Local homotopy products

by

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Abstract. This paper introduces a generalization of hereditary homotopy equivalences and provides the basic relationship with approximate fibrations (a generalization of cell-like maps). We also provide a number of results which highlight the similarity with fibre bundles.

The interrelated topics of cell-like maps, hereditary homotopy equivalences and near homeomorphisms have produced some very nice results in recent times. The reader is referred to the outstanding survey article, [La 2], of R. C. Lacher for an introduction and list of references.

The approximate fibrations of D. S. Coram and P. F. Duvall have been treated as a natural generalization of cell-like maps ([C–D1] and [C–D2]). The existence theorems of J. L. Bryant and R. C. Lacher ([B–L]) and D. Coram and P. Duvall ([C–D3] and [C–D4]) provide them with some intrinsic interest. Other related concepts include the shape fibrations and n-shape fibrations of S. Mardešić and T. B. Rushing ([M–R1], [M–R2], and [M–R3]) and the K-like maps of L. Mand [Ma]. Coram and Duvall have asked the natural question whether their maps are "near fibre bundles" or "near fibrations". There have been a number of partial results along this line ([Go 1], [Go 2], [Hus], [Q].

An examination of the methods used for approximating cell-like maps suggests that an analog of the notion of hereditary homotopy equivalence should be useful in studying the problem of approximating approximate fibrations. This paper presents one such generalization together with a theorem relating it to approximate fibrations. We also present a number of results which permit constructions similar to those used with coordinate bundles [St].

I would like to thank the referee whose suggestion, Proposition B5, improved the exposition.

A. 1. Notation and conventions. Unless otherwise specified, all spaces will be finite dimensional separable metric spaces and the metric will be such that closed and bounded sets are compact. The metric will be denoted by d.

An ANR will be a finite dimensional absolute neighborhood retract for separable metric spaces.

A map is a continuous function.

A map is proper if the inverse image of each compact set is compact.

$f \simeq g$ means f is homotopic to g.

$X \simeq Y$ means X is homeomorphic to Y.
$G \simeq H$ means G is isomorphic to H.
The particular meaning of $\simeq$ will be clear from context in each case.
J represents the natural numbers.
R represents the real numbers.
$R^k = \prod_{i=1}^{k} R$, a metric space with the Euclidean metric:

$$d((X_i),(Y_i)) = \Big(\sum_{i=1}^{k}(X_i - Y_i)^2\Big)^{1/2}.$$

If x is an element of X and $\delta > 0$, $N_\delta(x)$ is $\{x' \in X \mid d(x,x') < \delta\}$ and $\overline{N}_\delta(x)$ is $\{x' \in X \mid d(x,s') \leqslant \delta\}$.

$I = [0,1]$.

We provide the necessary definitions from [C-D 1] and [C-D 2]:

A.2. Definition. A surjective map $p\colon E \to B$ between metric spaces has the *approximate homotopy lifting property* (AHLP) with respect to a space X provided given an open cover ε of B and $g\colon X \to E$ and $H\colon X \times I \to B$ such that $pg = H_0$, there exists a map $G\colon X \times I \to E$ such that $G_0 = g$ and pG and H are ε-close. The map G is said to be an ε-*lift of* H.

It is equivalent for our spaces that given g and H and a continuous function $\varepsilon\colon Y \to (0,\infty)$, there exists a map $G\colon X \times I \to E$ such that $G_0 = g$ and $d\big(p(G(x,t)), H(x,t)\big) < \varepsilon\big(H(x,t)\big)$ for each (x,t) in $X \times T$.

A.3. Definition. A proper surjection $p\colon E \to B$ is an *approximate fibration* if 1) E and B are absolute neighborhood retracts and 2) p has the approximate homotopy lifting property for all metric spaces.

If $p\colon E \to B$ is an approximate fibration and b and b' lie in the same path component of B, then $p^{-1}(b)$ and $p^{-1}(b')$ have the same shape [C-D 1].

The following definitions generalize the notion of hereditary homotopy equivalence:

A.4. Definition. A map $p\colon X \to Y$ *has property* LHP-F, *local homotopy product structure with fibre* F, if for every y in Y there is a neighborhood U of y and a map $f\colon p^{-1}(U) \to F$ such that for every open V in U, the map $p \times f \mid p^{-1}(V)\colon p^{-1}(V) \to V \times F$ is a proper homotopy equivalence. If we may take $U = Y$, then p *has property* HP-F. If F is compact, we also require that p be proper. If $p\colon X \to Y$ has property HP-F and f is as above, we say p *has property* HP-F *via* f, or that f is a HP *trivilization of* p.

B. Basic relationships and constructions. To obtain a number of results which deal with modifications of LHP-F maps, we will need the following theorem. This theorem, in various forms, has appeared whenever one wishes to obtain global homotopy information from local information. A version appears in the 1966 Princeton Ph. D. Thesis of Dennis Sullivan [Su] and a more general version was stated by George Kozlowski in an address to the CBMS/NSF Regional Conference on Infinite Dimensional Topology held at Guilford College in October 1975 [Ko]. A version also appears in [Mi].

B.1. Theorem. *If F is a closed* ANR *subspace of the* ANR E *and* $U = \{U_j \mid j \in J\}$ *is a locally finite open cover of E such that for every nonempty finite subfamily S of U there is a strong deformation retraction of* $U_S = \bigcap_{U_i \in S} U_i$ *onto* $F \cap U_S$, *then there is a strong deformation retraction of E onto F which is limited by U.*

Proof. Although Milnor does not claim this result for locally finite covers, his proof yields this result. One must observe that the deformation which he constructs is indeed limited by the cover [Mi, Lemma 1].

The analogy of LHP-F maps with fibre bundles leads one to the following definitions and to B.3. B.3 is analogous to the fundamental result that the pullback of a trivial bundle is trivial.

Definition. If $p\colon E \to B$ and $p'\colon E' \to B$ are LHP-K maps, an LHP-K bundle map from p to p' is a pair (e,b) of maps $e\colon E \to E'$, $b\colon B \to B'$ such that $p'e = bp$ and for every y in B', there is a neighborhood U of y in B' such that $p' \mid p'^{-1}(U)$ is HP-K (via f') and $p \mid p^{-1}b^{-1}(U)\colon p^{-1}(b^{-1}(U)) \to b^{-1}(U)$ is HP-K via $f = f'e$.

The next proposition follows routinely from B.1; however, it is cast in terms which will be more useful in the sequel. B.2 and its Corollaries B.3 and B.4 are fundamental devices for working with LHP-F maps.

B.2. Proposition. *Suppose X, Y, and F are* ANR*'s*, $p\colon X \to Y$, $f\colon X \to F$ *are maps and* $\{U_i\}$ *is a locally finite open cover of Y such that for every* U_i, $p\colon p^{-1}(U_i) \to U_i$ *has property* HP-F *via* f. *Then* $p\colon X \to Y$ *has property* HP-F *via* f. *Further, we may find a homotopy inverse g for $p \times f$ so that* $g(U_i \times F) \subseteq p^{-1}(U_i)$ *for each* U_i *and homotopies* $h\colon g \circ (p \times f) = 1_X$, $k\colon (p \times f) \circ g \simeq 1_{Y \times F}$ *so that* $h(p^{-1}(U_i) \times I) \subseteq p^{-1}(U_i)$ and $k(U_i \times F \times I) \subseteq U_i \times F$.

Proof. Note that all the hypotheses are hereditary in Y. That is, if V is an open subset of Y, then $p \mid p^{-1}(V)\colon p^{-1}(V) \to V$, $f \mid p^{-1}(V)\colon p^{-1}(V) \to F$ and $\{V_i = U_i \cap V\}$ satisfy the hypotheses of the proposition. Thus it suffices to show that g, h, k can be found for p, f, and $\{U_i\}$. Also note that any refinement of $\{U_i\}$ will still satisfy the hypotheses and that if the conclusion is satisfied for this refinement, it is satisfied for $\{U_i\}$. Thus, without loss of generality, we may assume $\operatorname{diam}(U_i) \leqslant 1$ for every i.

Now, in order to apply the above theorem, we must convert the map $p \times f\colon X \to Y \times F$ to an inclusion. This is done in the usual way:

Consider the mapping cylinders M of $p \times f$ and M_i of

$$(p \mid p^{-1}(U_i)) \times f\colon p^{-1}(U_i) \to U_i \times F.$$

Note that if S is a nonempty finite intersection of elements of $\{M_i\}$ then S is the mapping cylinder of $\big(p \mid p^{-1}(\bigcap_{j=1}^{k} U_j)\big) \times f$ which is a proper homotopy equivalence. Thus there is a strong deformation retraction of S to $S \cap X = p^{-1}(\bigcap_{j=1}^{k} U_j)$. Note also that M is a finite dimensional ANR. Thus, according to the above theorem, there is a strong deformation retraction, $H\colon M\times I\to M$, of M to X which is limited by $\{M_i\}$.

This yields the proposition as follows: Let $i_Y\colon Y\times F\to M$ and $i_X\colon X\to M$ be the usual inclusions and let $r\colon M\to Y\times F$ be the usual retraction. Then $H_1\circ i_Y\colon Y\times F\to X$ is the desired homotopy inverse g for $p\times f=r\circ i_X$. Note that under projections to Y, no point is moved a distance more than 1. Now, the desired homotopies are $h\colon X\times I\to X$ where $h(x,t)=H_1(x,1-t)$ and $k\colon Y\times F\times I\to Y\times F$ where $k(y,f,t)=r\big(H(i_Y(y,f),t)\big)$. Note that h and k are proper, because i_X, i_Y, r and H are proper. This completes $B.2$.

B.3. Corollary. *If*

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "h"] & B' \end{tikzcd}$$

is an LHP-K *bundle map and p' is* HP-K, *then p is* HP-K.

B.4. Corollary. *If p has property* HP-F *via $f\colon X\to F$ and $\alpha\colon B\to Y$ is an open immersion, then the topological pullback, p', of p by α has property* HP-F *via $f\circ\alpha'$*

$$\begin{tikzcd} E \arrow[r, "\alpha'"] \arrow[d, "p'"'] & X \arrow[r, "f"] \arrow[d, "p"] & F \\ B \arrow[r, "\alpha"] & Y & \end{tikzcd}$$

Proof. The open sets openly embedded by α clearly have HP-F via $f\circ\alpha'$. A locally finite refinement of this cover provides the necessary cover of B.

The following proposition also follows routinely from B.1 by use of the mapping cylinder.

B.5. Proposition. *Let X and B be locally compact* ANR's *and let F be a compact* ANR. *Suppose $p\colon X\to B$ is proper and $f\colon X\to F$ is a map. If, for every open subset U of B, the map $p\times f|p^{-1}(U)\colon p^{-1}(U)\to U\times F$ is a homotopy equivalence, then $p\times f$ is a proper homotopy equivalence.*

Proof. Let $\emptyset=C_1\subseteq\mathring{C}_2\subset C_2\subset\ldots$ be a compact exhaustion of B. Note that $\{p^{-1}(C_i)\}$ is a compact exhaustion of x and $\{C_i\times F\}$ is a compact exhaustion of $B\times F$.

Define $U_i=B\setminus C_i$, $V_i=p^{-1}(U_i)=X\setminus p^{-1}(C_i)$ and $W_i=U_i\times F=(B\times F)\setminus(C_i\times F)$ (p is necessarily a surjection).

The hypothesis guarantees that $p\times f\colon V_i\to W_i$ is a homotopy equivalence for each i. Since these open sets are nested, or again from the hypothesis, we have that $p\times f$ restricts to a homotopy equivalence from $\bigcap_{j=1}^{n} V_{i_j}$ to $\bigcap_{j=1}^{n} W_{i_j}$ for every nonempty, finite set of indices $\{i_j\}_{j=1}^{n}$. Passing to mapping cylinders for $p\times f$ and its restrictions as in Proposition B.2, we obtain that $p\times f$ is a homotopy equivalence of X to $B\times F$ with homotopy inverse g and homotopies h and k of the compositions $g\circ(p\times f)$

to 1_X and $(p\times f)\circ g$ to $1_{B\times F}$ respectively which are limited by $\{V_i\}$ and $\{W_i\}$ respectively. Now recall that $V_i=X\setminus p^{-1}(C_i)$ and $W_i=(B\times F)\setminus(C_i\times F)$ for each i. Since $\{p^{-1}(C_i)\}$ and $\{C_i\times F\}$ are compact exhaustions of X and $B\times F$ respectively, we have that $p\times f$ is actually a proper homotopy equivalence of X to $B\times F$.

The next proposition shows that the approximate fibration property is hereditary.

B.6. Proposition. *If $p\colon X\to Y$ is an approximate fibration and U is an open subset of Y, then $p|p^{-1}(U)\colon p^{-1}(U)\to U$ is an approximate fibration.*

Proof. We will show that $p|p^{-1}(U)$ has the approximate homotopy lifting property for cells. By 2.6 of [C–D 2], this will imply that $p|p^{-1}(U)$ is an approximate fibration. To show this, let $h\colon D\times I\to U$ be continuous and suppose $H_0\colon D\times\{0\}\to p^{-1}(U)$ is a lifting of h_0 and that $\varepsilon\colon U\to(0,\infty)$ is given. Since $D\times I$ is compact, $h(D\times I)$ has a positive distance δ from $Y\setminus U$. Choose $\varepsilon'\colon Y\to(0,\infty)$ so that $\max_{x\in U}(\varepsilon'(x))$ is less than $\frac{1}{3}\delta$ and that $\varepsilon'(x)<\varepsilon(x)$ for $x\in U$. Apply the approximate homotopy lifting property of $p\colon X\to Y$ to the homotopy h, the lifting H_0 and the map ε', to obtain a near lifting H of h which extends H_0 and satisfies

$$d\big(pH(x,t),h(x,t)\big)<\varepsilon'h(x,t)\,.$$

Note that this last condition forces $pH(x,t)$ to be an element of U for each $(x,t)\in D\times I$. Thus H actually takes values in $p^{-1}(U)$. Hence, $p|p^{-1}(U)\colon p^{-1}(U)\to U$ has the approximate homotopy lifting property for cells and by 2.6 of [C–D 2], it is an approximate fibration.

For the next proposition, let $p\colon X\to Y$ and suppose $y\in Y$ and $e\in p^{-1}(y)$.

This result is analogous to the theorem of Lacher [La 1] that a cell-like map of Euclidean Neighborhood Retracts is a hereditary homotopy equivalence.

B.7. Proposition. *Suppose $p\colon X\to Y$ is an approximate fibration of absolute neighborhood retracts such that each fibre $F_y=p^{-1}(y)$ has the shape of a compact* ANR *F. Then p has property* LHP-F.

Proof. Since F and F_y are compact metric spaces, the shape theory described by Mardešić [Mar] agrees with that of Borsuk [Bor]. In particular, we will use the inverse system approach to Mardešić's theory described by Morita [Mo]. The constant inverse sequence $\{F_i=F, p_{i,i-1}=1_F\}$ is associated to F and the inverse system of open neighborhoods in X, connected by inclusion, is associated to F_y. There is a cofinal sequence of open neighborhoods $U_1\supseteq U_2\supseteq\ldots$ of F_y and, since F_y is shape equivalent to F, there are sequences of maps $S_i\colon U_i\to F$ and $r_i\colon F\to U_i$ determining shape maps S and r and satisfying:

1) for each k, there is a $k'\geqslant k$ so that

$$F\xrightarrow{p_{k',k}}F\xrightarrow{1}F$$

is homotopic to

$$F\xrightarrow{p_{k',k}}F\xrightarrow{r_k}U_k\xrightarrow{S_k}F$$

and

2) for each k, there is a $k'\geqslant k$ so that

$$U_{k'} \hookrightarrow U_k \xrightarrow{1} U_k$$

is homotopic to

$$U_{k'} \hookrightarrow U_k \xrightarrow{s_k} F \xrightarrow{r_k} U_k\,.$$

Thus, in particular, $s_1\colon\ U_1 \to F$ is a homotopy domination. Choose a neighborhood U of y in Y so that $p^{-1}(U)\subseteq U_1$ and define $f\colon\ p^{-1}(U) \to F$ by $f = S_1|p^{-1}(U)$. Then f is a homotopy domination.

Now, $p|p^{-1}(U)\colon\ p^{-1}(U) \to U$ is an approximate fibration, according to B.6. According to [C-D 1], the sequence in the top row below is exact. We build an exact sequence as indicated in the second row

$$\begin{tikzcd}
\ldots\,\pi_{k+1}(U,y) \arrow[r, "\partial_*"] \arrow[d, leftrightarrow] & \mathrm{Sh}_k(p^{-1}(y),e) \arrow[r, "i_*"] \arrow[d, shift right, "s_*"'] & \pi_k(p^{-1}(U),e) \arrow[r] \arrow[d, leftrightarrow] & \pi_k(U,y) \arrow[r, "\partial_*"] \arrow[d, leftrightarrow] & \ldots \\
\ldots\,\pi_{k+1}(U,y) \arrow[r, "s_*\partial_*"] & \mathrm{Sh}_k(F,f(e)) \arrow[r, "i_*\circ r_*"] \arrow[u, shift right, "r_*"'] & \pi_k(p^{-1}(U),e) \arrow[r, "p_*"] & \pi_k(U,y) \arrow[r, "s_*\partial_*"] & \ldots
\end{tikzcd}$$

Now, since F is an ANR, $\mathrm{Sh}_k(F,f(e)) = \pi_k(F,f(e))$ for every k and so we have the following exact sequence:

$$\ldots\,\pi_{k+1}(U,y) \to \pi_k(F,f(e)) \to \pi_k(p^{-1}(U),e) \to \pi_k(U,y) \to \ldots$$

Note that $i_*\circ r_*$ is split by f_* and so for every k, we have

$$\begin{tikzcd}
0 \arrow[r] & \pi_k(F) \arrow[r] & \pi_k(p^{-1}(U)) \arrow[r] \arrow[l, bend left, "f_*"] & \pi_k(U) \arrow[r] & 0\,.
\end{tikzcd}$$

Thus we have the following commutative diagram:

$$\begin{tikzcd}
\pi_k(p^{-1}(U)) \arrow[r, "f_*\oplus p_*", "\simeq"'] \arrow[dr, "(f\times p)_*"'] & \pi_k(F)\oplus\pi_k(U) \arrow[d, "\simeq"] \\
 & \pi_k(F\times U)
\end{tikzcd}$$

When $k = 1$, the groups in the exact sequence may not be abelian. However, since the splitting is on the left, one still obtains the direct sum decomposition [Hu 1].

Thus $f\times p$ is a homotopy equivalence by the Whitehead Theorem.

Observe that the homotopy equivalence is hereditary in U, for if V is an open subset of U containing y, then the corresponding exact sequence is still split by f. If y is not in U, one uses 2.9 of [C-D 1] to effect the change of base point.

The homotopy equivalence is proper, by B.5. This completes B.7.

Another fundamental technique for working with HP-F maps is the modification of HP-trivializations. According to B.8, one may replace one trivialization with another if the two are homotopic by a homotopy which is essentially proper. Proposition B.9 provides conditions under which one may find a trivialization which

agrees with a given one over a particular set and with a different one over another set. This will be particularly useful in attempting to approximate LHP-F maps with fibre bundle projections. It will be desirable to find an HP trivialization which is a bundle trivialization over some set. B.9 will provide a method of doing this.

B.8. Proposition. *Suppose* $p\colon\ X\to Y$ *has property* HP-F *via* $f\colon X\to F$. *If* H *is a homotopy from* f *to* f' *which is proper when restricted to* $p^{-1}(K)$ *for each compact subset* K *of* Y, *then* p *has* HP-F *via* f'.

Remark. Note that the condition on H is redundant when p is proper, for then $p^{-1}(K)$ is compact and any map whose domain is $p^{-1}(K)\times I$ is proper.

Proof. We must show that $(p\circ p_1)\times H\colon\ X\times I\to\ Y\times F$ is a proper homotopy from $p\times f$ to $p\times f'$. To see that $(p\circ p_1)\times H$ is proper, suppose that $K\subseteq Y\times F$ is compact. Choose compact sets $K_1\subseteq Y$ and $K_2\subseteq F$ such that $K\subseteq K_1\times K_2$. For instance, set $K_i = p_i(K)$. Now,

$$((p\circ p_1)\times H)^{-1}(K_1\times K_2) = p_1^{-1}p^{-1}K_1\cap H^{-1}(K_2) = p^{-1}(K_1)\times I\cap H^{-1}(K_2)\,.$$

This, however, is compact by hypothesis.

Now, $((p\circ p_1)\times H)^{-1}(K)$ is a closed subset of $((p\circ p_1)\times H)^{-1}(K_1\times K_2)$ and so it, too, is compact. Thus, since $p\times f$ is a proper homotopy equivalence, so is $p\times f'$.

Now, suppose V is an open subset of Y. Observe that $H|p^{-1}(V)\times I$ is a homotopy of $f|p^{-1}(V)$ to $f'|p^{-1}(V)$ and H is proper on $p^{-1}(K)$ for each compact subset K in V. Thus, by the above paragraph, $p\times f'$ is a proper homotopy equivalence of $p^{-1}(V)$ to $V\times F$. This completes B.8.

B.9. Proposition. *Suppose* $p\colon\ X\to\ Y$ *has property* HP-F *via* $f\colon\ X\to F$, *and that* H *is a homotopy from* f *to* f' *which is proper when restricted to* $p^{-1}(K)$, *for each compact subset* K *of* Y. *If* B_1 *and* B_2 *are two disjoint closed subsets of* X, *then there is a third map* $g\colon\ X\to F$ *such that*

1) p *has* HP-F *via* g *and*

2) $g|B_1 = f|B_1$ *and* $g|B_2 = f'|B_2$.

Proof. Choose a Urysohn function $u\colon\ X\to[0,1]$ such that $u(B_1) = 0$ and $u(B_2) = 1$. Set $g(x) = H(x,u(x))$. If x is in B_1, then $g(x) = H(x,0) = f(x)$ and if x is in B_2, then $g(x) = H(x,1) = f'(x)$. Then condition 2) above is satisfied. To verify condition 1) we must show that g is homotopic to f by a homotopy which is proper on each $p^{-1}(K)$. Set $L(x,t) = H(x,(1-t)u(x))$. Note that $L(x,0) = H(x,u(x)) = g(x)$ and that $L(x,1) = H(x,0) = f(x)$. We must show that L has the desired property, so let K be a compact subset of Y and consider $L' = L|p^{-1}(K)\times I\colon\ p^{-1}(K)\times I\to F$. Let C be a compact subset of F. $L'^{-1}(C)$ is a closed subset of $H^{-1}(C)\cap p^{-1}(K)$ which is a compact set, by hypothesis. Thus $L'^{-1}(C)$ is compact and L' is proper. Thus, by B.8, p has property HP-F via g. This completes the proof of B.9.

The next proposition allows one to combine HP trivializations which do not necessarily agree on their common domain.

To establish notation, suppose $p\colon X\to Y$ is a map and that Y is the union of two open sets A_1 and A_2. Set $B_1' = Y\setminus A_2$, $B_2' = Y\setminus A_1$.

$$\begin{aligned}
C_1' &= \{y\in A_1\cap A_2\colon 2d(y, B_1')\leqslant d(y, B_2')\}\,,\\
C_2' &= \{y\in A_1\cap A_2\colon 2d(y, B_2')\leqslant d(y, B_1')\}\,,\\
C_i &= p^{-1}(C_i'),\quad B_i = p^{-1}(B_i');\ i = 1, 2\,,\\
D_1' &= \{y\in Y\colon 2d(y, B_1')<d(y, B_2')\}\,,\\
D_2' &= \{y\in Y\colon 2d(y, B_2')<d(y, B_1')\}\,,\\
D_i &= p^{-1}(D_i')\,.
\end{aligned}$$

B.10. Proposition. *For $i = 1, 2$ suppose $p|p^{-1}(A_i)\colon p^{-1}(A_i)\to A_i$ has property* HP-F *via some f_i. If f_1 is homotopic to f_2 on $p^{-1}(A_1\cap A_2)$ by a homotopy H which is proper on $p^{-1}(K)$ for each compact K in $A_1\cap A_2$, then p has property* HP-F *via a map $g\colon X\to F$ such that, for $i\in\{1, 2\}$, $g|B_i = f_i|B_i$.*

Proof. We first show $C_1'\cap C_2' = \varnothing$:

Let y be an element of $C_1'\cap C_2'$, then $2d(y, B_1')\leqslant d(y, B_2')\leqslant d(y, B_1')$, so $d(y, B_1') = 0$. By the same argument, $d(y, B_2') = 0$ and so $y\in B_1'\cap B_2'$. Thus $C_1'\cap C_2'\subseteq B_1'\cap B_2'$. However, since

$$Y = A_1\cup A_2\,,\qquad B_1'\cap B_2' = (Y\setminus A_2)\cap(Y\setminus A_1) = Y\setminus(A_1\cup A_2) = \varnothing\,.$$

Thus $B_1\cap B_2 = \varnothing$ and $C_1\cap C_2 = \varnothing$. By B.9, there is a function $g'\colon p^{-1}(A_1\cap A_2)\to F$ which agrees with f_i on C_i and which is a HP trivialization for $p|p^{-1}(A_1\cap A_2)$. Now, observe that each D_i is an open neighborhood of B_i and that $D_i\cap p^{-1}(A_1\cap A_2)\subseteq C_i$. Again, note that D_1 and D_2 are disjoint because $D_i\subseteq B_i\cup C_i$. Thus $\{D_1, D_2, p^{-1}(A_1\cap A_2)\}$ is an open cover of X with the properties that $D_1\cap D_2 = \varnothing$, and $D_i\cap p^{-1}(A_1\cap A_2)\subseteq C_i$. Thus, setting

$$g(x) = \begin{cases} f_1(x)\,, & \text{if } x\in D_1\,,\\ g'(x)\,, & \text{if } x\in p^{-1}(A_1\cap A_2)\,,\\ f_2(x)\,, & \text{if } x\in D_2\,,\end{cases}$$

we obtain a continuous function, since each pair of the three component functions agree on their common domains. Thus, by B.2 p is HP-F via g. Note that since $B_i\subseteq D_i$, we have that $g|B_i = f_i|B_i$ for $i\in\{1, 2\}$.

The next theorem and its first corollary provide a partial converse for B.7. The result is that for ANR's, the notions of approximate fibration and proper LHP-F map coincide (B.13).

B.11. Theorem. *If the proper map $p\colon X\to Y$ has property* HP-F, *then p is an approximate fibration.*

Proof. We will show that p has the AHLP for discs. According to Theorem 2.6 of [C-D2], this implies that p is an approximate fibration. Let A be a disc and let

$$\begin{tikzcd} & & X \arrow[d, "p"] & \\ A\times\{0\} \arrow[urr, "h_0"] \arrow[r] & A\times I \arrow[r, "H"] & Y \arrow[r, "\varepsilon"] & (0, \infty) \end{tikzcd}$$

be given such that $ph_0 = H|A\times\{0\}$. We wish to extend h_0 to a map $h\colon A\times I\to X$ so that $d\big(ph(a, t), H(a, t)\big)<\varepsilon\big(H(a, t)\big)$. First choose a locally finite cover $\{U_i\}$ of Y so that

$$\operatorname{diam} U_i = \min_{\substack{U_j\cap U_i\neq\varnothing\\ y\in U_j}} \tfrac{1}{2}\varepsilon(y)$$

and suppose f is a HP trivialization of p.

Now apply Proposition B.2 to obtain maps $g\colon Y\times F\to X$, $h'\colon X\times I\to X$ and $k'\colon Y\times F\times I\to Y\times F$ so that:

$h'(x, 0) = x$ for each x in X,
$h'(x, 1) = g\big((p\times f)(x)\big)$ for each x in X,
$k'(y, f, 0) = (y, f)$ for each (y, f) in $Y\times F$,
$k'(y, f, 1) = (p\times f)g(y, f)$ for each (y, f) in $Y\times F$,
$h'\big(p^{-1}(U_i)\times I\big)\subseteq p^{-1}(U_i)$ for each i,
$k'(U_i\times F\times I)\subseteq U_i\times F$ for each i.

Now define $L\colon A\times I\to Y\times F$ by the formula $L(a, t) = \big(H(a, t), fh_0(a, 0)\big)$ and define $L'(a, t) = gL(a, t)\colon A\times I\to X$. Now, $L'(a, 0) = g\big(H(a, 0), fh_0(a, 0)\big) = g\big(ph_0(a, 0), fh_0(a, 0)\big) = g(p\times f)h_0(a, 0)$. Thus, to obtain h, we must insert the homotopy h' of $g(p\times f)$ to 1_X. First choose $0<\delta<1$ so small that for each a in A, there is some U_i such that $H(\{a\}\times[0, \delta])\subseteq U_i$. This can be done because A is compact.

Now, set

$$h(a, t) = \begin{cases} h'\left(h_0(a, 0), \dfrac{t}{\delta}\right), & \text{if } 0\leqslant t\leqslant\delta\,,\\[2ex] L'\left(a, \dfrac{t-\delta}{1-\delta}\right), & \text{if } \delta\leqslant t\leqslant 1\,.\end{cases}$$

To see that h is well defined, observe that $h'\left(h_0(a, 0), \dfrac{\delta}{\delta}\right) = h'\big(h_0(a, 0), 1\big)$ $= g(p\times f)\big(h_0(a, 0)\big) = L'(a, 0) = L'\left(a, \dfrac{\delta-\delta}{1-\delta}\right)$. Thus h is well defined and continuous. To see that h extends h_0, compute $h(a, 0) = h'\big(h_0(a, 0), 0\big) = h_0(a, 0)$. To compute the distance between ph and H, there are two cases:

Case 1. $t\leqslant\delta$: for each a in A, $ph(a, t)$ and $H(a, 0)$ are both contained in some U_i, by choice of h'. $H(a, 0)$ and $H(a, t)$ are both contained in some U_j, by choice of δ. Note that $H(a, 0)\in U_i\cap U_j$. Thus

$$\begin{aligned} d\big(ph(a, t), H(a, t)\big) &\leqslant d\big(ph(a, t), H(a, 0)\big)+d\big(H(a, 0), H(a, t)\big)\\ &\leqslant \operatorname{diam} U_i+\operatorname{diam} U_j< \frac{\varepsilon\big(H(a, t)\big)}{2}+\frac{\varepsilon\big(H(a, t)\big)}{2} = \varepsilon\big(H(a, t)\big)\,.\end{aligned}$$

Case 2. $t\geqslant\delta$: For each a in A, $ph(a, t)$ and $H(a, t)$ both lie in some U_i. Thus

$$d\big(ph(a, t), H(a, t)\big)\leqslant\operatorname{diam} U_i<\frac{\varepsilon\big(H(a, t)\big)}{2}<\varepsilon\big(H(a, t)\big)\,.$$

This completes B.11.

B.12. COROLLARY. *If a proper map $p\colon X\to Y$ has property* LHP-F, *then p is an approximate fibration.*

Proof (cf. [St] 11.3). Let D be a disc and suppose

$$\begin{tikzcd} & & X \arrow[d, "p"] & \\ D\times\{0\} \arrow[r] \arrow[urr, "h_0"] & D\times I \arrow[r, "H"] & Y \arrow[r, "\varepsilon"] & (0,\infty) \end{tikzcd}$$

is given. Choose a locally finite covering $\{U_i\}$ for Y such that

1) $p|p^{-1}(U_i)\colon p^{-1}(U_i)\to U_i$ has property HP-F and
2) $\operatorname{diam} U_i < \min_{y\in U_i} \varepsilon(y)$.

Choose a triangulation $\{V_i\}_{i=1}^k$ of D and $\{I_\lambda\}_{\lambda=1}^l$ of I where each I_λ is an interval $[t_\lambda, t_{\lambda+1}]$, so that for each V_j and I_λ, there is a U_i so that $H(V_j\times I_\lambda)\subseteq U_i$. Now, order $\{V_i\}$ so that simplicies of one dimension preceed those of higher dimensions and order $\{I_\lambda\}$ so that $I_\lambda<I'_\lambda$, if and only if $t_\lambda<t'_\lambda$. Now order the cells $V_i\times I_\lambda$ as follows:

$$V_i\times I_\lambda \text{ preceeds } V_j\times I_{\lambda'}, \text{ if } \begin{cases} \lambda \text{ preceeds } \lambda' \\ \text{or} \\ \lambda=\lambda' \text{ and } i \text{ preceeds } j. \end{cases}$$

We are now ready to lift H inductively.

For each $V_i\times I_\lambda$, let

$$N_{i\lambda} = \bigcap_{h(V_i\times I_\lambda)\subseteq U_j} U_j$$

and note that $N_{i\lambda}$ is an open neighborhood of $H(V_i\times I_\lambda)$, since $\{U_j\}$ is locally finite. Now, define $\varepsilon_{i\lambda}\colon N_{i\lambda}\to(0,\infty)$ so that $\varepsilon_{i\lambda}(x)<d(x, Y\setminus N_{i\lambda})$. Suppose inductively that we have constructed a map $h\colon \bigcup_{V_i\times I_\lambda<V_j\times I_\lambda} V_i\times I_\lambda\to X$ so that $ph(V_i\times I_\lambda)\subseteq N_{i\lambda}$. We wish to extend h to a map of $V_j\times I_{\lambda'}$. Note that h is already defined on $(\partial V_j\times I_{\lambda'})\cup(V_j\times\{t_{\lambda'}\})$ and that on this set it takes values in $p^{-1}(N_{j\lambda'})$ by the induction hypothesis. Choose a homeomorphism

$$k\colon (\sigma\times I, \sigma\times\{0\})\to\big(V_j\times I_{\lambda'}, (\partial V_j\times I_{\lambda'})\cup(V_j\times\{t_{\lambda'}\})\big)$$

and observe that we have:

$$\begin{tikzcd} & & p^{-1}N_{j\lambda'} \arrow[d, "p"] & \\ \sigma\times\{0\} \arrow[r] \arrow[urr, "hk"] & \sigma\times I \arrow[r, "Hk"] & N_{j\lambda'} \arrow[r, "\varepsilon_{j\lambda'}"] & (0,\infty) \end{tikzcd}$$

and since $N_{j\lambda'}\subseteq U_i$ for some i, we have that p has property HP-F. Applying B.11 yields a map $h'\colon \sigma\times I\to p^{-1}(N_{j\lambda'})$. Now, h may be extended to $V_j\times I_{\lambda'}$ by the map $h'k^{-1}$. This completes the induction and also serves as the initial step since V_1 is a o-simplex and $\partial V_1=\emptyset$.

Thus p has the AHLP for cells and so, by Theorem 2.6 of [C-D 2], p is an approximate fibration.

B.13. COROLLARY. If *$p\colon X\to Y$ is a proper map of* ANR's *and there is a y in Y such that $p^{-1}(y)$ has the shape of a compact* ANR *F, then p is an approximate fibration if and only if p has property* LHP-F.

B.14. PROPOSITION. *Let $X=X_1\cup X_2$ and $Y=Y_1\cup Y_2$ be* ANR's, *written as the union of open sets. Suppose $p_1\colon X_1\to Y_1$ and $p_2\colon X_2\to Y_2$ are maps so that $p\colon X\to Y$ defined by*

$$p(x)=\begin{cases} p_1(x), & \text{if } x\in X\setminus X_2, \\ p_2(x), & \text{if } x\in X_2 \end{cases}$$

is continuous. Then if each $p_i\colon X_i\to Y_i$ has property HP-F, *p has property* LHP-F.

Proof. According to B.7 and 2.6 of [C-D 1], it suffices to show that p has the AHLP for discs. Let

$$\begin{tikzcd} & & X \arrow[d, "p"] & \\ D^n\times\{0\} \arrow[r] \arrow[urr, "h_0"] & D^n\times I \arrow[r, "H"] & Y \arrow[r, "\varepsilon"] & (0,\infty) \end{tikzcd}$$

be given. Now, for each y in Y, choose a neighborhood U_y of y so that $\operatorname{diam} U_y<\min_{x\in U_y}\frac{\varepsilon(x)}{2(n+1)}$ and triangulate D and I so finely that $H(\sigma_j\times I_\lambda)\subseteq U_y$ for some y.

Set $N=\bigcup_{U_y\cap(X\setminus X_2)\neq\emptyset} U_y$ and note that if $H(\sigma_j\times I_\lambda)\subseteq N$ and $l\colon \sigma_j\times I_\lambda\to X$ is such that $d\big(p_1 l(x), H(x)\big)<\delta\big(H(x)\big)$ then

$$d\big(p_2 l(x), H(x)\big)<\frac{\varepsilon\big(H(x)\big)}{2(n+1)}+\delta\big(H(x)\big).$$

Thus, to obtain a near lifting of H, we can induct as in B.12, using p_1 if $H(\sigma\times I_\lambda)\subseteq N$ and p_2 if not.

If we require that $\delta(x)$ be less than $\frac{\varepsilon(x)}{2(n+1)}$, the cumulative error will be less than $\varepsilon(x)$, since each cell $\sigma_j\times I_\lambda$ has dimension at most $n+1$. This yields the near lifting, and so, the proposition.

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