On the winding number and equivariant homotopy classes of maps of manifolds with some finite group actions

by

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Abstract. The paper considers equivariant maps of a closed connected m-dimensional manifold M with an effective smooth action of a finite group G into a punctured linear $(m+1)$-dimensional space $E\setminus\{0\}$ with a smooth action of G on E such that O is a fixed point and every isotropy group of the action on M acts trivially on E. The following questions are investigated:

1. What numbers may be the winding numbers of such maps?
2. What are the equivariant homotopy classes of such maps?

The well-known Borsuk theorem asserts that any equivariant map of a sphere with the antipodic action of Z_2 into itself has an odd degree. In this paper we take up the question what winding numbers (degrees) have equivariant maps of a closed connected smooth G-manifold M into a linear G-space E of dimension greater by 1 with O removed when every isotropy group of the action of a finite group G on M acts trivially on E (Theorem 2.2).

Although these assumptions are very restrictive, they contain the case of free actions on M and the case of the trivial action on E. Without the imposed assumptions the results may be false (Example 2.4).

Moreover, Theorems 3.1, 4.3 and 5.1 give a complete equivariant homotopy classification of such maps and may be viewed as a generalization of the Hopf theorem.

The methods used are similar to those in Krasnoselski's paper [5]. Although the maps under consideration are continuous, they are treated by means of rather differential topology methods as in [3] or [6].

In the whole paper G is a finite group. By a manifold we mean a paracompact smooth manifold without boundary. All actions of a group G are assumed to be smooth.

1. Auxiliary results. We shall use a kind of mappings given by

1.1. Definition. Let P be a p-dimensional manifold and E a real $(m+1)$-dimensional vector space. A map $f\colon P\to E$ is called *good* iff f is continuous on P, f is smooth on some open set Q containing $f^{-1}(O)$ and O is a regular value of $f|Q$. If, in addition, G acts smoothly on P and E, f is a G-map and $f^{-1}(O)$ is contained in the part P_e of P consisting of all points with the trivial isotropy group $\{e\}$, then F is called a *G-good map*.

For a good map, $f^{-1}(O)$ is a $(p-m-1)$-dimensional submanifold of P (invariant if f is G-good and O is a fixed point of the action of G on E) or is empty.

The following facts concern extensions of good maps to good maps.

1.2. *Let P be a manifold, F a closed set contained in an open subset U of P and D a compact set in P. If $f\colon U\to E$ is a good map, then there exist an open set W containing $F\cup D$ and a good map $h\colon W\to E$ such that $h|F=f|F$.*

Proof. Choose open sets U_0, U_1 and U_2 such that $F\subset U_0\subset \overline{U}_0\subset U_1\subset \overline{U}_1\subset U_2\subset \overline{U}_2\subset U$ and a smooth function $\varphi\colon P\to[0,1]$ satisfying conditions $\varphi(x)=0$ for $x\in \overline{U}_1$ and $\varphi(x)=1$ for $x\in P\setminus U_2$. Choose open sets V_0 and V such that $D\subset V_0\subset \overline{V}_0\subset V$ with $\overline{V}$ compact. By 1.1 there exists an open set Q containing $f^{-1}(O)$ such that $f|Q$ is smooth and O is a regular value of $f|Q$. Let $\varepsilon>0$ be the minimum of $|f(x)|$ for x belonging to the compact set $K=(\overline{V}\cap \overline{U}_2)\setminus(U_1\cup Q)$. Let $f_0\colon P\to E$ be a smooth map such that $|f_0(x)-f(x)|<\varepsilon$ for $x\in K$. Define the map $f_1\colon U_1\cup V\to E$ by

$$f_1(x)=\begin{cases} f(x)+\varphi(x)\big(f_0(x)-f(x)\big) & \text{if } x\in U_1\cup(V\cap U),\\ f_0(x) & \text{if } x\in V\setminus \overline{U}_2.\end{cases}$$

f_1 is continuous, $f_1|U_1=f|U_1$, $f_1(x)\neq 0$ for $x\in K\cap V$ and therefore f_1 is smooth in some open set Z containing $f_1^{-1}(O)$ if $f_1^{-1}(O)\neq\emptyset$.

Let Z_0 and Z_1 be open sets such that $f_1^{-1}(O)\cap(\overline{V}_0\setminus U_0)\subset Z_0\subset\overline{Z}_0\subset Z_1\subset\overline{Z}_1\subset Z\setminus F$ with $\overline{Z}_1$ compact. Let $\psi\colon P\to[0,1]$ be a smooth function such that $\psi(x)=0$ for $x\in P\setminus Z_1$ and $\psi(x)=1$ for $x\in\overline{Z}_0$. There exists a compact set K_0 such that $f_1^{-1}(O)\cap(\overline{Z}_1\setminus Z_0)\cap\overline{U}_0\subset \operatorname{Int}K_0\subset K_0\subset U_1\setminus F$ and the tangent maps df_{1x} are epimorphisms for $x\in K_0$. The set $K_1=(\overline{Z}_1\setminus Z_0)\cap\overline{V}_0\setminus\operatorname{Int}K_0$ is compact and $f_1(x)\neq 0$ for $x\in K_1$. By the Sard lemma there exists a regular value $a\in E$ for $f_1|Z$ arbitrary close to O. Define $W=U_0\cup V_0$ and $h\colon W\to E$ by $h(x)=f_1(x)-\psi(x)a$. If $|a|$ is sufficiently small, then dh_x are epimorphisms for $x\in K_0$ and $h(x)\neq 0$ for $x\in K_1$. Therefore h is a good map and $h|F=f_1|F=f|F$.

1.3. *Let G act on a manifold P and on a vector space E with the fixed point O. Let U be a G-invariant open subset of P and V an H-invariant open subset of P for a subgroup H of G such that $gV\cap V=\emptyset$ for $g\in G\setminus H$. If $f\colon U\cup V\to E$ is a good map, $f^{-1}(O)\subset P_e$, $f|U$ is G-equivariant and $f|V$ is H-equivariant, then there is a unique extension of f to the G-good map $\bar{f}\colon U\cup GV\to E$.*

Proof. Set

$$\bar{f}(x)=\begin{cases} f(x) & \text{for } x\in U,\\ gf(g^{-1}x) & \text{for } x\in gV \text{ and } g\in G.\end{cases}$$

1.4. *Let B be a b-dimensional G-manifold with exactly one type of orbits corresponding to the conjugacy class of isotropy subgroups (H). Let G act on an $(m+1)$-dimensional vector space E with the fixed point O in such a way that H acts trivially on E and $b\leqslant m$ or $H=\{e\}$. If $f\colon W\to E$ is a G-good map on an open invariant subset W of B containing a closed invariant set A, then $f|A$ can be extended to a G-good map $h\colon B\to E$. (If $b\leqslant m$, this means that $h(x)\neq 0$ for $x\in B$ and h is equivariant.)*

Proof. We can assume that B is separable. At any point $x\in B\setminus A$ with the isotropy group $G_x=H$ there is a slice V (cf. [1] or [4]) which may be identified with R^b with an orthogonal action of H. Take a unit closed ball D in V. The set $B\setminus A$ can be covered by tubes $\operatorname{Int}GD_j$, where D_j are such disks, $j=1,2,\dots$ By induction we define G-good maps $f_k\colon W_k\to E$ on open invariant sets W_k containing $A\cup\bigcup_{j=1}^{k}GD_j=F_k$ such that $f_k|F_k=f_{k+1}|F_k$ for $k=0,1,2,\dots$ Put $W_0=W$ and $f_0=f$. Having f_k on W_k by 1.2, we choose a good map $h_k\colon U_k\cup V_{k+1}\to E$ where $U_k\supset F_k$ is open G-invariant, $V_{k+1}\supset D_{k+1}$ is open contained in a slice and $h_k|F_k=f_k|F_k$. The subgroup H acts trivially on V_{k+1} and on E, and so by 1.3 there is a G-good extension $f_{k+1}\colon W_{k+1}\to E$ of h_k on $W_{k+1}=U_k\cup GV_{k+1}\supset F_{k+1}$. The G-good extension $h\colon B\to E$ of $f|A$ is defined by $h(x)=f_k(x)$ for $x\in F_k$.

If G acts effectively on a connected manifold P, then the trivial group $\{e\}$ is principal (cf. [7]). The open and dense set P_e is called the *principal part of P* (cf. [1] or [4]). Its complement $P'=P\setminus P_e$ will be called the *singular part of P*. It is a finite union of submanifolds, and the dimension of P' is the greatest of the dimensions of those manifolds.

The following lemma will be important in our considerations.

1.5. Extension lemma. *Let G act on an $(m+1)$-dimensional vector space E with the fixed point O. Suppose also that G acts effectively on a connected manifold P in such a way that each isotropy group of the action on P acts trivially on E and $\dim P'\leqslant m$. If $f\colon U\to E$ is a G-good map on an open invariant set $U\subset P$ containing a closed invariant set F, then there exists a G-good extension $h\colon P\to E$ of $f|F$. If f is smooth, then h is also smooth.*

Proof. All conjugacy classes of isotropy groups of the G-manifold P are partially ordered. Thus they can be arranged in a sequence $(H_1),(H_2),\dots,(H_n)=(e)$ in such a way that whenever $(H_i)>(H_j)$ then $i<j$. The set $P_{(H_i)}$ of points of P with the isotropy groups belonging to H_i is a disjoint union of submanifolds of P by the existence of slices and $\dim P_{(H_i)}\leqslant m$ if $i<n$. Denote $F_k=F\cup\bigcup_{i=1}^{k}P_{(H_i)}$ for $k=0,1,\dots,n$. The sets F_k are closed because in a slice at a point x of P there are only points with isotropy groups not greater than G_x.

We shall construct G-good maps $f_k\colon U_k\to E$ on open invariant sets U_k containing F_k for $k=0,1,\dots,n$ such that $f_k|F_k=f_{k+1}|F_k$. Set $U_0=U$ and $f_0=f$. Suppose that U_k and f_k have been constructed and $k<n$.

Denote $B=P_{(H_{k+1})}$. Let W be an open invariant subset of P such that $F_k\subset W\subset\overline{W}\subset U_k$. Set $V_0=B\cap W$ and choose invariant sets V_1 and V_2 open in B and satisfying the condition $\overline{V}_0\subset V_1\subset\overline{V}_1\subset V_2\subset\overline{V}_2\subset B\cap U_k$, where closures are taken in B. The map $f_k|V_1$ is G-good because if $k+1<n$ then $f_k(x)\neq 0$ for $x\in V_1\subset B$

and if $k+1 = n$ then V_1 is open in P. By 1.4 there exists a G-good extension $h_k\colon B \to E$ of $f_k|\overline{V}_0$. If $k+1 = n$ set

$$f_{k+1}(x) = \begin{cases} f_k(x) & \text{if } x \in V_0\,, \\ h_k(x) & \text{if } x \in B = P_e\,. \end{cases}$$

Suppose that $k+1<n$. There exists an invariant tubular neighbourhood N of B, open in P, which can be identified with the equivariant normal bundle of B in P. Let $\pi\colon N \to B$ be the projection. We can assume that:

1. In this bundle we have an equivariant Riemannian metric.

2. The part of the bundle of the unit open disks N_1 over $\overline{V}_2$ denoted by $N_1|\overline{V}_2$ is contained in U_k.

3. The diameters $\delta(N_y)$ of the fibres of the vector bundle N over points $y \in B$ in some metric on P tend to O when y tends to infinity — the point added to B in the one-point compactification of B.

The set $B\setminus V_0 = F_{k+1}\setminus V_0$ is closed in P and the closure in P $\overline{N_1|B\setminus V_0}$ is the part of the bundle of the unit closed disks of N over $B\setminus V_0$ by condition 3. Let $\varphi\colon B \to [0, 1]$ be an equivariant smooth map such that $\varphi(x) = 1$ for $x \in \overline{V}_0$ and $\varphi(x) = 0$ for $x \in B\setminus V_1$. We can assume that $f_k(x) \neq 0$ for $x \in N_1|V_2$. Define $U_{k+1} = (U_k\setminus(\overline{N_1|B\setminus V_0})) \cup N_1$ and $f_{k+1}\colon U_{k+1} \to E$ by

$$f_{k+1}(x) = \begin{cases} f_k(x) & \text{for } x \in U_k\setminus(\overline{N_1|B\setminus V_0})\,, \\ f_k(\varphi(x)\cdot x) & \text{for } x \in N_1|V_2\,, \\ h_k \circ \pi(x) & \text{for } x \in N_1|B\setminus V_1\,. \end{cases}$$

f_{k+1} is a well-defined G-good map on U_{k+1}.

The last map $h = f_n$ is a G-good extension of $f|F$ on the whole manifold $P = U_n$.

All maps in 1.2, 1.3, 1.4 and 1.5 are smooth if f was smooth.

1.6. Corollary. *Let M be a closed connected manifold with an effective action of G. Let G act on the vector space E with the fixed point O in such a way that each isotropy group of the action on M acts trivially on E. Denote $E_0 = E\setminus\{0\}$.*

a) *If $\dim M < \dim E$, then there is a smooth G-map $f\colon M \to E_0$.*

b) *If $\dim M < \dim E - 1$, then any two continuous (smooth) G-maps $f_0, f_1\colon M \to E_0$ are G-homotopic (smoothly).*

For the proof we take in 1.5 $P = M$ and $U = \emptyset$ in case a) and

$$P = R\times M\,, \quad U = (R\setminus\{\tfrac{1}{2}\})\times M\,, \quad F = \{0, 1\}\times M$$

and

$$f(t, x) = \begin{cases} f_0(x) & \text{for } t \in (-\infty, \tfrac{1}{2}),\ x \in M \\ f_1(x) & \text{for } t \in (\tfrac{1}{2}, +\infty),\ x \in M \end{cases} \quad \text{in case b)}.$$

The following example shows that equivariant maps do not always exist.

1.7. Example. There is no equivariant map of the unit sphere $S^2 \subset R^3$ with the antipodic action of Z_2 into an orientable surface S_g of genus $g>0$ embedded

symmetrically with respect to O in R^3 with the action of Z_2 by symmetry with respect to O.

If such a map $f\colon S^2 \to S_g$ exists, then f is homotopic to a constant map because S^2 has a trivial fundamental group and the universal covering space of S_g is homeomorphic to an open disk. Therefore $\deg f = 0$. By 1.6 a) there exists an equivariant map $g\colon S_g \to S^2$ because S^2 is an equivariant deformation retract of $R^3\setminus O$. The map $g\circ f\colon S^2 \to S^2$ is equivariant with $\deg g\circ f = 0$, which contradicts Borsuk's theorem.

1.8. Homogeneity lemma. *Let G act effectively on a connected manifold P. If x, y belong to the same component C of the principal part P_e, then there exists an equivariant diffeomorphism $h\colon P \to P$ mapping x to y, equivariantly diffeotopic to id_P by the diffeotopy h_t, which does not move points beyond some compact invariant set and beyond P_e.*

The proof is similar to that in the non-equivariant case ([6]). We have in C the equivalence relation: $x\sim y$ iff the statement of 1.8 is true. Let V be a slice at x in C diffeomorphic to a Euclidean space and let $y \in V$. By the non-equivariant homogeneity there exists a diffeotopy $f_t\colon V \to V$ such that $f_0 = \mathrm{id}_V$, $f_1(x) = y$ and $f_t(z) = z$ beyond some compact set. We define the equivariant diffeotopy $h_t\colon P \to P$ by

$$h_t(z) = \begin{cases} z & \text{if } z \in P\setminus GV\,, \\ gf_t(g^{-1}z) & \text{if } z \in gV,\ g \in G\,. \end{cases}$$

Therefore the classes of the relation are open and C is the only class by connectivity.

1.9. Remark. If the component C is a nonorientable manifold and o_x and o_y are any orientations of the tangent spaces T_xP and T_yP, respectively, for $x, y \in C$, then the G-diffeomorphism h of 1.8 can be chosen in such a way that the tangent map dh_x maps o_x to o_y.

1.10. *There is a generalization of* 1.8 (*and* 1.9) *analogous to that in the non-equivariant case: If* $\dim P>1$ *and* x_i, y_i *for* $i = 1, \ldots, k$ *are two k-tuples of points of a component C belonging to different orbits, then there is a G-diffeomorphism $h\colon P \to P$ G-diffeotopic to id_P such that $h(x_i) = y_i$ for $i = 1, \ldots, k$.*

This follows by induction on k because a finite set does not separate a manifold of dimension greater than 1.

1.11. Remark. If x and y belong to different components of P_e, then a G-diffeomorphism $h\colon P \to P$ such that $h(x) = y$ does not always exist, e.g. if the subgroup of G preserving the component C of x denoted by G_C is different from the subgroup $G_{gC} = gG_Cg^{-1}$ for $y = gx$ (as in Example 3 of [7]). If g belong to the centre of G, then such an h exists. But there is no G-diffeotopy h_t from id_P to h because each h_t would map P_e onto P_e and C onto C.

2. Winding numbers of equivariant maps.

2.1. *Let M be a closed connected manifold of dimension $m\geq 1$ with an effective smooth action of a finite group G. Suppose that G acts smoothly on an $(m+1)$-dimen-*

sional Euclidean vector space E, O is a fixed point and every isotropy group of the action on M acts trivially on E. Denote $E_0 = E\setminus\{O\}$. *The above assumptions will always be observed in the sequel.*

If M is oriented, there are two possibilities:

a) *Every* $g \in G$ *simultaneously preserves the orientations of M and E or simultaneously reverses them.*

b) *Some* $g \in G$ *preserves the orientation of M and reverses the orientation of E or vice versa.*

In case a) we shall say that the actions of G on M and E are concordant and in case b) that they are discordant.

If M and E are oriented, then for a continuous map $f\colon M \to E_0$ the winding number $W(f)$ is by definition the degree of the map $f/|f|\colon M \to S^m \subset E$, where S^m is the unit sphere in E oriented as the boundary of the unit ball in E. If M is non-orientable, then the winding number modulo 2 denoted by $W_2(f)$ is defined similarly.

2.2. Theorem. *Let* G, M, E, E_0 *be as in* 2.1 *and let* M *be oriented.*

a) *If the actions of G are concordant, then for any continuous equivariant maps* $f_0, f_1\colon M \to E_0$ $W(f_0) \equiv W(f_1) \bmod |G|$.

b) *If the actions of G are discordant, then for any continuous equivariant map* $f\colon M \to E_0$ $W(f) = 0$ (*even without the assumptions about isotropy groups*).

Proof. b) Let θ_g and ψ_g denote the diffeomorphisms of M and E, respectively, corresponding to $g \in G$. The local degree at 0, $\deg_0 \psi_g$, is equal to 1 if ψ_g preserves the orientation of E and equal to -1 otherwise. Since the action of G is discordant, there exists a $g \in G$ such that $\deg\theta_g = -\deg_0\psi_g$. The map f is equivariant, and so $f \circ \theta_g = \psi_g \circ f$. Therefore $W(f)\deg\theta_g = \deg_0\psi_g W(f)$, $W(f) = -W(f)$ and $W(f) = 0$.

a) By the extension Lemma 1.5 applied to the manifold $P = R \times M$, the sets $U = (R\setminus\{\frac{1}{2}\}) \times M$, $F = (R\setminus(0, 1)) \times M$ and the mapping $f\colon U \to E$ defined by

$$f(t, x) = \begin{cases} f_0(x) & \text{if } t<\frac{1}{2}, \\ f_1(x) & \text{if } t>\frac{1}{2} \end{cases}$$

there is a G-good homotopy $h\colon I \times M \to E$ from f_0 to f_1. ($I \times M$ is a manifold with boundary, but h can be extended to a G-good map on the manifold P without boundary. Similarly we shall use the notion of G-good map in the sequel). $h^{-1}(O)$ is a finite equivariant subset of $(0, 1) \times M$, because $\dim I \times M = m+1 = \dim E$. Choose one point x_i in each orbit of $h^{-1}(O)$ for $i = 1, \ldots, k$.

It is known (cf. [3] or [6]) that $W(f_1) - W(f_0) = \sum_{x \in h^{-1}(O)} \deg_x h$, where $\deg_x h$ is the local degree at isolated zero x of h. If $\bar\theta_g$ and ψ_g denote the diffeomorphisms of $I \times M$ and E, respectively, corresponding to $g \in G$, then $\deg\bar\theta_g = \deg_0\psi_g$ because the actions of G are concordant. From the equality $h \circ \bar\theta_g = \psi_g \circ h$ we get $\deg_{gx} h \cdot \deg\bar\theta_g = \deg_0\psi_g \cdot \deg_x h$ and $\deg_{gx} h = \deg_x h$ for every $x \in h^{-1}(O)$ and $g \in G$.

So the local degrees of h at all points of one orbit of $h^{-1}(O)$ are equal. Therefore $W(f_1) - W(f_0) = \sum_{i=1}^{k} |G| \deg_{x_i} h$ and $W(f_1) \equiv W(f_0) \bmod |G|$.

2.3. Remark. If the action of G on E is orthogonal, we can consider equivariant maps $M \to S^m$ instead of $M \to E_0$ and the degrees of such maps instead of winding numbers. Since the sphere S^m is an equivariant deformation retract of E_0, this concerns also the results in sections 3–5.

Theorem 2.2 a) may be false if the assumptions on the isotropy groups are not satisfied.

2.4. Example. Consider the action of Z_2 on the unit circle $M = S^1 \subset R^2$ and $E = R^2$, in which the generator of Z_2 acts by symmetry with respect to a line. Those actions are concordant. The maps $f_0 = \mathrm{id}_M$ and $f_1 =$ constant map into one of two fixed points on S^1 are equivariant, but $\deg f_0 = 1$ and $\deg f_1 = 0$.

2.5. Corollary. *If the action of G on E is trivial, then for any action of G on M and mapping* $f\colon M \to E_0$ *constant on orbits* $W(f) \equiv 0 \bmod |G|$.

2.6. Remark. Theorem 2.2 can always be applied if the action of G on M is free. The proof in this case may be considerably simplified.

2.7. Example. Suppose that G acts on an $(m+1)$-dimensional Euclidean vector space with the fixed point O, N is a compact $(m+1)$-dimensional invariant submanifold of E with boundary $M = \partial N \subset E_0$ and the induced action of G on M is free. Then, for any equivariant map $f\colon M \to E_0$, $W(f) \equiv 0 \bmod |G|$ if $O \notin N$ and $W(f) \equiv 1 \bmod |G|$ if $O \in N$.

Indeed, if f_0 is the inclusion $M \to E_0$, then it is equivariant and has an extension to the inclusion $\bar f_0\colon N \to E$ without zeros if $O \notin N$ and with exactly one zero 0 with the local degree $\deg_0 \bar f_0 = 1$ if $O \in N$. Since $W(f_0) = \deg_0 \bar f_0$, this follows from 2.2 a).

If, in addition, the action of G on E is orthogonal, then the Gauss map $f_1\colon M \to S^m \subset E_0$, which assigns to a point $x \in M$ the unit vector normal to M at x directed outward of N, is equivariant. The degree of f_1 is equal to the Euler–Poincaré characteristic $\chi(N)$ of N (cf. [2]). For any equivariant map $f\colon M \to E_0$, $W(f) \equiv \chi(N) \bmod |G|$. If the number m is even, then $\chi(N) = \frac{1}{2}\chi(M)$ (by considering the double of N).

2.8. Example. Let E be an $(m+1)$-dimensional linear space, and N a compact $(m+1)$-dimensional manifold in E with boundary $M = \partial N$. Let $T\colon M \to M$ be a fixed point free smooth involution. T defines an action of Z_2 on M. Consider E with the action of Z_2 generated by symmetry with respect to O. Let $\bar T\colon N \to E$ be any continuous extension of T. The map $f_0\colon M \to E_0$ defined by $f_0(x) = x - T(x)$ is equivariant and $W(f_0) = \operatorname{ind}\bar T$, where $\operatorname{ind}\bar T$ is the fixed point index of $\bar T$ (cf. [2]) (the set of fixed points of $\bar T$ is compact and contained in $\operatorname{Int} N$). By 2.2, for any equivariant map $f\colon M \to E_0$, $W(f) \equiv \operatorname{ind}\bar T \bmod 2$.

2.9. Example. Let Z_n act on $E = C^p$ with the action of a generator g of Z_n defined by $\psi_g(z) = e^{2\pi i k/n} z$ for $z \in C^p$, where n and k are natural numbers. Let Z_n

act also on the unit sphere $M = S^{2p-1} \subset C^p$, the action defined by $\theta_g(z) = e^{2\pi il/n}z$ for $z \in S^{2p-1}$, where the natural numbers n and l are relatively prime. The action on S^{2p-1} is free. The class of $l \bmod n$ denoted by $[l]$ is an invertible element of Z_n. Let $[q] = [k]/[l]$ in Z_n, i.e. $ql \equiv k \bmod n$. The map $f_0\colon S^{2p-1} \to C^p \setminus \{O\}$ defined by $f_0(z_1, \ldots, z_p) = (z_1^q, \ldots, z_p^q)$ is equivariant and $W(f_0) = q^p$ because f_0 has an extension $\bar{f}_0\colon C^p \to C^p$ given by the same formula, O is the unique zero of $\bar{f}_0$ and $\deg_O \bar{f}_0 = q^p$. By 2.1, for any equivariant map $f\colon S^{2p-1} \to C^p \setminus \{O\}$, $W(f) \equiv q^p \bmod n$.

2.10. Remark. In case 2.2 a) if G_{p_i}, $i = 1, \ldots, k$, are Sylow subgroups of G and r_i is a number such that G_{p_i}-maps $f_i\colon M \to E_0$ have $W(f_i) \equiv r_i \bmod |G_{p_i}|$ for $i = 1, \ldots, k$, then the number r of 2.2 a) is uniquely $\bmod |G|$ determined by the numbers r_i by the conditions $r \equiv r_i \bmod |G_{p_i}|$ for $i = 1, \ldots, k$.

2.11. Corollary. *Under the conditions of Theorem* 2.2, *if in addition, the action of G on E is linear, then, for every continuous map $f\colon M \to E_0$ with $W(f) \not\equiv r \bmod |G|$ in the concordant case and $W(f) \neq 0$ in the discordant case, there is a point $x \in M$ such that $O \in \operatorname{conv}\{gf(g^{-1}x)\}_{g \in G}$.*

Indeed, if $O \notin \operatorname{conv}\{gf(g^{-1}x)\}_{g \in G}$, then the map $f_0\colon M \to E_0$ defined by

$$f_0(x) = \frac{1}{|G|} \sum_{g \in G} gf(g^{-1}x)$$

is equivariant and homotopic to f (by the standard homotopy). Therefore $W(f) = W(f_0) \equiv r \bmod |G|$ in the concordant case or $W(f) = W(f_0) = 0$ in the discordant case, which contradicts the assumptions.

In particular, if the action of G on E is trivial and if $W(f) \not\equiv 0 \bmod |G|$, then there is a point $x \in M$ such that $O \in \operatorname{conv}\{f(gx)\}_{g \in G}$.

In the case of the action of Z_2 by symmetry with respect to O on E and a free action of Z_2 on M we get

2.12. Corollary. *Let T be a fixed point free smooth involution on M and $f\colon M \to S^m$ a continuous map into the unit sphere in E.*

a) *If f has an odd degree, then there is a point $x \in M$ such that $f(Tx) = -f(x)$.*

b) *If $\deg f \not\equiv r \bmod 2$ in the concordant case or $\deg f \neq 0$ in the discordant case, then there is a point $x \in M$ such that $f(Tx) = f(x)$.*

From b) it follows that in the concordant case if $r = 1$, then every continuous map $\varphi\colon M \to R^m$ has a point $x \in M$ such that $\varphi(Tx) = \varphi(x)$ because R^m is homeomorphic to $S^m \setminus \{pt\}$.

3. Concordant actions. The following theorem gives the equivariant homotopy classification of maps in the concordant case.

3.1. Theorem. *Let G, M, E, E_0, r be as in* 2.1 *and* 2.2 a), *i.e. M is oriented and the actions of G on M and E are concordant. Then the function $W\colon [M, E_0]_G \to \{n = r + k|G| \text{ for } k \in Z\}$, assigning to an equivariant homotopy class $[f]$ represented by a continuous equivariant map $f\colon M \to E_0$ its winding number $W(f)$, is bijective.*

Proof. a) Surjectivity. Let $f_0\colon M \to E_0$ be equivariant continuous. Such

a map exists by 1.6 a). We may assume that $W(f_0) = r$. Let k be any integer different from O and $0 < a < 1$. Let p_i, $i = 1, \ldots, k$, be points of $(a, 1) \times C$ belonging to different orbits of the G-manifold $P = R \times M$, where C is a component of M_e. Let V_i be a slice at p_i contained in $(a, 1) \times C$ such that gV_i are disjoint for $i = 1, \ldots, k$ and $g \in G$. V_i can be mapped onto E by a diffeomorphism preserving the orientation if $k > 0$ and reversing the orientation if $k < 0$ such that p_i is mapped onto O. Those diffeomorphisms can be extended by 1.3 to a G-good map $f\colon U \to E$, where $U = (R \setminus [a, 2]) \times M \cup \bigcup_{i=1}^{k} GV_i$ such that $f(t, x) = f_0(x)$ for $t \in R \setminus [a, 2]$ and $x \in M$. By the extension Lemma 1.5 applied to $P = R \times M$ and $F = \{0, 3\} \times M \cup \bigcup_{i=1}^{k} GD_i$, where D_i are closed discs about p_i in V_i there exists a G-good map $h\colon [0, 3] \times M \to E$ such that $h|F = f|F$.

$h^{-1}(O)$ consists of points gp_i, $i = 1, \ldots, k$, $g \in G$ and additional points q_j, $j = 1, \ldots, l$. We may assume by 1.10 that $q_j \in (1, 2) \times M_e$. Define $f_1\colon M \to E_0$ by $f_1(x) = h(1, x)$. Then the restriction of h to $I \times M$ gives a G-good homotopy from f_0 to f_1. For $i = 1, \ldots, k$ $\deg_{p_i} h = \operatorname{sgn} k$, and for all $g \in G$ $\deg_{gp_i} h = \operatorname{sgn} k$, because the actions of G are concordant. Therefore $W(f_1) - W(f_0) = k|G|$ and $W(f_1) = r + k|G|$.

b) Injectivity. Suppose that for equivariant continuous maps $f_0, f_1\colon M \to E_0$ $W(f_0) = W(f_1)$. By the extension Lemma 1.5 there exists a G-good homotopy $h\colon I \times M \to E$ from f_0 to f_1. If $h^{-1}(O)$ is nonvoid, let $h^{-1}(O)$ consist of points gp_i, $i = 1, \ldots, k$, $g \in G$, where $p_i \in (0, 1) \times C$ and C is a component of M_e (cf. Proposition 2 of [7]). From the equalities $O = W(f_1) - W(f_0) = |G| \sum_{i=1}^{k} \deg_{p_i} h$ and $\deg_{p_i} h = \pm 1$ it follows that k is even and the points p_i can be arranged in such a way that $\deg_{p_i} h = (-1)^i$. Let $V \subset (0, 1) \times C$ be a slice at p_1 and D an open ball about p_1 in V. By 1.10 we may assume that $V \cap h^{-1}(O) = D \cap h^{-1}(O) = \{p_1, p_2\}$. By the Hopf theorem $h|V \setminus D$ can be extended to a continuous map $f\colon V \to E_0$. By 1.3 there is a G-good map $\bar{h}\colon I \times M \to E$ extending f and $h|I \times M \setminus G\bar{D}$. $\bar{h}^{-1}(O)$ consists of the orbits of p_i for $i > 2$ if $k > 2$. Proceeding further similarly, we get an equivariant homotopy $\bar{\bar{h}}\colon I \times M \to E_0$ from f_0 to f_1.

4. Discordant actions. Before formulating the general result in this case we give some examples. We still observe the assumptions of 2.1.

4.1. Example. Let M be an orientable manifold with a free action of G not preserving the orientation and let G act trivially on a linear space E. In this case the space of orbits M/G is a nonorientable manifold. There is a bijective correspondence between the set of equivariant homotopy classes $[M, E_0]_G$ and the set of non-equivariant homotopy classes $[M/G, E_0]$. By the Hopf theorem the degree mod 2 gives the bijective correspondence $[M/G, E_0] \approx Z_2$ and there are two different equivariant homotopy classes in $[M, E_0]_G$ although the winding number of any equivariant map $f\colon M \to E_0$ is 0 by 2.2 b). The same is true for equivariant maps $f\colon M \to S^m$.

In particular, there are two equivariant homotopy classes if M is an even-dimensional sphere with the action of Z_2 by antipodism (M/Z_2 is the nonorientable projective space) or if M is an orientable surface of genus g lying symmetrically with respect to O in R^3 with the action of Z_2 by symmetry with respect to O (M/Z_2 is the nonorientable surface of genus $g+1$).

4.2. Example. Let M be a G-manifold having a compact fundamental set in the sense of [7] and let G act trivially on E. There is a bijective correspondence between $[M, E_0]_G$ and $[F, E_0]$. If F is contractible, then there is only one homotopy class in $[F, E_0]$ and in $[M, E_0]_G$.

In particular, there is one equivariant homotopy class if M is the unit sphere of any orthogonal representation of G on V whose singular part is a union of hyperplanes (Corollary 9 in [7]). This is the case for the symmetry group of any Platon polyeder.

In the case of discordant actions of G on M and E the group G is the disjoint union of the subgroup G_+ and its coset G_-, where the actions of G_+ on M and E are concordant. The number $|G|$ is even. The equivariant homotopy classification of maps in this case gives

4.3. Theorem. *Let G, M, E, E_0 be as in* 2.1, *M being orientable, and let the actions of G on M and E be discordant. Denote by $M' = M\setminus M_e$ the singular part of M.*

a) *If $\dim M' = m-1$ then $[M, E_0]_G$ consists of one class.*

b) *If $\dim M' < m-1$ then $[M, E_0]_G$ consists of two classes.*

Proof. a) Let $f_0, f_1\colon M \to E_0$ be any equivariant maps (such maps exist by 1.6 a)). By the extension Lemma 1.5 there is a G-good homotopy $h\colon I\times M \to E$ from f_0 to f_1. The singular part $(0,1)\times M'$ of the $(m+1)$-dimensional manifold $P = (0,1)\times M$ has dimension m. Therefore there exists a $g_0 \in G\setminus\{e\}$ such that the fixed set P^{g_0} of g_0 has a component Q which is an m-dimensional submanifold of P. On a slice at any point from Q, g_0 acts by symmetry with respect to a hyperplane, and so $g_0^2 = e$. For any $g \in G$ different from e and g_0 the intersection $Q \cap P^g$ is a finite union of manifolds of dimensions less than m. Therefore there exists a point $x_0 \in Q$ with the isotropy group $G_{x_0} = \{e, g_0\}$. Let V be a slice at x_0. V may be identified with R^{m+1}, x_0 with 0 and $Q \cap V$ with a hyperplane H given by the equation $x_{m+1} = 0$. g_0 acts on V by symmetry with respect to this hyperplane. Let D be the unit open ball in V, $V_+ = \{x \in V\colon x_{m+1} \geqslant 0\}$, $V_- = \{x\in V\colon x_{m+1}\leqslant 0\}$ and let C be the component of P_e containing $\operatorname{Int} V_+$.

If $h^{-1}(O)$ is nonempty, it is a finite invariant subset of P_e. There exists a point $p \in C \cap h^{-1}(O)$. By 1.10 we can assume that $V_+ \cap h^{-1}(O) = D \cap V_+ \cap h^{-1}(O) = \{p\}$. There exists a continuous retraction $r\colon V_+ \to V_+\setminus D$. Define the map $f\colon V \to E_0$ by

$$f(x) = \begin{cases} h\circ r(x) & \text{for } x \in V_+,\\ h\circ r(g_0 x) & \text{for } x\in V_-.\end{cases}$$

f is G_{x_0}-equivariant because g_0 acts trivially on E. By 1.3 there is a G-good map $\bar{h}\colon I\times M \to E$ extending f and $h|I\times M\setminus G\bar{D}$. The number of orbits in $\bar{h}^{-1}(O)$ is less by 1 than that in $h^{-1}(O)$. By a similar procedure we get an equivariant homotopy $\tilde{h}\colon I\times M \to E_0$ from f_0 to f_1.

b) The condition $\dim M' < m-1$ implies that M_e is connected. Fix some equivariant map $f_0\colon M \to E_0$ (by 1.6 a)). As in part a) of the proof of 3.1, there is an equivariant map $f_1\colon M \to E_0$ and a G-good homotopy $H_0\colon I\times M \to E$ from f_0 to f_1 such that $H_0^{-1}(O)$ consists of exactly one orbit. We shall prove that $[M, E_0]_G$ consists of two different classes $[f_0]$ and $[f_1]$.

Let $f\colon M \to E_0$ be any continuous equivariant map. As in part b) of the proof of 3.1, there is a G-good homotopy $h\colon I\times M \to E$ from f_0 to f. Suppose that $h^{-1}(O)$ contains more than one orbit. Since the actions of G are discordant, for any $p\in h^{-1}(O)$ $\deg_{gp} h = \deg_p h$ if $g \in G_+$, $\deg_{gp} h = -\deg_p h$ if $g\in G_-$ and $\deg_p h = \pm 1$ because h is G-good. We can choose points $p_1, p_2 \in h^{-1}(O) \subset (0,1)\times M = P$, from different orbits in such a way that $\deg_{p_1} h = -1$ and $\deg_{p_2} h = 1$. Let V be a slice at p_1 and let D be an open unit ball in V. By 1.10 we may assume that

$$V\cap h^{-1}(O) = D\cap h^{-1}(O) = \{p_1, p_2\}$$

because $P_e = (0,1)\times M_e$ is connected. As in part b) of the proof of 3.1, we can modify h to a G-good homotopy $\bar{h}$ from f_0 to f without orbits or with one orbit in $\bar{h}^{-1}(O)$. In the first case $[f] = [f_0]$. In the second case there is a G-good homotopy h' from f_1 to f such that $h'^{-1}(O)$ consists of two orbits. Similarly, h' can be modified to a G-good homotopy $h''\colon I\times M \to E_0$ from f_1 to f, so $[f] = [f_1]$.

It remains to prove that the classes $[f_0]$ and $[f_1]$ are different. We have the G-good homotopy $H_0\colon I\times M \to E$ from f_0 to f_1 with $H^{-1}(O)$ consisting of one orbit. Suppose, on the contrary, that there exists also a continuous equivariant homotopy $H_1\colon I\times M \to E_0$ from f_0 to f_1. The homotopies H_0 and H_1 may be considered as G-good maps on the manifold without boundary $R\times M$ and we can suppose that there are numbers $0<a<b<1$ such that $H_0(t,x) = H_1(t,x) = f_0(x)$ for $t<a$ and $H_0(t,x) = H_1(t,x) = f_1(x)$ for $t>b$. The G-manifold $P = R\times R\times M$ has the singular part $P' = R\times R\times M'$ and $\dim P' \leqslant m$ by the assumption of $\dim M' \leqslant m-2$. Extension Lemma 1.5 applied to P, $F = P\setminus (0,1)\times(0,1)\times M$, $U = P\setminus [a,b]\times[a,b]\times M$ and to the G-good map $H\colon U \to E$ defined by

$$H(s,t,x) = \begin{cases} H_0(t,x) & \text{if } s<a,\\ H_1(t,x) & \text{if } s>b,\\ f_0(x) & \text{if } t<a,\\ f_1(x) & \text{if } t>b\end{cases}$$

gives a G-good map $\bar{H}\colon P\to E$ extending $H|F$.

The set $L = \bar{H}^{-1}(O) \cap I\times I\times M$ is a compact 1-dimensional invariant submanifold of P_e whose boundary is the orbit $\{O\}\times H_0^{-1}(O)$. So L is the disjoint union of arcs L_i, $i = 1, \ldots, |G|/2$ and a finite number of closed curves. The union $\tilde{L}$ of arcs L_i is invariant. The subgroup G_1 of G consisting of elements preserving L_1 consists of two elements. Let $g \in G_1\setminus\{e\}$. By the Brouwer fixed point theorem there exists an $x \in L_1 \subset P_e$ such that $gx = x$. But this is impossible because the action of G on P_e is free.

5. The nonorientable case. For a nonorientable manifold M we have the following equivariant homotopic classification of maps.

5.1. THEOREM. *Let G, M, E, E_0 be as in 2.1 and let M be nonorientable. Let M' be the singular part of M.*

a) *If G is odd, then the function W_2: $[M, E_0]_G \to Z_2$, assigning to an equivariant homotopy class $[f]$ represented by a continuous equivariant map f: $M \to E_0$ its winding number* mod2 *$W_2(f)$, is bijective.*

b) *If $|G|$ is even and* $\dim M' = m-1$, *then $W_2(f) = 0$ for every equivariant map f: $M \to E_0$ and $[M, E_0]_G$ consists of one class.*

c) *If G is even and* $\dim M' < m-1$, *then, for all equivariant maps f: $M \to E_0$, $W_2(f)$ is the same and $[M, E_0]_G$ consists of two classes.*

The proof of a) is similar to the proof of 3.1, using 1.9 and the fact that M_e is connected.

In cases b) and c) $W_2(f)$ are independent of f by arguments as in the proof of 2.2 a).

In case b) G contains an isotropy group G_{x_0} of the action on M of rank 2, which acts trivially on E. So the constant map is G_{x_0}-equivariant and $W_2(f) = W_2(\text{const}) = 0$ by the preceeding remark. The proof of the rest of b) is analogous to that of 4.3a).

The proof of c) is similar to that of 4.3b).

It can be seen by examples that all the cases in Theorem 5.1 are possible (in c) the winding number mod2 may be 0 and 1).

5.2. *Let G, M, E, E_0 be as in 5.1 and in addition let G act on E preserving the orientation. Denote by $\tilde{M}$ the double orientation covering manifold of M. The points of $\tilde{M}$ can be thought of as the orientations of the tangent spaces T_xM. The action of G on M lifts to the orientation preserving action of G on $\tilde{M}$: For $g \in G$ and an orientation o of T_xM, go is the image of the orientation o by the tangent map dg_p* (comp. [1], I. 9.4). *Let T: $\tilde{M} \to \tilde{M}$ be the involution on $\tilde{M}$ mapping an orientation o of T_xM into the opposite orientation $-o$ of T_xM. T commutes with the action of G on $\tilde{M}$ and reverses the orientation of $\tilde{M}$. Let π: $\tilde{M} \to M$ be the covering projection.*

The concordant actions of G on $\tilde{M}$ and E satisfy the assumptions of Theorem 2.2 a) *and every equivariant map f: $\tilde{M} \to E_0$ has the winding number $W(f) \equiv 0 \bmod |G|$.*

For the proof let g_0: $M \to E_0$ be any equivariant map. Set $f_0 = g_0 \circ \pi$. From the fact that $f_0 = f_0 \circ T$ we have $W(f_0) = -W(f_0)$ and therefore $W(f_0) = 0$. Then the result is a consequence of 2.2a).

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