Remark. The proof of the first theorem of this paper was found by the author during Spring 1979. Announcements in Notiziario della Unione Matematica Italiana, August–September 1979, N. 8-9, p. 19; and in Atti Accademia Nazionale dei Lincei (Rome), Rendiconti Cl. Sc. Fis. Mat. Nat., Ser. VIII, 67.6 (1979), pp. 383–386. We refer the reader to [MS1], as well as [MS] and [Mu1–6], for further information about Robinson's theorem, amalgamation, JEP and other soft model theoretical properties.

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A note on the isomorphic classification of spaces of continuous functions defined on intervals of ordinal numbers

by

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Abstract. Let ω_1 denote the first uncountable number, and let $\Gamma(\alpha)$ denote the interval of ordinal numbers not exceeding α, endowed with the order topology. For each natural number n an isomorphic classification of the space of continuous functions $C(\Gamma(\omega_1\cdot n))$ is given among the spaces $C(S)$ for which every point of S is either a P-point or a G_δ-point. For $n = 1$, this classification yields a characterization of $\Gamma(\omega_1)$.

Introduction. For each ordinal number α, let $\Gamma(\alpha)$ denote the topological space of non-zero ordinal numbers not exceeding α, endowed with the interval topology (cf. [5], p. 57). Let ω and ω_1 denote the smallest infinite ordinal number and the smallest uncountable ordinal number respectively. As customary, for any compact Hausdorff topological space S, $C(S)$ denotes the supremum-normed Banach space of continuous complex-valued functions defined on S. Two Banach spaces are said to be *isomorphic* provided there is a one-one bounded linear operator from one space onto the other space. A point p in a compact Hausdorff space S is called a *P-point* provided every G_δ-set containing p is a neighborhood of p (cf. [4], p. 63).

In [10], Semadeni showed that the Banach spaces $C(\Gamma(\omega_1\cdot n))$ for $1\leqslant n<\omega$ were mutually non-isomorphic. In this paper, we obtain an extension of this result by giving an isomorphic classification of these spaces among the spaces $C(S)$ for compact Hausdorff topological spaces S in which every point is either a G_δ-point or a P-point. A characterization of $\Gamma(\omega_1)$ in terms of isomorphisms of spaces of continuous functions is also thereby obtained.

Before stating our first result, we need to introduce a few more notions. A topological space is said to be *dispersed* (*scattered*) provided it contains no dense-in-itself non-empty subset (cf. [11], p. 147). Let S be a compact Hausdorff dispersed space and let $m(S)$ denote the Banach space of bounded complex-valued functions on S equipped with the supremum norm. Then, according to a theorem of Rudin [9], the conjugate space of $C(S)$ is isometric to the Banach space $l_1(S) = \{g: g\in m(S)$ and $\sum_s |g(s)|<\infty\}$ equipped with the usual l_1-norm so that the second conjugate space of $C(S)$ can be identified with $m(S)$. Let $X = C(S)$ and let X^* and X^{**} denote the first and second conjugate space of X, respectively. It then follows that for each x^{**} in X^{**} there corresponds a unique function h in $m(S)$ such that $x^{**}(x^*) = g(s)h(s)$ whenever g is a function in $l_1(S)$ which corresponds to x^* in X^*. As in [10], we let X_S denote the linear subspace of functionals in X^{**} which are sequentially continuous relative to the weak* topology of X^*. Then, defining $m_s(S)$, also as in [10], to be the linear subspace of functions in $m(S)$ which corresponds to linear functional in X_S, it follows that $m_s(S)$ is a norm-closed subspace of $m(S)$ which evidently contains $C(S)$. Thus, we can consider the quotient Banach space $\frac{m_s(S)}{C(S)}$. Since X_S is defined by isomorphic invariants, it follows that $\frac{m_s(S)}{C(S)}$ is isomorphic to $\frac{m_s(T)}{C(T)}$ whenever T is a compact Hausdorff space with $C(S)$ isomorphic to $C(T)$. The dimension of this quotient space is therefore also an isomorphic invariant and consequently plays a crucial role in the isomorphic classification of the space $C(\Gamma(\omega_1\cdot n))$ for each natural number n.

Our first result was proved in [10] in the case of intervals of ordinal numbers.

THEOREM 1. *Let S be a dispersed compact Hausdorff space in which every point is either a G_δ-point or a P-point. Then a function h in $m(S)$ belongs to $m_s(S)$ if and only if h is sequentially continuous on S.*

Proof. Suppose h belongs to $m_s(S)$ and let $\langle s_n\rangle$ be a sequence of points in S which converges to s_0. For each $n = 0, 1, 2, \ldots$ let g_n be the characteristic function of the singleton set $\{s_n\}$. Then g_n belongs to $l_1(S)$ for $n = 0, 1, 2, \ldots$ and the sequence $\langle g_n\rangle$ converges to g_0 relative to the weak* topology of $l_1(S)$. It follows that

$$\lim_{n\to\infty} h(s_n) = \lim_{n\to\infty}\Big(\sum_s h(s)g_n(s)\Big) = \Big(\lim_{n\to\infty}\sum_s h(s)g_n(s)\Big) = h(s_0)$$

whence $\lim_{n\to\infty} h(s_n) = (s_0)$. Thus h is sequentially continuous on S.

Conversely, suppose that h is sequentially continuous on S. To show that h belongs to $m_s(S)$ it suffices to show that the subspace

$$K = \{g\colon\ g\in l_1(S) \text{ and } \sum_{s\in S} f(s)h(s) = 0\}$$

of $l_1(S)$ is sequentially closed relative to the weak* topology of $l_1(S)$ as in Theorem 1 of [10]. Suppose that the sequence $\langle g_n\rangle$ of functions belonging to K converges to g_0 relative to the weak* topology of $l_1(S)$. Since $\sum_s |g_n(s)|$ converges for $n = 0, 1, 2, \ldots$ it follows that the set

$$A = \bigcup_{m=1}^{\infty} \{s\colon\ s\in S \text{ and } g_m(s)\neq 0\}$$

is countable. By hypothesis each point of S is either a G_δ-point or a P-point so that each point of $\mathrm{Cl}(A)$, the closure of A in S, must then either be a G_δ-point or a relatively isolated point in $\mathrm{Cl}(A)$. Hence, $\mathrm{Cl}(A)$ is a compact, dispersed, first countable (cf. Corollary 7.1.17 in [11]) space which must be homeomorphic to $\Gamma(\alpha)$ for some countable ordinal α by Theorem 8.6.10 of [11]. So h is continuous on $\mathrm{Cl}(A)$ since it is sequentially continuous on S and the relative topology on $\mathrm{Cl}(A)$ can be defined completely in terms of sequences. By the Tietze extension theorem, there exists a function f belonging to $C(S)$ whose restriction to $\mathrm{Cl}(A)$ coincides with h. It then follows from the definition of A that:

$$\sum_s g_0(s)h(s) = \sum_s g_0(s)f(s) = \lim_{n\to\infty}\Big(\sum_s g_n(s)f(s)\Big) = \lim_{n\to\infty}\Big(\sum_{s\in S} g_n(s)h(s)\Big) = 0\,.$$

Hence,

$$\sum_s g_0(s)h(s) = 0\,,$$

so that g_0 belongs to K, as desired.

LEMMA. *Let S be an uncountable compact Hausdorff space in which every point is either a G_δ-point or a P-point. Then there exists an open set I in S and a P-point p in S such that $\mathrm{Cl}(I)\setminus I = \{p\}$ and $\mathrm{Cl}(I)$ is homeomorphic to $\Gamma(\omega_1)$.*

Proof. S must contain at least one P-point since, otherwise every point of S is a G_δ-point and S, an uncountable space, is then homeomorphic to $\Gamma(\alpha)$ for some countable ordinal α by Corollary 7.1.17 and Theorem 8.6.10 of [11]. Since S is dispersed, there exists a P-point p in S which is isolated from all other P-points of S. By Theorem 8.54 of [11], S is zero-dimensional and so an easy application of transfinite induction yields a transfinite sequence $\langle W_\lambda\rangle_{1\leqslant\lambda<\omega_1}$ of clopen sets each containing p as its unique P-point such that W_λ is a proper subset of W_μ whenever $\lambda<\mu<\omega_1$.

Next, set $T = \bigcup_{\lambda<\omega_1}(W_1\setminus W_\lambda)$. Then p belongs to $\mathrm{Cl}(T)$ (an uncountable set) since otherwise every point of $\mathrm{Cl}(T)$ is a G_δ-point which implies that $\mathrm{Cl}(T)$ is homeomorphic to a countable interval of ordinals by an argument already used in the first paragraph of this proof. This contradiction shows that p belongs to $\mathrm{Cl}(T)$. Now, if $q\neq p$ and q belongs to $\mathrm{Cl}(T)\setminus T$, then q is a G_δ-point and so there exists a sequence of countable ordinals $\langle\lambda_n\rangle$ and a sequence of points $\langle S_n\rangle$ belonging to $S\setminus W_{\lambda_n}$ for $1\leqslant n<\omega$ such that $\lim_{n<\omega} s_n = q$.

Hence, setting $\lambda = \sup\{\lambda_n\colon\ 1\leqslant n<\omega\}$ which is smaller than ω_1, it follows that s_n belongs to $S\setminus W_\lambda$ for all $n<\omega$. The point q then belongs to the clopen set $S\setminus W_\lambda$ which is a subset of T. This contradiction together with what has been shown above establishes that $\mathrm{Cl}(T) = T\cup\{p\}$.

The desired subset I in S with $\mathrm{Cl}(T)$ homeomorphic to $\Gamma(\omega_1)$ will be constructed in T. Since T is open in S and $\mathrm{Cl}(T) = T\cup\{p\}$, the required topological properties of I relative to $\mathrm{Cl}(T)$ will be the same as the required topological properties of I considered as a subspace of S.

Set $U_\lambda = W_\lambda\cap \mathrm{Cl}(T)$ for each $\lambda<\omega_1$. Then the transfinite decreasing sequence of sets $\langle U_\lambda\rangle_{\lambda<\omega_1}$ is a base of clopen neighborhoods for p in the relative topology of $\mathrm{Cl}(T)$ since $\bigcap_{\lambda<\omega_1} U_\lambda = \{p\}$ and $\mathrm{Cl}(T)$ is compact. Moreover, for each $\lambda<\omega_1$, the set $\mathrm{Cl}(T)\setminus U_\lambda$ is a clopen subset of S all of whose points are G_δ-points of S. By Corollary 7.1.17 and Theorem 8.6.10 of [11], $\mathrm{Cl}(T)\setminus U_\lambda$ is homeomorphic to a compact space of ordinal numbers for each $\lambda<\omega_1$. Consequently, for each $\lambda<\omega_1$ there is a well-ordering $<_\lambda$ on $\mathrm{Cl}(T)\setminus U_\lambda$. The family of sets $\langle U_\lambda\rangle_{\lambda<\omega_1}$ will be used to construct a family of sets $\langle I_\lambda\rangle_{\lambda<\omega_1}$, and the required set I in T will be defined by setting $I = \bigcup_{\lambda<\omega_1} I_\lambda$.

Let $I_1 = \{s_1\}$ where s_1 is any isolated point of S contained in T. Suppose $1<\alpha<\omega_1$, and the set I_λ has been chosen for all $\lambda<\alpha$ satisfying the following:

(i) I_λ is clopen and $I_\lambda\subseteq I_\mu\subseteq T$ whenever $1\leqslant\lambda<\mu<\alpha$.

(ii) The set $I_\beta\setminus \bigcup_{\lambda<\beta} I_\lambda$ consists of precisely one point whenever $1\leqslant\beta<\alpha$.

(iii) $\mathrm{Cl}(\bigcup_{\lambda<\alpha} I_\lambda)$ is open and the set $\mathrm{Cl}(\bigcup_{\lambda<\alpha} I_\lambda)\setminus(\bigcup_{\lambda<\alpha} I_\lambda)$ consists of at most one point.

(iv) If $1\leqslant\lambda<\alpha$ and λ is not a limit ordinal, let $\mu[\lambda]$ denote the smallest ordinal μ such that I_λ is a subset of $\mathrm{Cl}(T)\setminus U_\mu$ and $(\mathrm{Cl}(T)\setminus U_\mu)\setminus I_\lambda$ is an infinite set. Then I_λ is an initial clopen interval in $\mathrm{Cl}(T)\setminus U_{\mu[\lambda]}$ relative to a well-ordering on $\mathrm{Cl}(T)\setminus U_{\mu[\lambda]}$ which induces the relative topology on $\mathrm{Cl}(T)\setminus U_{\mu[\lambda]}$. Moreover, if $\lambda = \lambda'+n$ for some natural number n and λ' is not a limit ordinal, then $\mu[\lambda] = \mu[\lambda']$ and $<_{\mu[\lambda]} = <_{\mu[\lambda']}$.

If $\alpha = \beta+1$ and β is a limit ordinal, let $\mu[\alpha]$ denote the smallest ordinal μ such that I_β is a subset of $\mathrm{Cl}(T)\setminus U_\mu$ and $(\mathrm{Cl}(T)\setminus U_\mu)\setminus I_\beta$ is an infinite set. Then since I_β is a clopen set by (i), there exists a well-ordering $<_{\mu[\alpha]}$ on $\mathrm{Cl}(T)\setminus U_{\mu[\alpha]}$ which induces the relative topology of $C(T)\setminus U_{\mu[\alpha]}$ and, relative to which, I_β is an initial clopen interval. Let s be the smallest element of the set $(\mathrm{Cl}(T)\setminus U_{\mu[\alpha]})\setminus I_\beta$ relative to the well-ordering $<_{\mu[\alpha]}$ and set $I_\alpha = I_\beta \cup \{s\}$. It is then a routine matter to verify that I_λ satisfies conditions (i)–(iv) for all $\lambda\leqslant\alpha$.

If $\alpha = \beta+1$ and β is not a limit ordinal, then there exists a natural number n such that $\alpha = \gamma+n$, and either $\gamma = 1$ or γ is a limit ordinal. In either case, it follows from (iv) that

$$\mu[\gamma+1] = \mu[\gamma+2] = \ldots = \mu[\gamma+(n-1)]$$

and

$$<_{\mu[\gamma+1]} = <_{\mu[\gamma+2]} = \ldots = <_{\mu[\gamma+(n-1)]}.$$

In addition, $I_{\gamma+(n-1)}$ is an initial clopen interval in $\mathrm{Cl}(T)\setminus U_{\mu[\gamma+1]}$ relative to the well-ordering $<_{\mu[\gamma+1]}$ on $\mathrm{Cl}(T)\setminus U_{\mu[\gamma+1]}$ again by (iv). Let t be the smallest element of $(\mathrm{Cl}(T)\setminus U_{\mu[\gamma+1]})\setminus I_{\gamma+(n-1)}$ relative to the well-ordering $<_{\mu[\gamma+1]}$ and then set $I_{\gamma+n} = I_{\gamma+(n-1)} \cup \{t\}$. As before, it is an easy matter to verify that I_λ satisfies conditions (i)–(iv) for all $\lambda\leqslant\alpha$.

Finally, to define I_α for α a limit ordinal, set $I_\alpha = \mathrm{Cl}(\bigcup_{\lambda<\alpha} I_\lambda)$. By (iii), I_α is clopen and $I_\alpha\setminus(\bigcup_{\lambda<\alpha} I_\lambda)$ consists of at most one point. Since $I_\lambda \neq I_\alpha$ for each $\lambda<\alpha$ by (ii), it follows that $I_\alpha\setminus(\bigcup_{\lambda<\alpha} I_\lambda)$ consists of precisely one point. Hence, I_λ satisfies conditions (i)–(iv) for all $\lambda\leqslant\alpha$.

After having defined I_λ as above for each $\lambda<\omega_1$ set $I = \bigcup_{\lambda<\omega_1} I_\lambda$. Then I is an open uncountable subset of S by (i) and (ii). So, p belongs to $\mathrm{Cl}(I)$ since otherwise all points of $\mathrm{Cl}(I)$ are G_δ-points and $\mathrm{Cl}(I)$ is homeomorphic to $\Gamma(\alpha)$ for some countable ordinal α by Corollary 7.1.17 and Theorem 8.6.10 of [11]. On the other hand, if s belongs to $\mathrm{Cl}(I)$ and $s \neq p$, then s is a G_δ-point (because $I\subset T$) and so there exist a sequence of points $\langle s_n\rangle$ and a sequence of countable ordinals $\langle \lambda_n\rangle$ with s_n belonging to I_{λ_n} for each n such that $\lim_{n<\omega} s_n = s$, from the definition of I. Since $\lambda = \sup_{1<n<\omega} \lambda_n<\omega_1$, it follows from (i) that s_n belongs to I_λ for each n. Consequently, s belongs to I_λ which is a clopen subset of I. Hence, $\mathrm{Cl}(I)\setminus I = \{p\}$, as desired.

Finally, we will show that $\mathrm{Cl}(I)$ is homeomorphic to $\Gamma(\omega_1)$ via a result due to Baker ([1], Theorem 2) which characterizes compact intervals of ordinals among dispersed compact Hausdorff spaces. It suffices to show that $\mathrm{Cl}(I)$ is homeomorphic to an interval of ordinals since $\mathrm{Cl}(I)$ is an uncountable compact space having p as its unique non-G_δ-point and $\Gamma(\omega_1)$ is homeomorphic to $\Gamma(\alpha)$ for $\omega_1\leqslant\alpha<\omega_1\cdot 2$ by Baker's theorem.

Each point in the dispersed compact Hausdorff space $\mathrm{Cl}(I)$ which is different from p has a countable base of neighborhoods by Corollary 7.1.17 of [11]. Consequently, to show that $\mathrm{Cl}(I)$ is homeomorphic to an interval of ordinals, it suffices by Baker's theorem cited above to show that the point p has a base of neighborhoods in $\mathrm{Cl}(I)$ which form a transfinite decreasing sequence $\langle V_\lambda\rangle_{\lambda<\omega_1}$ of sets clopen in the relative topology of $\mathrm{Cl}(I)$ such that the set $\bigcap_{\lambda<\beta} (V_\lambda\setminus V_\beta)$ contains at most one point for each limit ordinal $\beta<\omega_1$. In order to obtain the desired family of sets, set $V_\lambda = \mathrm{Cl}(I)\setminus I_\lambda$ for each $\lambda<\omega_1$. Then the transfinite sequence of sets $\langle V_\lambda\rangle_{\lambda<\omega_1}$ is decreasing and forms a base of clopen neighborhoods for p in the relative topology of $\mathrm{Cl}(I)$ by (i). Since

$$\bigcap_{\lambda<\beta} (V_\lambda\setminus V_\beta) = \bigcap_{\lambda<\beta} \big[(\mathrm{Cl}(I)\setminus I_\lambda)\setminus(\mathrm{Cl}(I)\setminus I_\beta)\big] = I_\beta\setminus \bigcup_{\lambda<\beta} I_\lambda \quad \text{for} \quad 1\leqslant\beta<\omega_1,$$

it follows from (ii) that the set $\bigcap_{\lambda<} (V_\lambda\setminus V_\beta)$ consists of precisely one point for each ordinal β less than ω_1.

THEOREM 2. *Let S be a compact Hausdorff space. Then S is homeomorphic to $\Gamma(\omega_1)$ if and only if $C(S)$ is isomorphic to $C(\Gamma(\omega_1))$ and each point of S is either a G_δ-point or a P-point.*

Proof. If S is homeomorphic to $\Gamma(\omega_1)$, then the conditions of the theorem are obviously satisfied.

Conversely, suppose that S is a compact Hausdorff space such that $C(S)$ is isomorphic to $C(\Gamma(\omega_1))$ and each point of S is either a G_δ-point or a P-point.

According to [8], S is dispersed if and only if every infinite-dimensional subspace of $C(S)$ contains an isomorphic copy of $C(\Gamma(\omega))$. Since this latter property is evidently preserved under isomorphisms, and $\Gamma(\omega_1)$ is dispersed, it follows that S is also dispersed. From Proposition 7.6.5 of [11], it follows easily that S is uncountable. Hence, by the lemma there exists an open set I in S and a P-point p in S such that $\mathrm{Cl}(I) = I \cup \{p\}$ and $\mathrm{Cl}(I)$ is homeomorphic to $\Gamma(\omega_1)$. It will next be shown that p is the unique P-point of S.

Suppose q is any P-point of S. Then the characteristic functions of the two singleton sets $\{p\}$ and $\{q\}$ each belong to $m_s(S)$ by Theorem 1. Moreover, these two functions are linearly dependent modulo $C(S)$ since the dimension of the quotient space $\dfrac{m_s(\Gamma(\omega_1))}{C(\Gamma(\omega_1))}$ is equal to 1 by [10], and $C(S)$ is isomorphic to $C(\Gamma(\omega_1))$. It then follows easily that $p = q$, so that p is the unique P-point of S.

Having obtained a subset $\mathrm{Cl}(I)$ in S homeomorphic to $\Gamma(\omega_1)$, it will finally be shown that $S\diagdown\mathrm{Cl}(I)$ is a clopen subset of S which is homeomorphic to $\Gamma(\alpha)$ for some countable ordinal α. It will then follow that $\Gamma(\omega_1)$ is homeomorphic to S since $\Gamma(\omega_1)$ is homeomorphic to the disjoint union of itself with any compact interval of countable ordinal numbers by Theorem 2 of [1].

Set $F = S\diagdown\mathrm{Cl}(I)$. If this open set is not also closed in S, then p belongs to $\mathrm{Cl}(F)$ since I is open in S and $\mathrm{Cl}(I) = I \cup \{p\}$. Hence, $\mathrm{Cl}(F) = F \cup \{p\}$. Let f and g be the characteristic functions of the sets F and I respectively. Then f and g are each sequentially continuous on S so that each of these functions belongs to $m_s(S)$ by Theorem 1. Since the dimension of $\dfrac{m_s(S)}{C(S)}$ is equal to 1 from the second paragraph of this proof, there exist scalars a and b not both zero such that the function $h = af+bg$ belongs to $C(S)$. Then, $a = h(p) = b$ since p belongs to $\mathrm{Cl}(F) \cap \mathrm{Cl}(I)$ and $F \cap I = \emptyset$. On the other hand, $h(p) = 0$ since p does not belong to $F \cup I$. This contradiction proves that F is closed in S. Thus, F is a dispersed compact Hausdorff space and every point of F is a G_δ-point. By Corollary 7.1.17 and Theorem 8.6.10 of [11], F is homeomorphic to $\Gamma(\alpha)$ for some countable ordinal α as desired.

The following definition will be used in a characterization of the compact Hausdorff spaces S, all of whose points are G_δ-points or P-points, for which $C(S)$ is isomorphic to $C(\Gamma(\omega_1))$.

Definition. For a natural number n, let $k_1<k_2< \dots <k_m$ be a sequence of m natural numbers not exceeding n. Set $k_0 = 0$. Define an equivalence relation on the space $\Gamma(\omega_1\cdot n)$ as follows. For α and β belonging to $\Gamma(\omega_1\cdot n)$, α is equivalent to β if and only if $\alpha = \beta$, or $\alpha = \omega_1\cdot i$ and $\beta = \omega_1\cdot j$ with $k_{l-1}<i, j\leqslant k_l$ for some natural number $l\leqslant m$. The resulting quotient space equipped with the quotient topology ([5], p. 94) will be denoted by

$$\frac{\Gamma(\omega_1\cdot n)}{[k_1, k_2, \dots, k_m]}.$$

Theorem 3. *Let S be a compact Hausdorff space in which every point is either a G_δ-point or a P-point and let n be a natural number. Then $C(S)$ is isomorphic to $C(\Gamma(\omega_1\cdot n))$ if and only if S is homeomorphic to the quotient space $\dfrac{\Gamma(\omega_1\cdot n)}{[k_1, \dots, k_m]}$ for a finite sequence of natural numbers $k_1<k_2< \dots <k_m\leqslant n$.*

Proof. In order to prove the sufficiency, suppose that S is homeomorphic to $\dfrac{\Gamma(\omega_1\cdot n)}{[k_1, \dots, k_m]}$ for a finite sequence of natural numbers $k_1<k_2< \dots <k_m\leqslant n$. According to [2], for any compact Hausdorff space T containing a convergent sequence having an infinite number of terms, the Banach space $C(T)$ is isomorphic to the linear subspace of functions in $C(T)$ which vanish at every point of any fixed finite subset of T. It follows immediately that the spaces $C(S)$ and $C(\Gamma(\omega_1\cdot n))$ are each isomorphic to the linear subspace of functions in $C(\Gamma(\omega_1\cdot n))$ which vanish at each P-point of $\Gamma(\omega_1\cdot n)$, and hence these spaces are isomorphic to each other.

In order to prove the converse, let S be a compact Hausdorff space such that $C(S)$ is isomorphic to $C(\Gamma(\omega_1\cdot n))$. Then S is dispersed (as in the proof of the preceding theorem) and the quotient space $\dfrac{m_s(S)}{C(S)}$ has dimension n since the corresponding quotient space for $\Gamma(\omega_1\cdot n)$ has dimension n by [10]. Furthermore, the characteristic function of a singleton set consisting of a P-point belongs to $m_s(S)$ by Theorem 1. Since such functions corresponding to distinct P-points are linearly independent modulo $C(S)$ (cf. the proof of the preceding theorem), it follows that S cannot contain more than n P-points. However, an argument also used in the preceding theorem shows that S is uncountable and must contain at least one P-point. Hence S has precisely m P-points $p_1, p_2, \dots, p_m$ for some natural number $m\leqslant n$.

Since S is 0-dimensional by [8], there exist m mutually disjoint clopen (uncountable) neighborhoods $S_1, S_2, \dots, S_m$ of the P-points $p_1, p_2, \dots, p_m$ respectively such that $S = \bigcup_{i=1}^{m} S_i$. Then S_i is a dispersed compact Hausdorff having p_i as its unique P-point for $i = 1, 2, \dots, m$. We will show that S_1 (and therefore each S_i) is homeomorphic to a quotient space of $\Gamma(\omega_1\cdot k)$ for some natural number k.

By the lemma and because S_1 is clopen in S, there exists a set I_1 in S which is open in S such that $\mathrm{Cl}(I_1) = I_1 \cup \{p_1\}$ and $\mathrm{Cl}(I_1)$ is homeomorphic to $\Gamma(\omega_1)$. Now, if the set $F_1 = S_1\diagdown\mathrm{Cl}(I_1)$ is not closed in S, then p_1 belongs to $\mathrm{Cl}(F_1)$ since I_1 is open and $\mathrm{Cl}(I_1) = I_1 \cup \{p_1\}$. In this case, $\mathrm{Cl}(F_1)$ is an uncountable dispersed compact Hausdorff space having p_1 as its unique P-point. An application of the lemma to this set then yields an open set I_2 in F_1 such that $\mathrm{Cl}(I_2) = I_2 \cup \{p_1\}$ and $\mathrm{Cl}(I_2)$ is homeomorphic to $\Gamma(\omega_1)$. If the set $F_2 = S_1\diagdown(\mathrm{Cl}(I_1 \cup I_2))$ is not closed in S, then as before, we see that $\mathrm{Cl}(F_2) = F_2 \cup \{p_1\}$ is an uncountable dispersed compact Hausdorff space having p_1 as its unique P-point. As above, there exists an open set I_3 contained in F_2 such that $\mathrm{Cl}(I_3) = I_3 \cup \{p_1\}$ and $\mathrm{Cl}(I_3)$

is homeomorphic to $\Gamma(\omega_1)$. This process is continued until k_1 open sets $I_1, I_2, \ldots, I_{k_1}$ have been obtained with $\mathrm{Cl}(I_j) = I_j \cup \{p_1\}$ homeomorphic to $\Gamma(\omega_1)$ for $j = 1, 2, \ldots, k_1$ such that the set $F_{k_1} = S_1 \setminus \mathrm{Cl}(\bigcup_{j=1}^{k_1} I_j)$ is closed in S. Note that this set is obtainable in a finite number of steps since $\dfrac{m_s(S)}{C(S)}$ has finite dimension and the characteristic functions corresponding to the sets I_j all belong to $m_s(S)$ by Theorem 1, and these functions are linearly independent modulo $C(S)$ (cf. the last paragraph of the proof of Theorem 2).

Now F_{k_1} is a compact set in S and each point of F_{k_1} is a G_δ-point. It follows that F_{k_1} is homeomorphic to a compact interval of countable ordinals by Corollary 7.1.17 and Proposition 8.6.10 of [11]. Thus, we have the following decomposition of S:

$$S_1 = \bigcup_{j=1}^{k_1} I_j \cup \{p_1\} \cup F_{k_1}$$

where

(i) $I_i \cap I_j = \emptyset$ for $i \neq j$;

(ii) $\mathrm{Cl}(I_j) = I_j \cup \{p_1\}$ is homeomorphic to $\Gamma(\omega_1)$ for $j = 1, 2, \ldots, k_1$;

(iii) F_{k_1} is a clopen set which is homeomorphic to $\Gamma(\alpha)$ for some countable ordinal α.

Since $\Gamma(\omega_1)$ is homeomorphic to $\Gamma(\omega_1+\alpha)$ (cf. [1], Theorem 2), it is an easy matter to deduce that S is homeomorphic to the quotient space obtained from $\Gamma(\omega_1\cdot k_1)$ by pinching the set $\{\omega_1, \omega_1\cdot 2, \ldots, \omega_1\cdot k_1\}$ to one point.

By applying the argument above to each of the sets $S_2, S_3, \ldots, S_m$, $(m-1)$ natural numbers $l_2, l_3, \ldots, l_m$ are obtained such that S_i is homeomorphic to the quotient space obtained from $\Gamma(\omega_1\cdot l_i)$ by pinching the set $\{\omega_1, \omega_1\cdot 2, \ldots, \omega_1\cdot l_i\}$ to a point for $i = 2, 3, \ldots, m$. Next, set $k_i = k_1 + \sum_{j=2}^{i} l_j$ for $i = 2, 3, \ldots, m$. Then S_i is homeomorphic to the quotient space obtained from the compact interval of ordinals $\Gamma(\omega_1\cdot k_i) \setminus \Gamma(\omega_1\cdot k_{i-1})$ by pinching the set $\{\omega_1\cdot(k_{i-1}+1), \omega_1\cdot(k_{i-1}+2), \ldots, \omega_1\cdot(k_{i-1}+l_i)\}$ to one point, for $i = 2, 3, \ldots, m$. Since S is partitioned by the m clopen sets $S_1, S_2, \ldots, S_m$, it follows that S is homeomorphic to the quotient space $T = \dfrac{\Gamma(\omega_1\cdot k_m)}{[k_1, \ldots, k_m]}$. It only remains to show that $k_m = n$. By arguing as in the sufficiency part of the proof of the present theorem, it is clear that $C(T)$ is isomorphic to $C(\Gamma(\omega_1\cdot k_m))$. Consequently, $C(\Gamma(\omega_1\cdot n))$ is isomorphic $C(\Gamma(\omega_1\cdot k_m))$ and it follows that $k_m = n$ by Theorem 2 of [10].

Remarks. 1. The condition in Theorem 3 stating that every point of S is either a G_δ-point or a P-point can be replaced by the (stronger) condition requiring that every point of S have a base of neighborhoods linearly ordered with respect to inclusion. Indeed, these two conditions are equivalent for compact Hausdorff spaces having the same weight ([11], page 105) as $\Gamma(\omega_1)$.

2. Theorem 3 is false for compact Hausdorff spaces in which there are points which are neither G_δ-points nor P-points. In fact, for each natural number n there are uncountably many mutually non-homeomorphic compact Hausdorff spaces S such that $C(S)$ is isomorphic to $\Gamma(\omega_1\cdot n)$ and S is not homeomorphic to any quotient space of the type given in Theorem 3. In order to see this, for each countable ordinal α let S_α^n denote the quotient space obtained from $\Gamma(\omega_1\cdot n)$ by pinching the set $\{\omega_1, \omega_1\cdot 2, \ldots, \omega_1\cdot n, \omega^{\alpha+1}\}$ to a point. Then $C(S_\alpha^n)$ is isomorphic to $C(\Gamma(\omega_1\cdot n))$ since each of these spaces is isomorphic to the linear subspace of functions in $C(\Gamma(\omega_1\cdot n))$ which vanish on the set $\{\omega_1, \omega_1\cdot 2, \ldots, \omega_1\cdot n, \omega^{\alpha+1}\}$ (cf. [2] and the proofs of Theorem 1 and Theorem 3). It is a routine matter to verify that S_α^n is not homeomorphic to S_β^n whenever $\alpha<\beta<\omega_1$ by a comparison of the derived sets ([11], p. 147) of these spaces.

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