The number of metrizable spaces

by

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Abstract. The theorems in this paper solve problems of the following sort. Given infinite cardinals $\boldsymbol{m}$ and $\boldsymbol{n}$ with $\boldsymbol{m} \leqslant \boldsymbol{n} \leqslant \boldsymbol{m}^{\omega}$ and a topological property P, find the number of topologically distinct metrizable spaces having weight $\boldsymbol{m}$, cardinality $\boldsymbol{n}$, and property P. The properties considered include connectedness, local compactness, and Čech completeness.

1. Introduction. Let X be an infinite metrizable space of weight $\boldsymbol{m}$. It is well known that $\boldsymbol{m}$ and $|X|$ satisfy the inequality $\boldsymbol{m} \leqslant |X| \leqslant \boldsymbol{m}^{\omega}$. This suggests the following problem. Given infinite cardinals $\boldsymbol{m}$ and $\boldsymbol{n}$ with $\boldsymbol{m} \leqslant \boldsymbol{n} \leqslant \boldsymbol{m}^{\omega}$, find the number of topologically distinct metrizable spaces having weight $\boldsymbol{m}$ and cardinality $\boldsymbol{n}$. The main result in this paper, Theorem 4.11 in § 4, states that the number of such spaces is $2^{\boldsymbol{n}}$. Lozier and Marty [LM] have proved that for each infinite cardinal $\boldsymbol{m}$ the number of topologically distinct continua of weight $\boldsymbol{m}$ is $2^{\boldsymbol{m}}$. (A continuum is a compact connected Hausdorff space; it need not be metrizable.) We obtain analogues of this result. For example, we show that for each infinite cardinal $\boldsymbol{m}$ the number of topologically distinct connected metrizable spaces of weight $\boldsymbol{m}$ is $2^{\boldsymbol{m}^{\omega}}$ and the number of topologically distinct connected completely metrizable spaces of weight $\boldsymbol{m}$ is $2^{\boldsymbol{m}}$. (See § 4 and § 5.) In § 6 we prove that for each infinite cardinal $\boldsymbol{m}$ the number of topologically distinct locally compact locally countable metrizable spaces of cardinality $\boldsymbol{m}$ is $\omega_1 \cdot \nu(\boldsymbol{m})$, where $\nu(\boldsymbol{m})$ is the number of cardinals $\leqslant \boldsymbol{m}$. This result can be regarded as an extension to higher cardinals of the classical result due to Mazurkiewicz and Sierpiński [MS] that the number of topologically distinct compact countable metrizable spaces is ω_1. In § 7 we show that the number of topologically distinct locally compact metrizable spaces of cardinality $\boldsymbol{m}$ is $\omega_1 \cdot \nu(\boldsymbol{m})$ if $\boldsymbol{m} < 2^{\omega}$ and $\nu(\boldsymbol{m})^{2^{\omega}}$ if $\boldsymbol{m} \geqslant 2^{\omega}$. Finally in § 8 we show that for each $\boldsymbol{m} \geqslant 2^{\omega}$ the number of topologically distinct connected paracompact Hausdorff spaces of cardinality $\boldsymbol{m}$ is the maximum possible, namely $2^{2^{\boldsymbol{m}}}$.

2. Notation, definitions, and known results. We adopt the following set-theoretic notation: k and n denote natural numbers; ω is the first infinite ordinal and also the set of natural numbers; ω_1 is the first uncountable ordinal; $\boldsymbol{m}$, $\boldsymbol{n}$, and $\boldsymbol{p}$ denote cardinal numbers and α, β, and γ denote ordinal numbers; $|E|$ is the cardinality of the set E; the set of real numbers is denoted by $\boldsymbol{R}$ and $\boldsymbol{R}^n$ denotes Euclidean n-space. Regarding separation axioms, we assume that regular, paracompact, locally compact, and compact spaces are always Hausdorff.

Let $m \geqslant 1$, let X be a connected space with more than one point, let $p \in X$. Then p is a *cut point of X of order m* if $X-\{p\}$ has m components. If p is a cut point of X of order $m>1$, then p is called a *cut point of X*; otherwise p is a *noncut point* of X. It is clear that if p is a cut point of X of order m, and h is a homeomorphism from X onto Y, then $h(p)$ is a cut point of Y of order m.

We shall make frequent use of the sum $\bigoplus_{s \in S} X_s$ of a collection $\{X_s: s \in S\}$ of topological spaces. The reader is referred to pp. 103–106 in [E] for the definition and a list of basic properties of $\bigoplus_{s \in S} X_s$. It is especially useful to remember that if $\{X_s: s \in S\}$ is a pairwise disjoint collection of open sets which covers a topological space X, then $X = \bigoplus_{s \in S} X_s$. We adopt the convention that in constructing $\bigoplus_{s \in S} X_s$, one tacitly assumes that the underlying sets for the spaces $\{X_s: s \in S\}$ are pairwise disjoint. The discrete space of cardinality m, denoted D_m, is the sum of m one-point spaces. Alexandroff [A] has proved that if X is a locally separable metrizable space, then $X = \bigoplus_{s \in S} X_s$, where each X_s is separable. (See p. 359 in [E].) It should be noted that if $X = \bigoplus_{s \in S} X_s$, then $X^{(\alpha)} = \bigoplus_{s \in S} X_s^{(\alpha)}$ for all $\alpha \geqslant 1$. (Here $X^{(\alpha)}$ denotes the derived set of X of order α.)

We now discuss a technique for constructing a new connected metrizable space from a given collection of connected metrizable spaces. (This construction generalizes the well known hedgehog with m spines.) Let $\{(X_s, \mathcal{T}_s): s \in S\}$ be a collection of connected metrizable spaces, each having more than one point. We tacitly assume that $\{X_s: s \in S\}$ is a pairwise disjoint collection. For each $s \in S$ let d_s be a metric on X_s compatible with $\mathcal{T}_s$, let $p_s \in X_s$, and let $Y_s = X_s - \{p_s\}$. Let $p(S)$ be a point not in $\bigcup_{s \in S} X_s$ and let $X(S) = \big(\bigcup_{s \in S} Y_s\big) \cup \{p(S)\}$. Define d: $X(S) \times X(S) \to R$ as follows: if $x, y \in Y_s$, $d(x, y) = d_s(x, y)$; if $x \in Y_s$, $y \in Y_t$, and $s \neq t$, $d(x, y) = d_s(x, p_s) + d_t(p_t, y)$; if $x \in Y_s$, $d\big(x, p(S)\big) = d\big(p(S), x\big) = d_s(x, p_s)$; $d\big(p(S), p(S)\big) = 0$. One can show that d is a metric on $X(S)$, that Y_s is an open subset of $X(S)$, and that the function from X_s into $X(S)$ which is the identity on Y_s and takes p_s to $p(S)$ is a homeomorphism from X_s onto the subspace $Y_s \cup \{p(S)\}$ of $X(S)$. We call $X(S)$ the *star-space determined by* $\{(X_s, d_s): s \in S\}$ *and* $\{p_s: s \in S\}$, and $p(S)$ is called the *adjunction point of* $X(S)$. It should be emphasized that if d_s is replaced by an equivalent metric d_s' for infinitely many $s \in S$, then the two star-spaces obtained need not be homeomorphic. However, in most applications of the star-space construction this is unimportant; in such cases we will omit all mention of the metrics d_s and refer to $X(S)$ as a star-space determined by $\{X_s: s \in S\}$ and $\{p_s: s \in S\}$.

For future reference we list some basic properties of the star-space construction. (1) The space $X(S)$ is connected. (2) $X(S) - \{p(S)\} = \bigoplus_{s \in S} Y_s$. (3) If $|S| = m \geqslant 1$ and each p_s is a noncut point of X_s, then $p(S)$ is a cut point of $X(S)$ of order m. (4) Let $m \geqslant 1$ and let $x \in Y_s$ for some $s \in S$. Then x is a cut point of $X(S)$ of order m if and only if x is a cut point of X_s of order m. (5) If d_s is a complete metric on X_s for all $s \in S$, then the star-space determined by $\{(X_s, d_s): s \in S\}$ and $\{p_s: s \in S\}$ is a complete metric space. (Thus, if each X_s is compact then every star-space determined by $\{X_s: s \in S\}$ and $\{p_s: s \in S\}$ is a complete metric space.)

Let $|S| = m \geqslant \omega$ and for $s \in S$ let $X_s = [0, 1] \times \{s\}$ and $p_s = (0, s)$. Let d_s be the "natural" metric on X_s obtained by using the Euclidean metric on $[0, 1]$. The star-space determined by $\{(X_s, d_s): s \in S\}$ and $\{p_s: s \in S\}$ is called the *hedgehog with m spines* and is denoted $J(m)$. (See p. 314 in [E] or p. 95 in [N].) Note that $J(m)$ is a pathwise connected complete metric space of weight m and cardinality $m \cdot 2^{\omega}$.

3. Constructing topologically distinct spaces. In this section we give several propositions which will be used in §§ 4 and 5 when constructing large collections of topologically distinct spaces. Most of these results belong to the folklore or generalize well known arguments. For example, Proposition 3.3 generalizes the argument given on p. 263 in [K] that the number of topologically distinct subsets of a separable metrizable space of cardinality 2^{ω} is $2^{2^{\omega}}$.

PROPOSITION 3.1. *Let $\{X_s: s \in S\}$ be a collection of topologically distinct connected metrizable spaces, each having more than one point, and let p_s be a noncut point of X_s for each $s \in S$. Let $S_1, S_2 \subseteq S$, and for $k = 1, 2$ let $X(S_k)$ be a star-space determined by $\{X_s: s \in S_k\}$ and $\{p_s: s \in S_k\}$. If h is a homeomorphism from $X(S_1)$ onto $X(S_2)$ which takes the adjunction point $p(S_1)$ of $X(S_1)$ to the adjunction point $p(S_2)$ of $X(S_2)$, then $S_1 = S_2$.*

PROPOSITION 3.2. *Let A_1 and A_2 be two spaces, each dense in itself, let B_1 and B_2 be scattered spaces. If $A_1 \oplus B_1$ is homeomorphic to $A_2 \oplus B_2$, then $A_1 \approx A_2$ and $B_1 \approx B_2$.*

PROPOSITION 3.3. *Let X be a T_1 space of weight m, let $\mathcal{A}$ be a collection of subsets of X such that $|\mathcal{A}| > 2^m$. Then there is a subcollection $\mathcal{A}_0$ of $\mathcal{A}$ such that $|\mathcal{A}_0| = |\mathcal{A}|$ and no two distinct elements of $\mathcal{A}_0$ are homeomorphic.*

Proof. First note that $|X| \leqslant 2^m$ and every subset of X has a dense subset of cardinality at most m. Define an equivalence relation $\sim$ on $\mathcal{A}$ as follows: $A \sim B$ if and only if A is homeomorphic to B. For each $A \in \mathcal{A}$ the number of continuous functions from A into X is at most $(2^m)^m = 2^m$; consequently each equivalence class of $\sim$ has cardinality at most 2^m. Since $|\mathcal{A}| > 2^m$, it follows that the number of distinct equivalence classes is $|\mathcal{A}|$. The desired subcollection $\mathcal{A}_0$ of $\mathcal{A}$ is obtained by choosing a representative element from each equivalence class of $\sim$.

4. The number of metrizable spaces. Let m and n satisfy $\omega \leqslant m \leqslant n \leqslant m^{\omega}$. In this section we show that the number of topologically distinct metrizable spaces having weight m and cardinality n is 2^n. It is easy to show that the number of such spaces is at most 2^n (see Proposition 4.1), and so the problem reduces to constructing the required number of spaces. The solution of this problem naturally factors into two cases: $n \geqslant 2^{\omega}$ and $n < 2^{\omega}$. For the case $n \geqslant 2^{\omega}$ it is convenient to construct connected metrizable spaces, since connectedness is the key property used in showing the spaces non-homeomorphic. For the case $n<2^\omega$ the key tool is the Cantor-Bendixson theorem and the Mazurkiewicz–Sierpiński result on the number of scattered subsets of R.

PROPOSITION 4.1. *The number of topologically distinct metrizable spaces of cardinality n is at most 2^n.*

Proof. Clearly we may assume that $n\geqslant\omega$. Let X be a set with $|X| = n$. A metric on X is a function from $X\times X$ into R. Hence the total number of metrics on X is at most $(2^\omega)^{n\cdot n} = 2^n$, and so there are at most 2^n metrizable topologies on X.

LEMMA 4.2. *There exist 2^ω topologically distinct metrizable spaces, each having these properties: compact, connected, weight ω, cardinality 2^ω, infinitely many noncut points, no cut point of order $>\omega$.*

Proof. For each $n\geqslant 2$ let $B_n = \{x\in R^n\colon \|x\|\leqslant 1\}$, let p_n be the origin of R^n, and let d_n be the usual Euclidean metric on B_n. As is well known, B_n is compact, connected, has no cut points, and B_n is not homeomorphic to B_k whenever $n\neq k$. For each $n\geqslant 2$ let ϱ_n be an equivalent metric on B_n defined by $\varrho_n(x, y) = d_n(x, y)/2^n$. Let $\mathscr{S} = \{S\colon S\subseteq\omega,\ S \text{ infinite and } S\cap\{0, 1\} = \emptyset\}$, and for each $S\in\mathscr{S}$ let $X(S)$ be the star-space determined by $\{(B_n, \varrho_n)\colon n\in S\}$ and $\{p_n\colon n\in S\}$. Then $X(S)$ is a connected metrizable space of weight ω and cardinality 2^ω, $p(S)$ is a cut point of $X(S)$ of order ω, and $p(S)$ is the only cut point of $X(S)$. Moreover, $X(S)$ is compact since each B_n is compact and $\lim\limits_{n\to\infty}(\text{diameter of } (B_n, \varrho_n)) = 0$. Finally, let S_1 and S_2 belong to $\mathscr{S}$ and suppose there is a homeomorphism h from $X(S_1)$ onto $X(S_2)$. Then h must take $p(S_1)$ to $p(S_2)$ and so $S_1 = S_2$ by Proposition 3.1. Consequently $\{X(S)\colon S\in\mathscr{S}\}$ is a collection of 2^ω topologically distinct spaces, and each space has the required properties.

Remark. The technique used by Lozier and Marty also gives a collection of 2^ω topologically distinct spaces satisfying all of the conditions in Lemma 4.2. (See p. 273 of [LM].)

LEMMA 4.3. *Let $\omega\leqslant m\leqslant 2^\omega$. Then there exist 2^m topologically distinct connected completely metrizable spaces of weight m and cardinality 2^ω, each having infinitely many noncut points.*

Proof. Let $\{X_\alpha\colon 0\leqslant\alpha<2^\omega\}$ be a collection of topologically distinct spaces as in Lemma 4.2. We may assume that $m>\omega$. For each $\alpha<2^\omega$ let p_α be a noncut point of X_α. Let $\mathscr{S} = \{S\colon S\subseteq 2^\omega,\ |S| = m\}$, and for each $S\in\mathscr{S}$ let $X(S)$ be a star-space determined by $\{X_\alpha\colon \alpha\in S\}$ and $\{p_\alpha\colon \alpha\in S\}$. Clearly $X(S)$ is a connected metrizable space of weight m and cardinality 2^ω with infinitely many noncut points, and $X(S)$ is complete since each X_α is compact. Since we are assuming that $m>\omega$, it follows that the adjunction point $p(S)$ is the only point of $X(S)$ which is a cut point of order m. Hence by Proposition 3.1, $X(S_1)$ is not homeomorphic to $X(S_2)$ whenever $S_1\neq S_2$. Thus $\{X(S)\colon S\in\mathscr{S}\}$ is a collection of 2^m topologically distinct spaces, each having the required properties.

Remark. In § 5 we shall make use of the fact that the spaces constructed in 4.3 are topologically complete.

LEMMA 4.4. *Let X and Y be connected spaces, each having more than one point. Then every point of $X\times Y$ is a noncut point of $X\times Y$.*

LEMMA 4.5. *Let m and n be cardinals with $2^\omega\leqslant m\leqslant n\leqslant m^\omega$. Then there is a connected metrizable space of weight m and cardinality n with no cut points.*

Proof. Let $J(m)$ be the hedgehog with m spines and let Y be the product of ω copies of $J(m)$. Then Y is a pathwise connected metrizable space of weight m and cardinality m^ω. Let $p\in Y$ and let S be a subset of Y with $|S| = n$. For each $x\in S$ let f_x be a continuous function from $[0, 1]$ into Y such that $f(0) = p$ and $f(1) = x$, and let $A_x = f_x([0, 1])$. Note that A_x is connected and $|A_x|\leqslant 2^\omega$. Let $Z = \bigcup\limits_{x\in S} A_x$; clearly Z is a connected metrizable space of weight $\leqslant m$ and cardinality n. Finally, $Z\times J(m)$ is a connected metrizable space of weight m and cardinality n with no cut points. (See Lemma 4.4.)

LEMMA 4.6. *Let m and n be cardinals with $2^\omega\leqslant m\leqslant n\leqslant m^\omega$. Then there exist 2^m topologically distinct connected metrizable spaces of weight m and cardinality n, each having infinitely many noncut points.*

Proof. The proof is by induction on m. The case $m = 2^\omega$ follows from Lemma 4.3. Now let $m>2^\omega$ and assume that the theorem is true for all p with $2^\omega\leqslant p<m$. Let n be a fixed cardinal with $m\leqslant n\leqslant m^\omega$; our objective is to construct 2^m topologically distinct connected metrizable spaces, each having weight m, cardinality n, and infinitely many noncut points. For each p with $2^\omega\leqslant p<m$ let $\mathscr{A}_p$ be a collection of 2^p topologically distinct connected metrizable spaces, each having weight p, cardinality p, and infinitely many noncut points. Let $\mathscr{A} = \bigcup\limits_{2^\omega\leqslant p<m}\mathscr{A}_p$; note that no two distinct elements of $\mathscr{A}$ are homeomorphic and $|\mathscr{A}|\geqslant m$. (If $|\mathscr{A}| = p<m$, then $\mathscr{A}_p\subseteq\mathscr{A}$ and $|\mathscr{A}_p| = 2^p$ gives a contradiction.) Let $\{X_\alpha\colon 0\leqslant\alpha<m\}$ be a collection of topologically distinct spaces such that $X_\alpha\in\mathscr{A}$ for $\alpha\geqslant 1$ and X_0 is a connected metrizable space with weight m, cardinality n, and no cut points. For each $\alpha<m$ let p_α be a noncut point of X_α. Let $\mathscr{S} = \{S\colon S\subseteq m,\ 0\in S \text{ and } |S| = m\}$, and for each $S\in\mathscr{S}$ let $X(S)$ be a star-space determined by $\{X_\alpha\colon \alpha\in S\}$ and $\{p_\alpha\colon \alpha\in S\}$. It is clear that each $X(S)$ is a connected metrizable space having weight m, cardinality n, and infinitely many noncut points. Moreover, the adjunction point $p(S)$ is the only point of $X(S)$ which is a cut point of $X(S)$ of order m. (Recall that X_0 has no cut points and $|X_\alpha|<m$ for $\alpha\geqslant 1$.) Hence by Proposition 3.1 $X(S_1)$ is not homeomorphic to $X(S_2)$ whenever $S_1\neq S_2$. Thus $\{X(S)\colon S\in\mathscr{S}\}$ is a collection of 2^m topologically distinct spaces, each having the required properties.

THEOREM 4.7. *Let m and n be infinite cardinals with $m\leqslant n\leqslant m^\omega$ and $n\geqslant 2^\omega$. Then the number of topologically distinct connected metrizable spaces of weight m and cardinality n is 2^n.*

Proof. By Proposition 4.1 the number of such spaces is at most 2^n, and so it remains to prove the existence of 2^n such spaces. First assume $2^m = 2^n$. If $m\leqslant 2^\omega$ then the only allowable value for n is 2^ω and we are finished by Lemma 4.3. If $m\geqslant 2^\omega$ then we are finished by Lemma 4.6.

Now assume $2^m < 2^n$. Let Y be a connected metrizable space of weight m and cardinality n. (If $m \geqslant 2^\omega$ use Lemma 4.5; if $m \leqslant 2^\omega$ use Lemma 4.3.) Let $X = Y \times Y$, let p be a point of Y, let $A = \{p\} \times Y$, and for each $y \in Y$ let $A_y = Y \times \{y\}$. Finally for each non-empty subset S of Y let $X(S) = A \cup (\bigcup_{y \in S} A_y)$. It is clear that each $X(S)$ is a connected metrizable space of weight m and cardinality n, and that $X(S_1) \neq X(S_2)$ whenever $S_1 \neq S_2$. Let $\mathscr{A} = \{X(S)\colon S \subseteq Y, S \neq \emptyset\}$. Then $|\mathscr{A}| = 2^n > 2^m$, so by 3.3 there is a subset $\mathscr{A}_0$ of $\mathscr{A}$ with $|\mathscr{A}_0| = 2^n$ such that no two distinct elements of $\mathscr{A}_0$ are homeomorphic. This completes the proof of the case $2^m < 2^n$.

Corollary 4.8. *For each cardinal $m \geqslant \omega$ the number of topologically distinct connected metrizable spaces of weight m is 2^{m^ω}. In particular the number of topologically distinct connected metrizable spaces of weight ω is 2^{2^ω}.*

Corollary 4.9. *For each cardinal $n \geqslant 2^\omega$ the number of topologically distinct connected metrizable spaces of cardinality n is 2^n.*

Lemma 4.10 (Mazurkiewicz and Sierpiński). *The number of topologically distinct scattered subsets of R is 2^ω.*

Theorem 4.11. *Let m and n be infinite cardinals with $m \leqslant n \leqslant m^\omega$. Then the number of topologically distinct metrizable spaces of weight m and cardinality n is 2^n.*

Proof. By 4.1, 2^n is an upper bound. By Theorem 4.7 we may assume that $n < 2^\omega$, and by Lemma 4.10 we may assume that $n > \omega$. (Recall that every scattered subset of R is countable.) In summary we have $\omega < n < 2^\omega$ and $m \leqslant n$, and we want to construct 2^n topologically distinct metrizable spaces, each of weight m and cardinality n. We consider two cases, namely $2^\omega = 2^n$ and $2^\omega < 2^n$.

First suppose $2^\omega = 2^n$. We begin by constructing a metrizable space X of weight m and cardinality n which is dense in itself. Let $A \subseteq R$, $|A| = n$. By the Cantor–Bendixson theorem, $A = B \cup C$, where B is dense in itself and C is countable. Since $|A| = n > \omega$, $|B| = n$. Let X be the sum of m copies of B; clearly X is a metrizable space of weight m and cardinality n which is dense in itself. Now let $\{C_\alpha\colon 0 \leqslant \alpha < 2^\omega\}$ be a collection of 2^ω topologically distinct scattered subsets of R, and for each $\alpha < 2^\omega$ let $X_\alpha = X \oplus C_\alpha$. Clearly each X_α is a metrizable space of weight m and cardinality n, and from Proposition 3.2 it follows that X_α is not homeomorphic to X_β for $\alpha \neq \beta$.

Now suppose $2^\omega < 2^n$. Let $\mathscr{A}$ be all subsets of R of cardinality n; $|\mathscr{A}| = 2^n$. Let $\mathscr{B}$ be all subsets of R of cardinality n which are dense in themselves, and let $\mathscr{C}$ be all countable subsets of R. By the Cantor-Bendixson theorem,

$$\mathscr{A} \subseteq \{B \cup C\colon B \in \mathscr{B}, C \in \mathscr{C}\}.$$

Since $|\mathscr{A}| = 2^n$, $|\mathscr{C}| = 2^\omega$, and $2^n > 2^\omega$, it follows that $|\mathscr{B}| = 2^n$. By Proposition 3.3 there is a subcollection $\mathscr{B}_0 = \{B_\alpha\colon 0 \leqslant \alpha < 2^n\}$ of $\mathscr{B}$ such that B_α is not homeomorphic to B_β whenever $\alpha \neq \beta$. For each $\alpha < 2^n$ let $X_\alpha = B_\alpha \oplus D_m$. Then $\{X_\alpha\colon 0 \leqslant \alpha < 2^n\}$ is a collection of 2^n metrizable spaces, each of weight m and cardinality n, and by Proposition 3.2 X_α is not homeomorphic to X_β whenever $\alpha \neq \beta$.

Corollary 4.12. *For each cardinal $n \geqslant \omega$ the number of topologically distinct metrizable spaces of cardinality n is 2^n.*

5. The number of connected completely metrizable spaces. By Corollary 4.8 we know that the number of topologically distinct connected metrizable spaces of weight m is 2^{m^ω}, and Lozier and Marty have proved that the number of topologically distinct continua of weight m is 2^m. In this section we show that the number of topologically distinct connected completely metrizable spaces of weight m is 2^m. We begin by stating a result due to A. H. Stone [St] on the possible values of $|X|$ whenever X is a completely metrizable space of weight m.

Proposition 5.1 (Stone). *Let X be an infinite completely metrizable space of weight m. Then $|X| = m$ or $|X| = m^\omega$.*

Proposition 5.2. *The number of topologically distinct Čech-complete spaces of weight m is at most 2^m.*

Proof. Clearly we may assume that $m \geqslant \omega$. Let X be a Čech-complete space of weight m. Then there is a homeomorphism c taking X into I^m. (See p. 115 in [E].) Now $\overline{c(X)}$ is a compactification of X and X is Čech-complete, so $\overline{c(X)} - c(X)$ is an $F\sigma$-set in $\overline{c(X)}$ and hence in I^m. Thus the complement of $c(X)$ is the union of an open set (namely $I^m - \overline{c(X)}$) and an $F\sigma$-set in I^m. Let $\mathscr{G} = \{G\colon G \subseteq I^m;\ G$ is the union of an open set and an $F\sigma$-set$\}$. The proof is complete if we can show that $|\mathscr{G}| \leqslant 2^m$. Since the weight of I^m is m, the number of open sets is at most 2^m and the number of $F\sigma$-sets is at most 2^m, so $|\mathscr{G}| \leqslant 2^m$.

Lemma 5.3. *For each cardinal $m \geqslant 2^\omega$ there is a connected completely metrizable space of weight m and cardinality m^ω with no cut points.*

Proof. Such a space can be obtained by taking the product of ω copies of $J(m)$, the hedgehog with m spines.

Theorem 5.4. *For each cardinal $m \geqslant \omega$ the number of topologically distinct connected completely metrizable spaces of weight m is 2^m. If $m < 2^\omega$ each such infinite space has cardinality 2^ω; if $m \geqslant 2^\omega$ each such space has cardinality m or m^ω and the number of each type is 2^m.*

Proof. By Proposition 5.2 we know that 2^m is an upper bound. First suppose $m < 2^\omega$. The existence of 2^m spaces with the desired properties follows from Lemma 4.3. Moreover one can easily show that an infinite connected metrizable space of weight $m < 2^\omega$ has cardinality 2^ω. (See p. 433 in [E].)

Now assume $m \geqslant 2^\omega$. By Proposition 5.1 we know that a completely metrizable space of weight m must have cardinality m or m^ω. The proof is complete if we can prove the following. Let $m \geqslant 2^\omega$, let $n = m$ or $n = m^\omega$; then there exist 2^m topologically distinct connected completely metrizable spaces, each having weight m, cardinality n, and infinitely many noncut points. The proof, which is similar to that of Lemma 4.6, is by induction on m. For $m = 2^\omega$, $m = m^\omega = 2^\omega$ and we are finished by Lemma 4.3. Now let $m > 2^\omega$ and assume true for all p with $2^\omega \leqslant p < m$. We shall construct 2^m topologically distinct connected completely metrizable spaces, each having weight m, cardinality m^{ω}, and infinitely many noncut points. (The proof for "cardinality m" is similar but simpler.) For each p with $2^{\omega}\leqslant p<m$ let $\mathscr{A}_p$ be a collection of 2^p topologically distinct connected completely metrizable spaces, each having weight p, cardinality p, and infinitely many noncut points, and let $\mathscr{A}=\bigcup_{2^{\omega}\leqslant p<m}\mathscr{A}_p$. Let $\{X_\alpha\colon 0\leqslant\alpha<m\}$ be a collection of topologically distinct spaces such that $X_\alpha\in\mathscr{A}$ for $\alpha\geqslant 1$ and X_0 is a connected completely metrizable space with weight m, cardinality m^{ω} and no cut points. For each $\alpha<m$ let p_α be a noncut point of X_α and let d_α be a complete metric on X_α. Let $\mathscr{S}=\{S\colon S\subseteq m, 0\in S \text{ and } |S|=m\}$, and for each $S\in\mathscr{S}$ let $X(S)$ be the star-space determined by $\{(X_\alpha, d_\alpha)\colon \alpha\in S\}$ and $\{p_\alpha\colon \alpha\in S\}$. Then $\{X(S)\colon S\in\mathscr{S}\}$ is the desired collection of 2^m topologically distinct spaces.

Remark. Theorem 5.4 and the result of Lozier and Marty [LM] suggest the problem of counting the number of connected locally compact metrizable spaces. It turns out that such spaces always have weight ω and the total number of such spaces is 2^{ω}.

THEOREM 5.5. *The number of topologically distinct connected locally compact metrizable spaces is 2^{ω}. Moreover, each such infinite space has weight ω and cardinality 2^{ω}.*

Proof. Let X be an infinite connected locally compact metrizable space. By Alexandroff's Theorem $X=\bigoplus_{s\in S}X_s$, where each X_s is a locally compact separable metrizable space. Since X is connected, $|S|=1$ and so X is a locally compact separable metrizable space. Note that $|X|=2^{\omega}$. By 5.2 the number of topologically distinct locally compact separable metrizable spaces is at most 2^{ω}, and the existence of 2^{ω} such spaces which are also connected follows from Lemma 4.2.

6. The number of locally compact locally countable metrizable spaces. In this section we show that the number of topologically distinct locally compact locally countable metrizable spaces of cardinality m is $\omega_1\cdot\nu(m)$, where $\nu(m)$ denotes the number of cardinal numbers $\leqslant m$. There are two reasons for considering this enumeration result. First, it is a reasonable extension to higher cardinals of the classical result of Mazurkiewicz and Sierpiński [MS] that the number of topologically distinct compact countable metrizable spaces is ω_1. Second, the result is used in § 7 where we enumerate topologically distinct locally compact metrizable spaces of cardinality m.

The following notation is used in this section. The space consisting of a single point is denoted by W_0, and for $1\leqslant\beta<\omega_1$ W_β denotes the space of all ordinals $\leqslant\omega^{\beta}$ with the order topology. Thus each W_β is a compact countable metrizable space. See Lemma 4.5 in [MP] for a proof that $W_\beta^{(\beta)}=\{\omega^{\beta}\}$ for $\beta\geqslant 1$; note that $W_\beta^{(\alpha)}=\emptyset$ for $\alpha>\beta$. For $0\leqslant\beta<\omega_1$ and for any cardinal m we let $K(\beta, m)$ denote the sum of m copies of W_β. Mazurkiewicz and Sierpiński [MS] have proved that every compact countable metrizable space is homeomorphic to $K(\beta, n)$ for some $\beta<\omega_1$ and some natural number n.

LEMMA 6.1. *Let X be an infinite locally countable metrizable space. Then the weight of X and the cardinality of X are the same.*

Proof. Let the weight of X be m; it suffices to show that $|X|\leqslant m$. Let $\mathscr{G}$ be an open cover of X, each element of which is countable. Since the weight of X is m, there is a subcollection $\mathscr{G}_0$ of $\mathscr{G}$ with $|\mathscr{G}_0|\leqslant m$ such that $\mathscr{G}_0$ covers X. Hence $|X|\leqslant m\cdot\omega=m$.

LEMMA 6.2 (Mazurkiewicz and Sierpiński). *The number of topologically distinct compact countable metrizable spaces is ω_1. Moreover, each such space is homeomorphic to the sum of a finite number of copies of some W_β $(0\leqslant\beta<\omega_1)$.*

LEMMA 6.3. *The number of topologically distinct locally compact countable metrizable spaces is ω_1.*

Proof. By Lemma 6.2 it suffices to prove that ω_1 is an upper bound. Let $\{X_\alpha\colon 0\leqslant\alpha<\omega_1\}$ be all topologically distinct compact countable metrizable spaces. For each $\alpha<\omega_1$ let $\mathscr{A}_\alpha$ be all subsets of X_α obtained by removing at most one point from X_α, and let $\mathscr{A}=\bigcup_{0\leqslant\alpha<\omega_1}\mathscr{A}_\alpha$; note that $|\mathscr{A}|=\omega_1$. Now let X be a locally compact countable metrizable space. To complete the proof it suffices to show that X is homeomorphic to some element of $\mathscr{A}$. We may assume that X is not compact. Let X^* be the Alexandroff one-point compactification of X. Clearly X^* is a compact countable metrizable space, so $X^*\approx X_\alpha$ for some $\alpha<\omega_1$. Hence X is homeomorphic to a subset of X_α obtained by removing one point of X_α.

LEMMA 6.4. *Let X be a locally compact locally countable metrizable space of cardinality $m\geqslant\omega_1$. Then X is homeomorphic to $\bigoplus_{0\leqslant\beta<\omega_1}K(\beta, m_\beta)$, where $0\leqslant m_\beta\leqslant m$ for all β and $\sum_{0\leqslant\beta<\omega_1}m_\beta=m$.*

Proof. By Alexandroff's Theorem $X=\bigoplus_{s\in S}X_s$, where each X_s is a separable metrizable space. By Lemma 6.1 each X_s is countable. We are going to show that each X_s is the sum of a countable number of spaces, each of which is homeomorphic to some W_β. The proof is then completed by taking m_β to be the total number of copies of W_β as s ranges over S. (Since each W_β is countable and $m\geqslant\omega_1$, $\sum_{0\leqslant\beta<\omega_1}m_\beta=m$.)

Fix $s\in S$ and let $X_s=\{p_n\colon n<\omega\}$. Now X_s is zero-dimensional, so each point p_n has a neighborhood V_n which is both open and closed. We may assume that each V_n is also compact. Let $U_1=V_1$ and for $n>1$ let $U_n=V_n-\bigcup_{k<n}V_k$. Clearly $\{U_n\colon n<\omega\}$ is a pairwise disjoint collection of open sets and $X_s=\bigcup_{n<\omega}U_n$, so $X_s=\bigoplus_{n<\omega}U_n$. Now each U_n is also closed and hence compact. By Lemma 6.2 U_n is homeomorphic to the sum of a finite number of copies of some W_β.

LEMMA 6.5. *Let $\beta<\gamma<\omega_1$. Then $W_\gamma=G\oplus H$, where $G\approx W_\beta$ and $H\approx W_\gamma$.*

Proof. Since $W_\gamma=[0, \omega^{\beta}]\oplus(\omega^{\beta}, \omega^{\gamma}]$, it suffices to show that $(\omega^{\beta}, \omega^{\gamma}]\approx W_\gamma$. Recall that

$$\omega^{\beta}=\omega^{\beta}\cdot 1<\omega^{\beta}\cdot 2<\ldots<\omega^{\beta}\cdot n<\ldots<\omega^{\beta+1}\leqslant\omega^{\gamma}.$$

Now

$$(\omega^\beta, \omega^\gamma] = (\omega^\beta, \omega^{\beta+1}] \oplus (\omega^{\beta+1}, \omega^\gamma] \quad \text{and} \quad W_\gamma = [0, \omega^{\beta+1}] \oplus (\omega^{\beta+1}, \omega^\gamma],$$

so it suffices to show that $(\omega^\beta, \omega^{\beta+1}] \approx [0, \omega^{\beta+1}]$. This easily follows from the fact that $(\omega^\beta \cdot n, \omega^\beta \cdot (n+1)] \approx (\omega^\beta \cdot (n+1), \omega^\beta \cdot (n+2)]$ for all $n < \omega$. (See p. 298 in [S].)

LEMMA 6.6. *Let m be an infinite cardinal, let α be an ordinal with $1 \leq \alpha \leq \omega_1$, let X be a topological space. Suppose there is a subset U of X such that (1) $X = U \oplus (X-U)$; (2) $U \approx \bigoplus_{\beta<\alpha} K(\beta, m)$; (3) $X - U = \bigoplus_{s \in S} X_s$, where $|S| \leq m$ and for each $s \in S$ there exists $\beta < \alpha$ such that $X_s \approx W_\beta$. Then $X \approx \bigoplus_{\beta<\alpha} K(\beta, m)$.*

Proof. Since $U \approx \bigoplus_{\beta<\alpha} K(\beta, m)$, $U = \bigoplus_{\beta<\alpha} \big(\bigoplus_{t \in T} W(\beta, t)\big)$, where $|T| = m$ and $W(\beta, t) \approx W_\beta$ for each $t \in T$. For each $\beta < \alpha$ let $S_\beta = \{s\colon s \in S, X_s \approx W_\beta\}$; then $X - U = \bigoplus_{\beta<\alpha} \big(\bigoplus_{s \in S_\beta} X_s\big)$. Since $X = U \oplus (X-U)$, it easily follows that

$$X = \bigoplus_{\beta<\alpha} \Big[\big(\bigoplus_{t \in T} W(\beta, t)\big) \oplus \big(\bigoplus_{s \in S_\beta} X_s\big)\Big]$$

and so $X \approx \bigoplus_{\beta<\alpha} K(\beta, m)$.

LEMMA 6.7. *Let m be a cardinal with $m \geq \omega_1$, let α be an ordinal with $1 \leq \alpha \leq \omega_1$, let $\{m_\beta\colon 0 \leq \beta < \alpha\}$ be a sequence of cardinals such that for all $\beta < \alpha$, $\sum_{\beta \leq \gamma < \alpha} m_\gamma = m$. If X is homeomorphic to $\bigoplus_{\beta<\alpha} K(\beta, m_\beta)$, then X is homeomorphic to $\bigoplus_{\beta<\alpha} K(\beta, m)$.*

Proof. It suffices to construct a subset U of X satisfying (1)–(3) in Lemma 6.6. We first consider a special case, namely $m = \omega_1$ and $\alpha = \omega_1$. Let $X = \bigoplus_{0 \leq \beta < \omega_1} \big(\bigoplus_{t \in E_\beta} W(\beta, t)\big)$, where $|E_\beta| = m_\beta$ and $W(\beta, t) \approx W_\beta$ for all $t \in E_\beta$. Let $A = \{\beta\colon 0 \leq \beta < \omega_1, m_\beta \neq 0\}$; since $\sum_{\beta \leq \gamma < \omega_1} m_\gamma = \omega_1$ for all $\beta < \omega_1$, A is uncountable. Let $\{A_\beta\colon 0 \leq \beta < \omega_1\}$ be a pairwise disjoint collection of subsets of A such that each A_β has cardinality ω_1 and $\gamma \in A_\beta$ implies $\gamma > \beta$. Now let $\beta < \omega_1$ be fixed. For each $\gamma \in A_\beta$ and each $t \in E_\gamma$ let $W(\gamma, t) = G(\beta, \gamma, t) \oplus H(\beta, \gamma, t)$, where $G(\beta, \gamma, t) \approx W_\beta$ and $H(\beta, \gamma, t) \approx W_\gamma$. (Use Lemma 6.5.) Let

$$X_\beta = \bigcup \{G(\beta, \gamma, t)\colon \gamma \in A_\beta, t \in E_\gamma\}.$$

Then X_β is the union of a pairwise disjoint collection of ω_1 open sets, each of which is homeomorphic to W_β, and so $X_\beta \approx K(\beta, \omega_1)$. Now let $U = \bigcup_{0 \leq \beta < \omega_1} X_\beta$; since $\{X_\beta\colon 0 \leq \beta < \omega_1\}$ is a pairwise disjoint collection of open sets, and $X_\beta \approx K(\beta, \omega_1)$ for all $\beta < \omega_1$, $U \approx \bigoplus_{0 \leq \beta < \omega_1} K(\beta, \omega_1)$. It is easy to check that U also satisfies (1) and (3) of Lemma 6.6.

We now construct U under the assumption that $\alpha < \omega_1$ or $m > \omega_1$. Let $X = \bigoplus_{0 \leq \beta < \alpha} \big(\bigoplus_{t \in E_\beta} W(\beta, t)\big)$, where $|E_\beta| = m_\beta$ and $W(\beta, t) \approx W_\beta$ for each $t \in E_\beta$. Let $A = \{\gamma\colon 0 \leq \gamma < \alpha, m_\gamma \geq \omega\}$, and for each $\gamma \in A$ let $\{E(\gamma, \beta)\colon 0 \leq \beta \leq \gamma\}$ be a partition of E_γ such that $|E(\gamma, \beta)| = m_\gamma$ for all $\beta \leq \gamma$. For $\gamma \in A$, $\beta < \gamma$, and $t \in E(\gamma, \beta)$ let $W(\gamma, t) = G(\beta, \gamma, t) \oplus H(\beta, \gamma, t)$, where $G(\beta, \gamma, t) \approx W_\beta$ and $H(\beta, \gamma, t) \approx W_\gamma$. Let β be fixed, $0 \leq \beta < \alpha$. Let

$$Y_\beta = \bigcup \{G(\beta, \gamma, t)\colon \beta < \gamma < \alpha, \gamma \in A, t \in E(\gamma, \beta)\},$$

and let $X_\beta = Y_\beta$ for $\beta \notin A$ and $X_\beta = Y_\beta \cup \big(\bigcup_{t \in E(\beta, \beta)} W(\beta, t)\big)$ for $\beta \in A$. It is clear that X_β is the union of a pairwise disjoint collection of open sets, each of which is homeomorphic to W_β; we now want to show that the number of spaces in this union is m. Note that

$$\sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \in A}} m_\gamma + \sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \notin A}} m_\gamma = m.$$

First suppose $\alpha = \omega_1$. Then $m > \omega_1$ and

$$\sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \notin A}} m_\gamma \leq \omega_1 \quad \text{so} \quad \sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \in A}} m_\gamma = m.$$

Next suppose $\alpha < \omega_1$. Then

$$\sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \notin A}} m_\gamma \leq \omega \quad \text{so} \quad \sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \in A}} m_\gamma = m.$$

Thus in either case

$$\sum_{\substack{\beta \leq \gamma < \alpha \\ \gamma \in A}} m_\gamma = m$$

and from this it easily follows that the number of copies of W_β is m. Thus $X_\beta \approx K(\beta, m)$. Now let $U = \bigcup_{0 \leq \beta < \alpha} X_\beta$; since $\{X_\beta\colon 0 \leq \beta < \alpha\}$ is a pairwise disjoint collection of open sets and $X_\beta \approx K(\beta, m)$ for all $\beta < \alpha$, $U \approx \bigoplus_{\beta<\alpha} K(\beta, m)$. It is easy to check that U also satisfies (1) and (3) of Lemma 6.6.

THEOREM 6.8. *For each cardinal $m \geq \omega$ the number of topologically distinct locally compact locally countable metrizable spaces of cardinality m is $\omega_1 \cdot \nu(m)$, where $\nu(m)$ is the number of cardinals $\leq m$.*

Proof. We begin by showing that for each $m \geq \omega$ there exist $\omega_1 \cdot \nu(m)$ topologically distinct locally compact locally countable metrizable spaces of cardinality m. First suppose $\omega_1 \cdot \nu(m) = \omega_1$. Then $\{K(\beta, m)\colon 0 \leq \beta < \omega_1\}$ is a collection of ω_1 locally compact locally countable metrizable spaces, each of cardinality m. Moreover if $\beta < \gamma$ then $\big(K(\beta, m)\big)^{(\beta+1)} = \emptyset$ and $\big(K(\gamma, m)\big)^{(\beta+1)} \neq \emptyset$, and so $K(\beta, m)$ is not homeomorphic to $K(\gamma, m)$ whenever $\beta < \gamma$. Now suppose $\omega_1 \cdot \nu(m) = \nu(m)$. For each cardinal $p \leq m$ let $X(p) = K(W_1, p) \oplus D_m$. Then $\{X(p)\colon p \leq m\}$ is a collection of $\nu(m)$ locally compact locally countable metrizable spaces, each of cardinality m. Moreover, if $p < n \leq m$ then $|X(p)^{(1)}| = p$ and $|X(n)^{(1)}| = n$ and so $X(p)$ cannot be homeomorphic to $X(n)$.

To complete the proof we must show that $\omega_1\cdot\nu(m)$ is an upper bound. The proof is by induction on m. If $m=\omega$ we are finished by Lemma 6.3. Now let $m>\omega$ and assume true for all cardinals p with $\omega\leqslant p<m$. Let $S=\{p\colon \omega\leqslant p<m\}$, for each $p\in S$ let $\mathscr{A}_p$ be all topologically distinct locally compact locally countable metrizable spaces of cardinality p, and let $\mathscr{A}=\bigcup_{p\in S}\mathscr{A}_p$. Since $|\mathscr{A}_p|\leqslant\omega_1\cdot\nu(p)\leqslant\omega_1\cdot\nu(m)$ for all p, it follows that $|\mathscr{A}|\leqslant|S|\cdot\omega_1\cdot\nu(m)=\omega_1\cdot\nu(m)$. Let

$$\mathscr{B}=\{(\bigoplus_{0\leqslant\beta<\alpha}K(\beta,m))\oplus A\colon 1\leqslant\alpha\leqslant\omega_1,\ A\in\mathscr{A}\text{ or }A=\emptyset\}.$$

It is clear that $|\mathscr{B}|\leqslant\omega_1\cdot\nu(m)$ and that each $B\in\mathscr{B}$ is a locally compact locally countable metrizable space of cardinality m.

Now let X be a locally compact locally countable metrizable space of cardinality m. To complete the proof it suffices to show that X is homeomorphic to some $B\in\mathscr{B}$. By Lemma 6.4 $X\approx\bigoplus_{0\leqslant\beta<\omega_1}K(\beta,m_\beta)$, where $\sum_{0\leqslant\beta<\omega_1}m_\beta=m$. We now consider two cases: (1) $\sum_{\beta\leqslant\gamma<\omega_1}m_\gamma=m$ for all $\beta<\omega_1$; (2) there is an ordinal α, $1\leqslant\alpha<\omega_1$, such that $\sum_{\beta\leqslant\gamma<\alpha}m_\gamma=m$ for all $\beta<\alpha$ and $\sum_{\alpha\leqslant\gamma<\omega_1}m_\gamma<m$. If (1) holds, then $X\approx\bigoplus_{0\leqslant\beta<\omega_1}K(\beta,m)$ by Lemma 6.7 and so $X\approx B$ for some $B\in\mathscr{B}$. Suppose (2) holds, and note that $X\approx(\bigoplus_{\beta<\alpha}K(\beta,m_\beta))\oplus(\bigoplus_{\alpha\leqslant\beta<\omega_1}K(\beta,m_\beta))$. By Lemma 6.7 $\bigoplus_{\beta<\alpha}K(\beta,m_\beta)\approx\bigoplus_{\beta<\alpha}K(\beta,m)$, and since $\sum_{\alpha\leqslant\beta<\omega_1}m_\beta<m$ it follows that $\bigoplus_{\alpha\leqslant\beta<\omega_1}K(\beta,m_\beta)\approx A$ for some $A\in\mathscr{A}$. Consequently $X\approx B$ for some $B\in\mathscr{B}$.

COROLLARY 6.9. *The number of topologically distinct locally compact locally countable metrizable spaces of cardinality* ω_1 *is* ω_1.

COROLLARY 6.10. *The number of topologically distinct locally compact locally countable metrizable spaces of cardinality* $\aleph_\omega$ *is* ω_1.

COROLLARY 6.11. *For each cardinal* $m\geqslant\omega$ *the number of topologically distinct locally compact locally countable metrizable spaces of weight* m *is* $\omega_1\cdot\nu(m)$.

Remark. There exist cardinals m for which $\nu(m)=m$. Indeed, if m is a fixed point of the aleph function (i.e., $\aleph_m=m$), then $\nu(m)=m$.

7. The number of locally compact metrizable spaces. In this section we find the number of locally compact metrizable spaces of cardinality m and also the number of locally compact metrizable spaces of weight m. The solution is obtained by considering two cases, namely $m\geqslant2^\omega$ and $m<2^\omega$. Recall that $\nu(m)$ is the number of cardinals $\leqslant m$.

THEOREM 7.1. *Let* m *be an infinite cardinal. The number of topologically distinct locally compact metrizable spaces of cardinality* m *is* $\omega_1\cdot\nu(m)$ *if* $m<2^\omega$ *and* $\nu(m)^{2^\omega}$ *if* $m\geqslant2^\omega$.

Proof. First suppose $m<2^\omega$. By Theorem 6.8 it suffices to show that every locally compact metrizable space X of cardinality m (where $m<2^\omega$) is locally countable. Let $p\in X$, let K be a compact neighborhood of p. Then $|K|\leqslant\omega$ or $|K|=2^\omega$. Since $|X|=m<2^\omega$, K must be countable.

Now suppose $m\geqslant2^\omega$. Let $\mathscr{F}$ be the collection of all functions from 2^ω into the set of all cardinal numbers $\leqslant m$. Note that $|\mathscr{F}|\leqslant\nu(m)^{2^\omega}$. First we prove the existence of $\nu(m)^{2^\omega}$ topologically distinct locally compact metrizable spaces, each of cardinality m. Let $\{X_\alpha\colon 0\leqslant\alpha<2^\omega\}$ be a collection of 2^ω topologically distinct compact connected metrizable spaces, each of cardinality 2^ω. (Use Lemma 4.2.) Let $f\in\mathscr{F}$, for each $\alpha<2^\omega$ let $X(f,\alpha)$ be the sum of $f(\alpha)$ copies of X_α, and let

$$X(f)=(\bigoplus_{\alpha<2^\omega}X(f,\alpha))\oplus D_m,$$

where D_m is the discrete space of cardinality m. Clearly each $X(f)$ is a locally compact metrizable space of cardinality m. Moreover, if $f\neq g$ then $X(f)$ is not homeomorphic to $X(g)$. (Let $f\neq g$, say $f(\alpha)>g(\alpha)$ for some $\alpha<2^\omega$, and suppose $X(f)\approx X(g)$. Each copy of X_α in the sum defining $X(f)$ must map homeomorphically onto a copy of X_α in the sum defining $X(g)$. Since $f(\alpha)>g(\alpha)$ this is impossible.)

Next we show that $\nu(m)^{2^\omega}$ is an upper bound. Let $\{X_\alpha\colon 0\leqslant\alpha<2^\omega\}$ be all topologically distinct locally compact separable metrizable spaces. (Use Proposition 5.2.) For each $f\in\mathscr{F}$ let $X(f)=\bigoplus_{\alpha<2^\omega}X(f,\alpha)$, where $X(f,\alpha)$ is the sum of $f(\alpha)$ copies of X_α. The cardinality of $\{X(f)\colon f\in\mathscr{F}\}$ is $\nu(m)^{2^\omega}$. Now let X be a locally compact metrizable space of cardinality m, where $m\geqslant2^\omega$. To complete the proof, it suffices to show that X is homeomorphic to some $X(f)$. By Alexandroff's Theorem $X=\bigoplus_{s\in S}X_s$, where each X_s is a separable metrizable space. Note that X_s is also locally compact and $|S|\leqslant m$. For each $\alpha<2^\omega$ let $S_\alpha=\{s\in S\colon X_s\approx X_\alpha\}$ and let f be the element of $\mathscr{F}$ defined by $f(\alpha)=|S_\alpha|$. Then $X=\bigoplus_{\alpha<2^\omega}(\bigoplus_{s\in S_\alpha}X_s)$ and so $X\approx X(f)$.

COROLLARY 7.2. *The number of topologically distinct locally compact metrizable spaces of cardinality* 2^ω *is* 2^{2^ω}.

COROLLARY 7.3. *The number of topologically distinct locally compact metrizable spaces of cardinality* $\aleph_{2^\omega}$ *is* 2^{2^ω}.

LEMMA 7.4. *Let* X *be an infinite locally compact metrizable space of weight* m. *If* $m\geqslant2^\omega$ *then* $|X|=m$, *and if* $m<2^\omega$ *then* $|X|=m$ *or* $|X|=2^\omega$.

Proof. First assume $m\geqslant2^\omega$. If suffices to show that $|X|\leqslant m$. Let $\mathscr{G}$ be an open cover of X such that for each $G\in\mathscr{G}$, $\bar{G}$ is compact. Note that $|\bar{G}|\leqslant2^\omega$. Since the weight of X is m, there is a subcollection $\mathscr{G}_0$ of $\mathscr{G}$ with $|\mathscr{G}_0|\leqslant m$ which covers X. Hence $|X|\leqslant m\cdot2^\omega=m$. Now suppose $m<2^\omega$. By Proposition 5.1, $|X|=m$ or $|X|=m^\omega$. Since $\omega\leqslant m<2^\omega$, $m^\omega=2^\omega$.

THEOREM 7.5. *Let* m *be an infinite cardinal. If* $m\geqslant2^\omega$ *the number of topologically distinct locally compact metrizable spaces of weight* m *is* $\nu(m)^{2^\omega}$, *and each such space has cardinality* m. *If* $m<2^\omega$ *the number of topologically distinct locally compact metrizable spaces of weight* m *is* 2^m, *and each such infinite space has cardinality* m *or* 2^ω. *Moreover the number of cardinality* m *is* $\omega_1\cdot\nu(m)$ *and the number of cardinality* 2^ω *is* 2^m.

Proof. First suppose $m \geq 2^\omega$. By 7.4 every locally compact metrizable space of weight $m \geq 2^\omega$ has cardinality m, and so we are finished by 7.1.

Now assume $m < 2^\omega$. It follows from 5.2 that 2^m is an upper bound for the number of topologically distinct locally compact metrizable spaces of weight m. Moreover by 7.4 it follows that each such infinite space has cardinality m or 2^ω, and it follows from 7.1 that the number of such spaces of cardinality m is $\omega_1 \cdot \nu(m)$. Consequently the proof is complete if we can construct 2^m topologically distinct locally compact metrizable spaces, each of weight m and cardinality 2^ω. Let $\{X_\alpha: 0 \leq \alpha < 2^\omega\}$ be a collection of 2^ω topologically distinct compact connected metrizable spaces, each of cardinality 2^ω. Let $\mathscr{S} = \{S: S \subseteq 2^\omega, |S| = m\}$, and for each $S \in \mathscr{S}$ let $X(S) = \bigoplus_{\alpha \in S} X_\alpha$. Clearly $\{X(S): S \in \mathscr{S}\}$ is a collection of 2^m locally compact metrizable spaces, each of weight m and cardinality 2^ω, and $X(S_1)$ is not homeomorphic to $X(S_2)$ whenever $S_1 \neq S_2$.

8. The number of connected paracompact spaces. For each cardinal $m \geq 2^\omega$ the number of connected compact spaces of cardinality m is 2^m and the number of connected metrizable spaces of cardinality m is 2^m. (See [LM] and § 4.) Each of these classes of spaces is contained in the class of connected paracompact spaces of cardinality m. Is this latter class larger? Yes. More precisely, we show that the number of spaces in this class is the maximum possible, namely 2^{2^m}. The proof makes use of the following facts about ultrafilters. Let S be a set with $|S| = m \geq \omega$, let p and q be free ultrafilters on S. Then p and q are of the *same type* if there is a permutation π of S such that $q = \{\pi(E): E \in p\}$. It is well known that p and q are of the same type if and only if the subspaces $\{p\} \cup S$ and $\{q\} \cup S$ of βS are homeomorphic. Also, there exist 2^{2^m} free ultrafilters on S, no two of which are of the same type. (See [R].)

The following result, which is an easy consequence of Lemma 1 in [M], will be useful.

LEMMA 8.1. *Let X be a regular space such that $X = K \cup M$, where K is compact and M is metrizable. Then X is paracompact.*

THEOREM 8.2. *For each cardinal $m \geq 2^\omega$ the number of topologically distinct connected paracompact spaces of cardinality m is the maximum possible, namely 2^{2^m}.*

Proof. It suffices to construct 2^{2^m} such spaces. Let S be a set with $|S| = m$. For each $s \in S$ let $X_s = [0,1] \times \{s\}$, let d_s be the Euclidean metric on X_s, let $q_s = (1, s)$, and let $\mathscr{W}_s$ be a fundamental system of open neighborhoods of q_s, no one of which contains $(0, s)$. Let $X(S)$ be the star-space determined by $\{(X_s, d_s): s \in S\}$ and $\{(0, s): s \in S\}$. Let $\{p_\alpha: 0 \leq \alpha < 2^{2^m}\}$ be a collection of 2^{2^m} free ultrafilters on S, no two of which are of the same type. Now let $\alpha < 2^{2^m}$ be fixed; we are going to construct a connected paracompact space X_α of cardinality m. Let $X_\alpha = X(S) \cup \{\alpha\}$, and take as a base for X_α the collection of all open subsets of $X(S)$ together with all sets of the form $\{\alpha\} \cup (\bigcup_{s \in E} W_s)$, where $E \in p_\alpha$ and $W_s \in \mathscr{W}_s$. Note that the subspace $\{\alpha\} \cup \{q_s: s \in S\}$ of X_α is homeomorphic to the subspace $\{p_\alpha\} \cup S$ of βS. It is not difficult to check that X_α is T_1, regular, connected, and has cardinality m; that X_α is paracompact follows easily from 8.1.

Now suppose h is a homeomorphism from X_α onto X_β. Then h must take α to β and $\{q_s: s \in S\}$ to $\{q_s: s \in S\}$. (Note that $\{q_s: s \in S\} \cup \{\alpha\}$ is the set of noncut points of X_α and α is the only point of X_α which does not have a countable local base.) Hence $\{\alpha\} \cup \{q_s: s \in S\} \approx \{\beta\} \cup \{q_s: s \in S\}$ and so p_α and p_β are of the same type and $\alpha = \beta$. Consequently $\{X_\alpha: 0 \leq \alpha < 2^{2^m}\}$ is a collection of 2^{2^m} topologically distinct spaces, each having the desired properties.

9. Concluding remarks. The results in this paper can be viewed as giving a rough measure of the "niceness" of a class of topological spaces. For example, the number of compact manifolds, with or without boundary, is ω (see [CK]); the number of locally compact connected (separable) metrizable spaces is 2^ω; the number of connected separable metrizable spaces is 2^{2^ω}. Similarly, for each cardinal $m \geq 2^\omega$ the number of continua of cardinality m is 2^m and the number of connected metrizable spaces of cardinality m is 2^m but the number of connected paracompact spaces of cardinality m is 2^{2^m}.

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