hierarchies from a modern standpoint

Part III. *BP*-sets

by

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Abstract. The Borel programmable or *BP*-sets recently introduced by Blackwell [29] are shown to lie strictly the *C*-sets and the *R*-sets. Iteration of the operation that takes one from the Borel sets to the *BP*-sets produces a family which is shown to coincide with the *R*-sets.

Chapter F. Set-theoretic programming

§ 13. Generalities.

(a) Introductory. In a paper [29] recent in date but classical in spirit, D. Blackwell has introduced a new family of Lebesgue measurable sets of reals, obtained by enlarging the Borel family through a process he calls *programming*. This new family, the *Borel-programmable* or *BP-sets*, can itself be further enlarged by iterating the programming process, to produce the *programmable* or *P-sets*. In a review [34], B. V. Rao & A. Maitra raise "a whole spectrum" of problems about *BP*- and *P*-sets, the most pressing being to determine the relationship between these two new families and the older families of *C*-sets and *R*-sets studied in Parts I and II of this series. Some partial results on this last problem have been presented in [32]. As the Abstract indicates, we now possess a complete solution:

$$C\text{-sets} \subsetneq BP\text{-sets} \subsetneq R\text{-sets} = P\text{-sets}\,.$$

The proof will be presented below, along with some conjectures and suggestions for further research.

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(b) Definitions. Let I be the set ω of natural numbers or some other denumerable set, and give the power set $\mathscr{P}(I)$ its usual topology, making it a standard space (uncountable Polish space). A Borel function e from any space $\mathscr{X}$ to $\mathscr{P}(I)$ will be called an *encoder*. A *program* is a function $p\colon \mathscr{P}(I)\to\mathscr{P}(I)$ such that $x\subseteq p(x)$ for all x. For ordinal α we define the αth *iterate* p_α by:

$$p_0(x) = x\,,\quad p_{\beta+1}(x) = p\big(p_\beta(x)\big)\,,\quad p_\alpha(x) = \bigcup_{\beta<\alpha} p_\beta(x) \text{ at limits}\,.$$

Cardinality considerations reveal that at Ω — first uncountable ordinal — we reach the ultimate iterate of p. Indeed, for each particular x there is some $\alpha<\Omega$ depending on x such that $p_\beta(x) = p_\alpha(x)$ for all $\beta>\alpha$. A Borel function d from $\mathscr{P}(I)$ to any space $\mathscr{Y}$ will be called a *decoder*. For encoder e, program p, and decoder d, $\mathscr{J}(e,p,d)$ denotes the composition $dp_\Omega e$.

Let $\mathscr{H}\subseteq\mathscr{P}(I)$ be a uniform family, i.e. a σ-field containing the Borel sets and stable under inverse image by $\mathscr{H}$-measurable functions. A function $f\colon\mathscr{X}\to\mathscr{Y}$ will be called *$\mathscr{H}$-programmable* iff it is of form $\mathscr{J}(e,p,d)$ where the program p is $\mathscr{H}$-measurable. A set $A\subseteq\mathscr{X}$ will be called *$\mathscr{H}$-programmable* iff its characteristic function is, or equivalently iff its *revised* characteristic function $\varrho_A\colon\mathscr{X}\to\mathscr{P}(I)$ is, where $\varrho_A(x) = I$ if $x\in A$ and $=\emptyset$ otherwise. When $\mathscr{X} = \mathscr{P}(I)$ we can iterate:

$$\begin{aligned}\mathscr{H}P^0 &= \mathscr{H},\\ \mathscr{H}P^{\beta+1} &= (\mathscr{H}P^\beta)\text{-programmable sets},\\ \mathscr{H}P^\alpha &= \text{smallest uniform family containing } \bigcup_{\beta<\alpha}\mathscr{H}P^\beta \text{ at limits } \alpha<\Omega,\\ \mathscr{H}P^\Omega &= \bigcup_{\alpha<\Omega}\mathscr{H}P^\alpha.\end{aligned}$$

(c) below implies that these form an increasing sequence of uniform families, culminating in a family that cannot be further enlarged by programming; also, that the notion of $\mathscr{H}P^\alpha$-set can be transferred from spaces of form $\mathscr{P}(I)$ to arbitrary standard spaces, making use of Borel isomorphisms. When $\mathscr{H}$ = Borel we write *BP-sets* $= BP^1$, *P-sets* $= BP^\Omega$.

(c) Uniformity Lemma. *Let $\mathscr{H}$ be a uniform family of subsets of $\mathscr{P}(I)$, $\mathscr{K} = \mathscr{H}$-programmable sets. Then $\mathscr{K}$ is a uniform family containing $\mathscr{H}$, and $\mathscr{K}$-measurable functions $=$ $\mathscr{H}$-programmable functions.*

Proof. For $\mathscr{H}$ = Borel the main contentions of (c) (expressed in slightly different terminology) are elegantly proved in [29]; as the arguments there are in fact valid for general $\mathscr{H}$, they need not be repeated here. ∎

Beyond the Uniformity Lemma, [29] contains two main results: (i) that the BP-sets are Lebesgue measurable (and measurable w.r.t. many other measures); (ii) that the BP-sets (are stable under operation $\mathscr{A}$ and hence) contain the C-sets. Both admit of considerable generalization.

For any σ-field $\mathscr{F}$ of subsets of $\mathscr{P}(I)$, let $\mathscr{F}^*$ denote the set of $F\in\mathscr{F}$ such that every subset of F belongs to $\mathscr{F}$. E.g. if $\mathscr{F}$ is the family of Lebesgue measurable sets, $\mathscr{F}^*$ = sets of Lebesgue measure zero; while if $\mathscr{F}$ is the family of almost open (Baire property) sets, $\mathscr{F}^*$ = meager (1st category) sets. $\mathscr{F}$ will be called a *regularity class* iff every uncountable family of pairwise disjoint elements of $\mathscr{F}$ contains an element of $\mathscr{F}^*$. Both examples have this property. (d) below can be applied to the family of universally measurable sets (those measurable w.r.t. any σ-finite complete measure) and to the family of universally almost open (those A such that $f^{-1}[A]$ is almost open for all Borel f) to show that these families cannot be enlarged by programming.

(d) Regularity Lemma ([32, I 5.1]). *Let $\mathscr{F}$ be a regularity class, $\mathscr{H}$ a uniform family, of subsets of $\mathscr{P}(I)$. If $\mathscr{H}\subseteq\mathscr{F}$, then $\mathscr{H}$-programmable sets $\subseteq\mathscr{F}$ as well.*

Proof. Let A be an $\mathscr{H}$-programmable set, $\varrho_A = \mathscr{J}(e,p,d)$ with p $\mathscr{H}$-measurable. It is readily seen that $\mathscr{E} = \{B\colon e^{-1}[B]\in\mathscr{F}\}$ is a regularity class containing $\mathscr{H}$, and that for $\alpha<\Omega$, $D_\alpha = \{x\colon d(p_\alpha(x)) = I\}$ and $P^i_\alpha = \{x\colon i\in p_{\alpha+1}(x)-p_\alpha(x)\}$ belong to $\mathscr{H}$ and hence to $\mathscr{E}$. Now since for any α, A can be written as $e^{-1}[D_\alpha\cup N]$ where $N\subseteq\bigcup_{i\in I}P^i_\alpha$, viz. $N = \{x\colon d(p_\Omega(x)) = I\ \&\ p_\Omega(x)\neq p_\alpha(x)\}$, it suffices to show that for some α each P^i_α belongs to $\mathscr{E}^*$. Well, if this were not the case, for each α there would be some i for which $P^i_\alpha\notin\mathscr{E}^*$. But then some one i would have to work for uncountably many α. Since the P^i_α for the same i and different α are disjoint, this contradicts the fact that $\mathscr{E}$ is a regularity class. ∎

Recall that $Q(I)$ = finite sequences from I, $Q^*(I)$ = finite sequences of even length, $\#s$ = code number of s for $s\in Q(\omega)$. The notions of $\mathscr{H}$-operation and of R-transform were introduced in §§ 2(c), 9(b) respectively. The following implies $R^{\beta+1}\subseteq BP^{\beta+1}$ for all $\beta<\Omega$ and R-sets$\subseteq P$-sets:

(e) Transform Lemma ([32, V 2.1]). *Let $\mathscr{H}$ be a uniform family of subsets of $\mathscr{P}(I)$, Φ an $\mathscr{H}$-operation, $\mathscr{K}$ = $\mathscr{H}$-programmable sets, $\Psi = R\Phi$. Then Ψ is a $\mathscr{K}$-operation.*

Proof. We may assume that $I=\omega$ and that Φ acts on ω-indexed systems. Let B be the truth table of Φ. A coded version of the truth table of $R\Phi$ is:

$$B' = \{y\subseteq\omega\colon \exists S\subseteq\{s\in Q(\omega)\colon \#s\in y\}[(\ )\in S\ \&\ \forall t\in S\{i\colon t\oplus i\in S\}\in B]\}\,.$$

Translated, the inductive analysis of § 9(b) shows $\varrho_{B'} = \mathscr{J}(e,p,d)$ for:

$$\begin{aligned} e(x) &= \omega - x,\\ d(z) &= \begin{cases}\omega & \text{if } \#(\ )\notin z,\\ \emptyset & \text{otherwise.}\end{cases}\\ p(y) &= y\cup\{\#t\colon \{i\colon \#(t\oplus i)\notin y\}\notin B\}.\end{aligned}$$

(f) Preview. In the next section we show that C-sets$\subsetneqq BP$-sets. The proof generalizes to show: (i) $\mathscr{R}^{\beta+1}\subsetneqq BP^{\beta+1}$ for all $\beta<\Omega$; (ii) for no operation Γ is the family of BP-sets the smallest uniform family stable under Γ.

In the following section we show that BP-sets $\subsetneq \mathscr{R}^2$ by showing them bi-primitive R-sets (and one could improve this to a strict inclusion). This generalizes to show $BP^{\beta+1} \subsetneq \mathscr{R}^{\beta+2}$ for all $\beta < \Omega$. Hence $BP^\alpha = \mathscr{R}^\alpha$ at limits and P-sets $= R$-sets. This also shows that the BP^α are all distinct, since the $\mathscr{R}^\alpha$ are known to be; a fact which can also be shown directly by a universal set argument (as in [32, III]).

§ 14. First inclusion. We wish to prove that for any standard space $\mathscr{X}$, the C-sets are properly included in the BP-sets. We begin by adapting the construction of § 5(b) to produce an $\mathscr{R}^2$-set universal for the C-subsets of $\mathscr{X}$. Let $B \subseteq \mathscr{X} \times \omega^\omega$ be an open set universal for the open subsets of $\mathscr{X}$. Let D consist of all triples $(x, \xi, T) \in \mathscr{X} \times \omega^\omega \times \mathscr{P}(Q^*(Q(\omega)))$ such that T is a wellfounded $Q(\omega)$-tree and:

(i) $\exists \xi_0 \in \omega^\omega\, \forall n_0 \in \omega\, \forall \zeta_0 \in \omega^\omega\, \exists m_0 \in \omega\, \exists \xi_1\, \forall n_1\, \forall \zeta_1\, \exists m_1 \ldots$

$$\forall k[\sigma = (\xi_0|n_0, \zeta_0|m_0, \ldots)|2k \text{ terminal for } T \rightarrow (x, \pi(\#\sigma, \xi)) \in B]$$

where we assume some suitable assignment of code numbers $\#\sigma$ to $\sigma \in Q^*(Q(\omega))$ has been introduced. The arguments of § 5(b) show D is universal for the C-sets (and itself an $\mathscr{R}^2$-set); we wish to show it is a BP-set. To that end we restate the kind of inductive analysis of the set of wellfounded trees and of D used in § 5(b) in slightly different language. For any $Q(\omega)$-tree T and any $W \subseteq Q^*(Q(\omega))$ define:

$$T_0 = \{\sigma\colon \sigma \text{ terminal for } T\}, \qquad W_0 = T_0 \cap W,$$
$$T_{\beta+1} = T_\beta \cup \{\sigma\colon \forall s, t\ \sigma \oplus s \oplus t \in T_\beta\},$$
$$W_{\beta+1} = W_\beta \cup \{\sigma\colon \exists\xi \forall n \forall\zeta \exists m \sigma \oplus \xi|n \oplus \zeta|m \in W_\beta\},$$
$$T_\alpha = \bigcup_{\beta<\alpha} T_\beta, \qquad W_\alpha = \bigcup_{\beta<\alpha} W_\beta \text{ at limits.}$$

We then have for $W = \{\sigma\colon (x, \pi(\#\sigma, \xi)) \in B\}$:

(ii) T is wellfounded $\leftrightarrow (\ ) \in T_\Omega$,
(i) above holds $\leftrightarrow (\ ) \in W_\Omega$,

$$T_\alpha \cap W_\Omega \subseteq W_\alpha.$$

In view of (ii), to achieve our goal it will suffice to produce a Borel encoder e, program p, and decoder d such that $f = \mathscr{J}(e, p, d)$ satisfies:

(iii) $\forall$ wellfounded tree $T \forall W[f(T, W) = (T_\alpha, W_\alpha)$ for the least α with $(\ ) \in T_\alpha]$.

To this end we consider the functions used in passing from β to $\beta+1$:

$$g(T) = T \cup \{\sigma\colon \forall s, t\ \sigma \oplus s \oplus t \in T\},$$
$$h(W) = W \cup \{\sigma\colon \exists\xi \forall n \forall\zeta \exists m\ \sigma \oplus \xi|n + \zeta|m \in W\}.$$

The former is Borel, the latter only C-measurable. But as we have already seen that C-measurable functions are BP, we may set $h = \mathscr{J}(e', p', d')$ where p' is a Borel function $\mathscr{P}(\omega) \rightarrow \mathscr{P}(\omega)$.

We now let $I = Q^*(Q(\omega))$ and let J be the disjoint union of:
two copies of I called the *primary registers*,
a copy of ω for each $\sigma \in I$, called the *σ-auxiliary register*.

Thus any $y \subseteq J$ consists of several parts which we call the *entries* of y in the various registers. $T(y)$, $W(y)$ will denote the entries in the primary registers. $R(\sigma, y)$ will denote entry in the σ-auxiliary register. Define the Borel encoder e by:

$$T(e(T, W)) = T_0, \qquad W(e(T, W)) = W_0, \qquad R(\sigma, e(T, W)) = \varnothing.$$

Define the Borel decoder d simply by:

$$d(y) = (T(y), W(y)).$$

Call y *acceptable* if $(\ ) \notin T(y)$ and $g(T(y)) - T(y) \neq \varnothing$; for acceptable y let $\sigma(y)$ be the element σ of the latter set for which $\#\sigma$ is least.

The Borel program $p\colon \mathscr{P}(J) \rightarrow \mathscr{P}(J)$ will be defined by cases. In each case, for $z = p(y)$, any register of z not explicitly mentioned agrees with y:
Case y unacceptable: Let $z = y$.
Case y acceptable, $R(\sigma(y), y) = \varnothing$: Let $R(\sigma(y), z) = e'(W(y))$.
Case y acceptable, $R(\sigma(y), y) \neq \varnothing$, $p'(R(\sigma(y), y)) \neq R(\sigma(y), y)$:
Let $R(\sigma(y), z) = p'(R(\sigma(y), y))$.
Case y acceptable, $R(\sigma(y), y) \neq \varnothing$, $p'(R(\sigma(y), y)) = R(\sigma'(y), y)$:
Let $T(z) = g(T(y))$, $W(z) = d'(R(\sigma(y), y))$.

If T is a wellfounded tree, what is the effect of iterated application of p to $e(T, W)$? Well, we pass through a sequence of $y_\alpha \subseteq J$ with $T(y_\alpha) = T_\alpha$, $W(y_\alpha) = W_\alpha$. p cannot take us directly from y_β to $y_{\beta+1}$, but it takes us there through a sequence of z^γ_β during which the computations needed to compute $W_{\beta+1}$ from W_β are carried out in an auxiliary register. So long as $(\ ) \notin T_\beta$, $T_{\beta+1} - T_\beta$ will be nonempty and an appropriate blank auxiliary register will be available for these computations. When we reach an α with $(\ ) \in T_\alpha$, p halts and d prints out (T_α, W_α). Thus (iii) is satisfied, completing the construction. ∎

§ 15. Second inclusion. We wish to prove that for any standard space the BP-sets are included among what we called in § 9(b) the bi-primitive R-sets, and hence are properly included in $\mathscr{R}^2$. The problem quickly reduces to showing that for any Borel program $p\colon \mathscr{P}(\omega) \rightarrow \mathscr{P}(\omega)$ the inverse image $(p_\Omega)^{-1}[U]$ is a coprimitive R-set for all U in some basis $\mathscr{U}$ for $\mathscr{P}(\omega)$.

As $\mathscr{U}$ we choose the family of all $U(s) = \{y \subseteq \omega\colon \forall k < \text{length } s\, (k \in y \leftrightarrow s(k) = 1)\}$ for $s \in Q(\{0, 1\})$. For $V \in \mathscr{U}$ let $Y(V) = \{x \subseteq \omega\colon p(x) \in V\}$, $Z(V) = Y(V) \cap \cap \{x\colon p(x) = x\}$. As these sets are Borel, we can certainly represent them in the form:

(i) $y \in Y(V) \leftrightarrow \exists a_0 \in \omega\, \forall b_0 \in \omega\, \exists a_1\, \forall b_1 \ldots \forall n[y \in Y(V, (a_0, b_0, \ldots)|2n)]$,

(ii) $z \in Z(V) \leftrightarrow \exists a_0\, \forall b_0\, \exists a_1\, \forall b_1 \ldots \forall n[z \in Z(V, (a_0, b_0, \ldots)|2n)]$

where the $Y(V, s)$ and $Z(V, s)$ come from $\mathscr{U}$.

Using (i) and (ii) we will associate to each $U \in \mathscr{U}$ and $x \subseteq \omega$ a game G, and prove that PRO has a winning strategy in this game iff $p_\Omega(x) \in U$. Since G will have, modulo some tedious but routine coding, the form of the game associated with x being a member of $\Gamma(W)$ where Γ is the operation co-$(R(\text{co-}\mathscr{G}))$ and each $W(\sigma) \in \mathscr{U} \cup \{\emptyset\}$, this will show that $(p_\Omega)^{-1}[U]$ is indead a co-primitive set.

(a) The auxiliary game. G is best thought of as consisting of a potentially infinite sequence of subgames, each of length $\omega+1$. At the end of each subgame, PRO has an opportunity to win the whole game G. If he does, the game ends at that point. If not, the players go on to the next subgame. If CON manages to get through all the subgames without PRO winning, this counts as a win for her.

0-*subgame*: PRO and CON alternately choose c_k^0, $d_k^0 \in \omega$ for $k \in \omega$; recalling that $\pi(i,j) = 2^i(2j+1)-1$, we let $a_{ij}^0 = c_{\pi(i,j)}^0$, $b_{ij}^0 = d_{\pi(i,j)}^0$. CON then chooses $n_0 \in \omega$, and we let W_0 be the intersection of all $Z(U, (a_{i0}^0, b_{i0}^0, a_{i1}^0, b_{i1}^0 \ldots)|2j)$ for $\pi(i,j)<n_0$. If $x \in W_0$, PRO wins at this point; otherwise, the players proceed to the next subgame.

m-subgame, $m>0$: PRO and CON alternately choose c_k^m, d_k^m giving rise to a_{ij}^m, b_{ij}^m. CON then chooses n_m. Let W_m be the intersection of all

$$Y(W_{m-1}, (a_{i0}^m, b_{i0}^m, a_{i1}^m, b_{i1}^m, \ldots)|2j)$$

for $\pi(i,j)<n_m$. If $x \in W_m$, PRO wins at this point; otherwise the players proceed to the next subgame.

(b) Who wins?

CLAIM. *If* $p_\Omega(x) \in U$, PRO *has a winning strategy in* G.

Proof. Assuming $p_\Omega(x) \in U$, let PRO play as follows in G: Let $\beta(0)$ be the least α such that $p_\alpha(x) = p_\Omega(x)$. PRO has a winning strategy φ_0 for (ii) with $z = p_{\beta(0)}(x)$, $V = U$. Let PRO choose his c_k^0 in the 0-subgame so that for each i the sequence $(a_{i0}^0, b_{i0}^0, a_{i1}^0, b_{i1}^0, \ldots)$ constitutes a play agreeing with φ_0. Then no matter what n_0 CON chooses, we will have $p_{\beta(0)}(x) \in W_0$. If it happens that $\beta(0) = 0$, this means that PRO wins already at the end of the 0-subgame.

Suppose the players have reached the beginning of the m-subgame, $m>0$, without PRO having won yet, and that we have $p_{\beta(m-1)}(x) \in W_{m-1}$. Then for some $\beta(m)<\beta(m-1)$ we have $p_{\beta(m)+1}(x) \in W_{m-1}$. This means that PRO has a winning strategy φ_m for (i) with $y = p_{\beta(m)}(x)$, $V = W_{m-1}$. Let PRO choose his c_k^m in the m-subgame so that for each i, $(a_{i0}^m, b_{i0}^m, a_{i1}^m, b_{i1}^m, \ldots)$ agrees with φ_m. Then no matter what n_m CON chooses, $p_{\beta(m)} \in W_m$. Since when PRO plays this way the decreasing sequence $\beta(0)>\beta(1)>\beta(2)>\ldots$ must soon reach an m with $\beta(m) = 0$ and $x \in W_m$, the strategy just described is a winning one for PRO as required. ∎

CLAIM. *If* $p_\Omega(x) \notin U$, CON *has a winning strategy in* G.

Proof. Assuming $p_\Omega(x) \notin U$, let CON play as follows in G: Let α be least such that $p_\alpha(x) = p_\Omega(x)$, and let $(\beta(i)\colon i \in \omega)$ enumerate the ordinals $\leqslant\alpha$, with $\beta(0) = \alpha$. It is immediately seen that $p_{\beta(i)}(x) \notin Z(U)$ for all i. Hence CON has a winning strategy ψ_i^0 for (ii) with $z = p_{\beta(i)}(x)$, $V = U$. Let CON choose her d_k^0 in the 0-subgame so that for each i, $(a_{i0}^0, b_{i0}^0, a_{i1}^0, b_{i1}^0, \ldots)$ agrees with ψ_i^0. We then have:

$$(*) \qquad \forall i\, \exists j\, p_{\beta(i)}(x) \notin Z(U(a_{i0}^0, b_{i0}^0, a_{i1}^0, b_{i1}^1, \ldots)|2j).$$

Using this fact we define inductively certain $i(m)$, $j(m)$, $X(m)$:

$i(0) = 0$,

$j(m) = $ least j such that $(*)$ holds for $i = i(m)$ and j,

$$X(m) = \bigcap_{m'<m} X_{m'} \cap Z(U, (a_{i(m),0}^0, b_{i(m),0}^0, a_{i(m),1}^0, b_{i(m),1}^0, \ldots)|2j(m))),$$

$i(m+1) = $ the i such that $\beta(i) = \sup\{\beta\colon p_\beta(x) \in X(m)\}$ if this set is nonempty; $i(m+1)$ undefined if this set is empty.

Remark. If β is a limit and $X \in \mathscr{U}$, then whether $p_\beta(x) \in X$ or not depends on the answer to the question, "Is $k \in p_\beta(x)$?" for *finitely many* k. But for each of these k there is a $\delta_k<\beta$ such that for all γ with $\delta_k \leqslant \gamma \leqslant \beta$ the answer to the question, "Is $k \in p_\gamma(x)$?" is the same. If δ is the sup of these δ_k, then for all γ with $\delta \leqslant \gamma \leqslant \beta$ the answer to the question whether $p_\gamma(x) \in X$ or not is the same.

In view of the foregoing remark, if $i(m+1)$ is defined, then $p_{\beta(i(m+1))}(x) \in X(m)$. In the decreasing sequence $\beta(i(0))>\beta(i(1))>\beta(i(2))>\ldots$ we must soon reach an m_0 for which $i(m_0)$ is undefined. If CON chooses n_0 so that $\pi(i(m), j(m))<n_0$ for all $m<m_0$, then we will have $p_\beta(x) \notin W_0$ for all β. If in the subsequent subgames CON plays the same way, but with Y in place of Z, we will have $p_\beta(x) \notin W_m$ for all β and *a fortiori* $x \notin W_m$. Thus the strategy described is a winning one for CON as required. ∎

Chapter G. Marginalia

§ 16. Regularity. The measurability of BP-sets follows from their inclusion in the R-sets, and more economically from the lemma of § 13(d). S. Shreve [35] has asked whether his Measure Duality Theorem (cf. § 6(b)) holds for BP-sets. We will sketch an affirmative answer, omitting purely measure-theoretic details and concentrating on programming aspects of the problem.

We recall from § 6 that for any standard space $\mathscr{X}$ the set of complete regular probability measures on $\mathscr{X}$ can be made into a standard space $\mathscr{M}$, and that if $f\colon \mathscr{X} \to \mathscr{X}$ is Borel, so is $f^*\colon \mathscr{M} \to \mathscr{M}$ defined by $f^*(\mu)(U) = \mu(f^{-1}[U])$. The problem of proving the Measure Duality Theorem for BP-sets quickly reduces (by a purely analytic argument) to showing for $\mathscr{X} = \mathscr{P}(\omega)$ that if $p\colon \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ is a Borel program, then $(p_\Omega)^*$ is BP.

Two observations about this situation will be useful. First, note that for a finite $\{0, 1\}$-sequence ending in a 1, $s\oplus 1$, whenever $y \subseteq \omega$ belongs to the basic clopen set $U(s\oplus 1)$, then by the definition of program, $p(y) \in U(s)$ implies $p(y) \in U(s\oplus 1)$. It follows that if $p^*(\mu)(U(t)) = \mu(U(t))$ for all $t \lhd s$, then $p^*(\mu)(U(s\oplus 1)) \geqslant \mu(U(s\oplus 1))$. Second, writing χ_α^s for the characteristic function of $(p_\alpha)^{-1}[U(s)]$, note that by the Remark of the preceding section, for a limit α, $\chi_\alpha^s = \lim\limits_{\beta\to\alpha} \chi_\beta^s$. Since $(p_\alpha)^*(\mu) = (U(s)) = \int \chi_\alpha^s d\mu$, it follows by Lebesgue's Theorem that $(p_\alpha)^*(\mu)(U(s)) = \lim\limits_{\beta\to\alpha} (p_\beta)^*(\mu)(U(s))$.

Now the topology on the space $\mathcal{M}$ of measures on $\mathcal{P}(\omega)$ is such that we may safely identify $\mu \in \mathcal{M}$ with its *signature*, the element of $[0, 1]^\omega$ whose ith term is $\mu(U(s \oplus 1))$ where $\# s = i$. Introducing an enumeration $(r_i\colon i \in \omega)$ of the rationals in $[0, 1)$, we may identify a real ϱ in $[0, 1]$ with the set of i for which $r_i < \varrho$. These identifications and the two foregoing observations reduce the original problem of proving the Measure Duality Theorem for BP-sets to that of proving Theorem (a) below.

Define the *lexicographic* and *termwise* partial orders on $(\mathcal{P}(\omega))^\omega$ by setting, for $x = (x(i)\colon i \in \omega)$, $y = (y(i)\colon i \in \omega)$:

$$x \leqslant_{\mathrm{lex}} y \leftrightarrow \forall i(\neg x(i) \subseteq y(i)) \to \exists j < i(x(i) \subsetneqq y(i))) ,$$
$$x \leqslant_{\mathrm{term}} y \leftrightarrow \forall i(x(i) \subseteq y(i)) .$$

Given $x^\beta = (x^\beta(i)\colon i \in \omega)$ for $\beta < \alpha$, define the *limes superior* by:

$$\limsup_{\beta < \alpha} x^\beta = (x(i)\colon i \in \omega)$$

where

$$k \in x(i) \leftrightarrow \forall \gamma < \alpha \, \exists \gamma < \beta < \alpha \; k \in x^\beta(i) .$$

Call $p\colon (\mathcal{P}(\omega))^\omega \to (\mathcal{P}(\omega))^\omega$ a *pseudo-program* iff $x \leqslant_{\mathrm{lex}} p(x)$ for all x. Define iterates for pseudo-programs as for programs, at limits setting:

$$p_\alpha(x) = \limsup_{\beta < \alpha} p_\beta(x) .$$

(a) PSEUDO-PROGRAMMING THEOREM. *If p is a Borel pseudo-program, then the ultimate iterate p_Ω is BP.*

As a start on the proof, let Y_i be the set of $y = (y(j)\colon j \in (\omega) \in (\mathcal{P}(\omega))^\omega$ such that $y(j)$ is a singleton or $\emptyset$ for $j \leqslant i$, and is $\emptyset$ for $j > i$. Let $Y_i(x) = \{y \in Y_i\colon y \leqslant_{\mathrm{term}} x\}$. Let $Y = \bigcup_i Y_i$. In this notation we state:

(b) COMBINATORIAL LEMMA. *If p is a pseudo-program and for some α, x, i it is true that i is least such that $p_{\alpha+1}(x)(i) \neq p_\alpha(x)(i)$, then there exists an element of $H_i(p_{\alpha+1}(x))$ not belonging to $H_i(p_\beta(x))$ for any $\beta \leqslant \alpha$.*

Proof of Lemma. By induction on i, the case $i = 0$ being trivial. Suppose true for $j < i$, and p, α, x being as in the statement of the lemma, fix $n \in p_{\alpha+1}(x)(i) - p_\alpha(x)(i)$. Let β be least such that for all $j < i$, $p_\beta(x)(j) = p_\alpha(x)(j) (= p_{\alpha+1}(x)(j))$. We consider the case β a limit (treatment of the opposite case being even easier). Let $\gamma^* < \beta$ and $j^* < i$ be such that:

(i) $$\forall j < j^* \, \forall \gamma^* < \gamma < \beta \; p_\gamma(x)(j) = p_\alpha(x)(j),$$

(ii) $$\neg \exists \gamma' < \beta \, \forall \gamma' < \gamma < \beta \; p_\gamma(x)(j^*) = p_\alpha(x)(j^*).$$

Since $p_\beta(x) = \limsup_{\gamma < \beta} p_\gamma(x)$ there is a $\delta^* < \beta$ such that for all $\delta^* < \gamma < \beta$, $n \notin p_\gamma(x)$. Apply (ii) above to $\max(\gamma^*, \delta^*)$ to obtain a γ with $\gamma^*, \delta^* < \gamma < \beta$ and $p_{\gamma+1}(x)(j^*) \neq p_\gamma(x)(i^*)$. By (i), the lemma applies to this γ and j^* to yield a certain $z \in H_{j^*}(p_{\gamma+1}(x))$. Now let $y(j) = z(j)$ for $j < i$, $y(i) = \{n\}$, $y(j) = \emptyset$ for $j > i$. Then $y \in H_i(p_{\alpha+1}(x))$ is readily verified to satisfy the condition of the lemma. ■

Proof of Theorem modulo Lemma. Given our Borel pseudo-program p, we need to find Borel encoder e, program q, and decoder d with $\mathcal{J}(e, q, d) = p_\Omega$. To this end let J be the union of:

one copy of H^2, called the *order register*,

a copy of ω^2 for each $y \in H$, called the y-*auxiliary register*.

For $K \subseteq J$, the entry of K in the order register is a binary relation $R(K)$ whose field is a subset $D(K)$ of H. We call K *acceptable* iff $R(K)$ is a linear order on $D(K)$, and the y-auxiliary register is blank for $y \notin D(K)$. For acceptable K and $y \in D(K)$, let $P(K, y) \in (\mathcal{P}(\omega))^\omega$ be the z with $z(i) = \{j\colon (i, j) \in y\text{-auxiliary register of } K\}$. Let y_0 be the degenerate sequence, $y_0(i) = \emptyset$ for all i.

We define Borel encoder $e\colon (\mathcal{P}(\omega))^\omega \to \mathcal{P}(J)$ and decoder $d\colon \mathcal{P}(J) \to (\mathcal{P}(\omega))^\omega$ by:

$$D(e(x)) = \{y_0\} , \quad R(e(x)) = \{(y_0, y_0)\},$$
$$P(e(x), y_0) = x , \quad y\text{-auxiliary register of } e(x) \text{ blank for } y \neq y_0,$$
$$d(K)(i) = \bigcap_{y \in D(K)} \bigcup_{(y, y') \in R(K)} P(K, y')(i)$$
$$(= P(K, y) \text{ if there is an } R(K)\text{-largest } y \in D(K)).$$

It remains to define a Borel program $q\colon \mathcal{P}(J) \to \mathcal{P}(J)$ satisfying $d(q_\alpha(e(x))) = p_\alpha(x)$ for all α including Ω.

Intuitively, the idea is that iterated application of q to $e(x)$ results in $p_{\beta+1}(x)$ for larger and larger values of β being recorded in various auxiliary registers. The order register keeps track of the sequence in which the auxiliary registers were filled. The Combinatorial Lemma guarantees the existence of blank auxiliary registers for recording purposes.

Formally, let $q(K) = K$ unless K is acceptable, $p(d(K)) \neq d(K)$, and there exist a $j \in \omega$ and a $y \in H_j(p(d(K))) = D(K)$. In that case, take the least j and first y (in some wellordering of H) and let $q(K) = L$ where:

$$D(L) = D(K) \cup \{y\} , \quad R(L) = R(K) \cup \{(y', y)\colon y' \in D(L)\},$$
$$P(L, y) = p(d(K)) , \quad y'\text{-auxiliary registers of } K \text{ and } L \text{ agree for } y' \neq y.$$

A little thought shows that this formal definition does what it should according to the informal description above. ■

Of the results established for C-sets in Chapter C we have now seen that those pertaining to measure carry over to BP-sets (and indeed, they carry over to all BP^α). What of the material on category and selections? Here we have only partial results, with many questions still open. We cite:

(c) CONJECTURE. *The "Hard" Selection Theorem of* § 8(f) *holds for BP-sets.*

§ 17. **Effectivity.** Logicians will have noticed that set-theoretic programming is merely the topological or "boldface" side of a subject whose recursion-theoretic or "lightface" side is the theory of (nonmonotone) inductive definitions surveyed in [28]. In view of the connection we have established between programming and the R-transform it should be noted that there is an effective theory of the latter as well, developed by P. Hinman [31]. No doubt a thorough-going application of the methods and results of definability theory to the class of sets we have been considering (R-sets $= (G_{\delta\sigma} \cap F_{\sigma\delta})$-game sets $= P$-sets) would yield much information. Here we will offer but a sample of what can be expected.

We restrict our attention to $\mathscr{P}(\omega)$. For a field $\mathscr{H}$ of subsets of that space, we define a map f from the space to itself to be *$\mathscr{H}$-measurable* iff the inverse image of any clopen set belongs to $\mathscr{H}$. For σ-fields this agrees with the definition we have been using throughout. We let $\mathscr{H}'$ be the class of sets whose (revised) characteristic functions are of form $dp_\Omega e$ for d and e $\mathscr{H}$-measurable and p an $\mathscr{H}$-measurable program. Define fields $\mathscr{F}^\alpha$ by: $\mathscr{F}^0 =$ clopen sets, $\mathscr{F}^{\beta+1} = (\mathscr{F}^\beta)'$, $\mathscr{F}^\alpha = \bigcup_{\beta<\alpha} \mathscr{F}^\beta$ at limits.

(a) Proposition. (i) *$\mathscr{F}^1 =$ field generated by open sets,*

(ii) *analytic sets $\subseteq \mathscr{F}^2$,*

(iii) *primitive R-sets $\subseteq \mathscr{F}^3$,*

(iv) *$\mathscr{F}^\Omega = P$-sets $= R$-sets.*

(b) Proposition. *$\mathscr{F}^2 \subseteq \sigma$-field generated by analytic sets.*

Proof. (a) will be left as an exercise. For (b) we make use of a result attributed to Gandy in [28], plus a result from Miller's thesis [33].

Let p be an $\mathscr{F}^1$-measurable program. By (i) of (a), for any n, $p^{-1}[\{x \subseteq \omega\colon n \in x\}]$ is an F_σ set. Translating from the language of programming to the language of inductive definability, p is a Σ^0_2 boldface inductive definition. Let it be Σ^0_2 lightface in parameter $t \subseteq \omega$. Gandy's result is:

closure ordinal of Σ^0_2-in-t inductive definitions
$=$ closure ordinal of Π^0_1-in-t inductive definitions
$=$ least ordinal not recursive in t.

Translating back into the language of programming, we have $p_\Omega(x) = p_{\lambda(x)}(x)$ where $\lambda(x) =$ least ordinal not recursive in t and x.

Let L be a vocabulary containing the binary predicate $\in$ of the language of set theory, two constants $\bar{t}$ and $\bar{x}$, and a binary predicate P. For our fixed parameter t and any $x \subseteq \omega$, let $\varphi(x)$ be the sentence of the infinitary logic $L(\omega_1, \omega)$ which is the conjunction of (i) the Kripke-Platek axioms for admissible sets, (ii) the statement that "the natural numbers are standard", (iii) complete descriptions of t and x (i.e. all statements to the effect that $n \in \bar{t}$ for $n \in t$, $n \notin \bar{t}$ for $n \notin t$, and similarly for x), (iv) the statement that $P(\alpha, n)$ holds iff α is an ordinal and n is a natural number and $n \in p_\alpha(x)$ (written out using the name $\bar{x}$ for x, the name $\bar{t}$ for t, and the Σ^0_2-in-t definition of p). Miller's result is a selection theorem implying:

there exists a Borel function assigning each $x \subseteq \omega$ a binary relation $R(x)$ on ω such that $(\omega, R(x))$ is a model of $\varphi(x)$.

Now it is a well-known fact ("Ville's Lemma") that the wellfounded part of

a model of the Kripke-Platek axioms is again a model of those axioms; hence the wellfounded part of $(\omega, R(x))$ is a model of $\varphi(x)$. Since $\{(R, k)\colon k \in$ wellfounded part of $(\omega, R)\}$ is a co-analytic subset of $\mathscr{P}(\omega^2) \times \omega$, we see:

there exists a function measurable w.r.t. the σ-field generated by analytic sets, assigning each $x \subseteq \omega$ a binary relation $Q(x)$ on ω such that $(\omega, Q(x))$ is a wellfounded model of $\varphi(x)$.

Now when a wellfounded model of $\varphi(x)$ is replaced by the isomorphic "admissible set", the latter contains all ordinals recursive in t and x. Hence we have $n \in p_{\lambda(x)}(x) = \bigcup_{\alpha<\lambda(x)} p_\alpha(x)$ iff $(\omega, Q(x))$ is a model of the statement that $\exists \alpha P(\alpha, n)$. Since the class of models of the latter is a Borel set, we see putting everything together that the inverse image under p of $\{x\colon n \in x\}$ is the inverse image under Q of a Borel set, completing the proof. ∎

The following suggests itself, but lacking sufficient knowledge of inductive definability theory the authors have been unable to settle it:

(c) Conjecture. *$\mathscr{F}^3 \subseteq$ smallest σ-field stable under operation $\mathscr{A}$ and containing the primitive R-sets.*

§ 18. A glimpse beyond. We consider what looks at first glance like a slight modification of the pseudo-programming of § 16. Let $\mathscr{X} = (\mathscr{P}(\omega))^{\Omega(\omega)}$. For $x, y \in \mathscr{X}$ define:

$$x \leqslant_{\mathrm{lex}} y \leftrightarrow \forall s\big(\neg(x(s) \subseteq y(s)) \to \exists t \lhd s(x(t) \subsetneqq y(t))\big).$$

Define a *meta-program* to be a $p\colon \mathscr{X} \to \mathscr{X}$ satisfying $x \leqslant_{\mathrm{lex}} p(x)$ for all x. Define iterates of p by treating limits as follows:

$$p_\alpha(x)(s) = \begin{cases} \bigcap_{\gamma<\alpha} \bigcup_{\gamma<\beta<\alpha} p_\beta(x)(s) & \text{if} \\ \exists \gamma<\alpha\, \forall t, \beta\big(t \lhd s \;\&\; t \neq s \;\&\; \gamma<\beta<\alpha \to p_\beta(t) = p_\gamma(t)\big), \\ \varnothing \text{ otherwise.} \end{cases}$$

A function is *Borel-meta-programmable* (*BMP*) iff it is of form $dp_\Omega e$ for some Borel e and d and some Borel meta-program p. A set is *BMP* iff its characteristic function is. We state without proof:

(a) Proposition. *The class of BMP sets is stable under (plain) programming.*

(b) Proposition. *R-sets $= P$-sets $\subsetneqq$ BMP-sets.*

We guess:

(c) Conjecture. *BMP-sets $\subsetneqq (F_{\sigma\delta\sigma} \cap G_{\delta\sigma\delta})$-game sets.*

We have considered the following classes, in order of size: (i) the C-sets of Selivanovskii; (ii) the Borel-programmable or BP-sets of Blackwell; (iii) the R-sets of Kolmogorov; (iv) the Borel-game sets of Vaught, which (by Martin's Borel Determinateness Theorem) are included in: (v) the absolutely Δ^1_2 sets of Solovay [30]. It is worth remarking that if Γ is an absolutely Δ^1_2 operation, so is $R\Gamma$. Also (as

D. Normann has pointed out to one of the authors) it is not hard to see that the absolutely Δ_2^1 sets cannot be enlarged by programming; the same is true of meta-programming.

Postscript

All of the results presented above are illuminated, and some were anticipated, by developments in *effective* descriptive set theory. Under this heading should be listed the work of: Aczel on so-called next-quantifiers, Cenzer on monotone *vs.* nonmonotone Σ_1^1-induction, Harrington & Kechris on monotone *vs.* nonmonotone Δ_2^1-induction, Harrington on the R-transform and the first strongly admissible or nonprojectible ordinal, John on winning strategies for Σ_3^0-games, Moschovakis on the preservation of the prewellordering and other properties by game-quantifiers, and Solovay on Σ_1^1-induction and winning strategies for Σ_2^0-games. Unfortunately, most of the most relevant material was unpublished and unavailable to us while we were engaged in the work reported above. Now some of it has at last appeared in the long-awaited volume: Y. N. Moschovakis, *Descriptive Set Theory*, North Holland, Amsterdam 1980. See especially parts 6E and 7C.

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Sur les classes de Baire des fonctions de deux variables

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Résumé. Dans cet article on introduit la définition de certaines propriétés d'une famille de fonctions d'une variable réelle qui sont les propriétés bien connues d'une fonction ("être mesurable", "être de classe 1 de Baire", "être ponctuellement discontinue" et "avoir la propriété de Baire") de la même façon pour toute fonction de la famille considérée et on démontre quelques théorèmes sur les fonctions de deux variables dont les sections par rapport à l'une de deux variables ont l'une des propriétés considérées.

Dans cet article j'introduis la définition suivante:

DÉFINITION 1. Soient R l'espace des nombres réels et T un ensemble d'indices. On dit que *la famille de fonctions* f_t: $R \to R$ $(t \in T)$ *a la propriété*:

(A_1) lorsque, quels que soient le nombre $\varepsilon > 0$ et l'ensemble mesurable (au sens de Lebesgue) A de mesure positive, il existe un ensemble mesurable $B \subset A$ tel que $m(B) > 0$ (m désigne la mesure de Lebesgue dans R) et $\operatorname{osc}_B f_t \leqslant \varepsilon$ pour tout $t \in T$;

(A_2) lorsqu'il existe pour tout ensemble fermé, non vide $A \subset R$ un point $x_0 \in A$ tel que les fonctions partielles f_t/A $(t \in T)$ sont équicontinues au point x_0; c'est-à-dire qu'il existe pour tout nombre $\varepsilon > 0$ un nombre $\delta > 0$ tel que $|f_t(x) - f_t(x_0)| < \varepsilon$ pour tout $x \in A \cap (x_0 - \delta, x_0 + \delta)$ et pour tout $t \in T$;

(A_3) lorsque, quel que soit l'intervalle ouvert $J \subset R$, il existe un point $x_0 \in J$ auquel les fonctions f_t $(t \in T)$ sont équicontinues; et

(A_4) lorsque, quels que soient le nombre $\varepsilon > 0$ et l'ensemble $A \subset R$ ayant la propriété de Baire et étant de deuxième catégorie, il existe un ensemble $B \subset A$ de deuxième catégorie, ayant la propriété de Baire et tel que $\operatorname{osc}_B f_t \leqslant \varepsilon$ pour tout $t \in T$.

Je démontre ensuite quelques théorèmes sur les fonctions de deux variables dont les sections par rapport à l'une de deux variables ont l'une des propriétés (A_i) $(i = 1, 2, \ldots, 4)$. En particulier, je donne dans cet article la résolution négative du problème suivant:

PROBLÈME (Problème 2, [1]). Existe-t-il une fonction f: $X \times Y \to R$ qui n'est pas de première classe de Baire et dont les coupes f^y sont de première classe de Baire et les coupes f_x sont semi-équicontinues supérieurement? (X et Y étant ici des espaces métriques, séparables et complets).