Proof. By Theorem 8 it suffices to prove that $I_0'(A)$ is a band. To this end, suppose that $\{x_i|i\}\subset I_0'(1)$ and $x = \bigvee_i x_i$ exists in A; we must prove that $x \in I_0'(1)$. It follows easily by convexity that $x^- \in I_0'(1)$. On the other hand, $|a|x^+ = \bigvee_i |a|x_i^+$ holds for every $a \in A$ by p-distributivity. But $|a|x_i^+ \leqslant 1$, $\forall i$, $\forall a \in A$, and, therefore, $x^+ \in I_0'(1)$.

Let us recall that an ordered ring with unity 1 is of *bounded inversion* if every element greater than 1 is a unit. If A is a commutative f-ring with unity and S is the set of non-zero-divisors, $(S_0^{-1}A, +, \cdot, \leqslant)$ will denote the *total* ring of fractions, ordered by the cone $(S_0^{-1}A)^+ = \{a/s|\ as \geqslant 0\}$. Then we have

COROLLARY 8. *Given a commutative f-ring A with unity, each of the following conditions is sufficient for $J(A)$ to be a band.*

(a) *The mapping $A \mapsto S_0^{-1}A$, $x \mapsto x/1$ preserves all suprema of subsets of A.*

(b) *A is of bounded inversion.*

(c) *Every non-unit is a zerodivisor.*

Proof. Each of these conditions implies p-distributivity [6].

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Intersections of separators and essential submanifolds of I^N

by

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Abstract. A compactum X in $I^N = I^m \times I^n$ is *essential* in the first m directions if and only if the projection of X to I^m is a stable map. Similarly define Y to be essential in the last n directions. We discuss conditions under which X and Y must have nonempty intersection. If $m \in \{1, 2\}$ then $X \cap Y \neq \emptyset$, while for $m, n > 2$ examples of disjoint essential compacta are constructed. We give applications, including an apparently new characterization of dimension in terms of mappings into R^n, and a generalization of the Cantor manifold concept.

1. Introduction. The boundary S^{N-1} of I^N can be written as the non-singular join of two distinct canonical lower dimensional spheres S^{m-1} and S^{n-1} for each choice of m, n with $m+n = N$. These spheres bound convex balls D^m and D^n whose intersection is nonempty. The balls are examples of compacta which are essentially embedded in the sense that D^k does not retract to S^{k-1}, $k \in m, n$. Suppose we replace D^m and D^n by different essentially embedded compacta X and Y; then is it possible that $X \cap Y = \emptyset$? Indeed this is possible as we shall show in Section 4, while in Section 3 we shall show it is impossible whenever $m \in \{1, 2\}$. A final result in Section 3 is an apparently new characterization of dimension in terms of mappings into R^n. In Section 5, we will generalize the Cantor manifold concept.

It is not known whether all infinite dimensional compacta have infinite cohomological dimension. A solution to this longstanding problem would be equivalent to a solution of the CE-map dimension raising problem and related problems [E1]. In 3.1 of [W] it was shown that any compactum which can be written as the intersection of separators of co-infinitely many faces of the Hilbert cube has infinite cohomological dimension. In Section 4 we shall show that, at least in finite dimensional cubes, there are essentially embedded manifolds which cannot be written as intersections of separators in the non-essential directions. This situation is related to the one described above, concerning non-intersecting essentially embedded compacta. It may shed light on the question of which compacta in the Hilbert cube can be written as the intersection of co-infinitely many separators of faces of the Hilbert cube. We note that quite recently Roman Pol [P] proved the existence of a compact metric space X which is neither countable dimensional nor strongly

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infinite dimensional. The latter implies X cannot be shown to have infinite cohomological dimension by methods of [W]; nevertheless since X contains closed subspaces of arbitrarily high finite dimension, its cohomological dimension is infinite anyway.

2. Preliminaries.

2.1. NOTATION. Let $I^N = \prod \{I_k \mid 1 \leqslant k \leqslant N\}$, where $I_k = [-1, 1]$. In the N-cube I^N, let A_k be the set of points whose kth coordinate is -1, and let B_k be the set of points whose kth coordinate is 1, $1 \leqslant k \leqslant N$. $S^{N-1} \subset I^N$ will denote the $(N-1)$-sphere boundary of I^N, and $S^{N-1} \times [0, 1]$ will be a fixed collar of S^{N-1} in I^N, $S^{N-1} \times \{0\}$ being identified with S^{N-1}. For $0 \leqslant \varepsilon \leqslant 1$, let $I_\varepsilon^N = I^N - (S^{N-1} \times [0, \varepsilon))$, with $A_k^\varepsilon = A_k \times \{\varepsilon\}$, $B_k^\varepsilon = B_k \times \{\varepsilon\}$ taken to be the faces of the cube I_ε^N.

2.2. DEFINITION. Let Σ be a finite or countably infinite set and

$$\mathscr{F} = \{(\mathscr{A}_k, \mathscr{B}_k) \mid k \in \Sigma\}$$

be a collection of disjoint pairs of closed subsets of a space X. By a *separator* of $(\mathscr{A}_k, \mathscr{B}_k)$ we mean a closed subset of X which separates $\mathscr{A}_k$ and $\mathscr{B}_k$ in X. We say $\mathscr{F}$ is an *essential family* in X if for each choice $\{S_k \mid k \in \Sigma\}$, where S_k is a separator of $(\mathscr{A}_k, \mathscr{B}_k)$, $\bigcap \{S_k \mid k \in \Sigma\} \neq \emptyset$. Let $Y \subset X$, $\Gamma \subset \Sigma$, and suppose $\{(Y \cap \mathscr{A}_k, Y \cap \mathscr{B}_k) \mid k \in \Gamma\}$ is an essential family in Y. Then we say Y is *essential in the directions* Γ. Note that if $X = I^N$, then $\{(A_k, B_k) \mid 1 \leqslant k \leqslant N\}$ is an essential family for X by 2.4 of [R–S–W].

2.3. DEFINITION. If $Y \subset I^N$ is essential in the directions $\{1, \ldots, k\}$, i.e., the first k directions, then we say Y is *properly* embedded if $Y \cap (S^{N-1} \times [0, \varepsilon]) = S^{k-1} \times [0, \varepsilon]$ for some $\varepsilon > 0$. If Y is essential in k directions different from the first k, then we also say Y is *properly* embedded if the above is true under canonical exchange of coordinates.

2.4. LEMMA (see 5.2 of [R–S–W]). *If $X \subset I^N$ is a compactum which is essential in the direction k, then X contains a continuum from A_k to B_k.*

2.5. LEMMA (see 4.3 of [W]). *Let X be a compactum, let $\{(A_k', B_k'): 1 \leqslant k \leqslant n\}$ be a family of disjoint pairs of closed subsets of X, and let $f_k: X \to I_k$ with $A_k' = f_k^{-1}(1)$, $B_k' = f_k^{-1}(-1)$. The family $\{(A_k', B_k') \mid 1 \leqslant k \leqslant n\}$ is essential if and only if the mapping $f: X \to I^n$ defined by $f = (f_1, \ldots, f_n)$ is stable.*

2.6. LEMMA. *Suppose $X \subset I^N$ is a compactum and $X \cap S^{N-1} = S^{m-1}$. Then X is essential in the first m directions if and only if there does not exist a retraction of X to S^{m-1}.*

Proof. This can easily be derived from 2.5.

2.7. LEMMA. *Suppose X is a polyhedron embedded in I^N so that $X \cap \partial I^N = S^{n-1}$. Then the n-skeleton $X^{(n)}$ is essential in the first n directions if and only if the homomorphism*

$$i_*: H_n(X, S^{n-1}) \to H_n(I^N, S^{n-1})$$

induced by inclusion is not the zero map.

Proof. Suppose $X^{(n)}$ is not essential; then by Lemma 2.6 there is a retraction $r: X^{(n)} \to S^{n-1}$. Therefore in the commutative diagram

$$\begin{tikzcd}
 & & & H_{n-1}(X^{(n)}) \arrow[dl, "r_*"'] \arrow[d, equal] & \\
\to H_n(X) \arrow[r] \arrow[d] & H_n(X, S^{n-1}) \arrow[r, "\partial"] \arrow[d, "i_*"] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "="] & H_{n-1}(X) \arrow[r] \arrow[d] & {} \\
0 = H_n(I^N) \arrow[r] & H_n(I^N, S^{n-1}) \arrow[r, "\cong"] & H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(I^N) = 0 &
\end{tikzcd}$$

the map ∂ is the zero map. Therefore i_* is also the zero map.

Conversely, suppose i_* is the zero map; then so is ∂. We will use obstruction theory to construct a retraction $r: X^{(n)} \to S^{n-1}$. Since S^{n-1} is $(n-2)$-connected, we can construct a retraction $r: X^{(n-1)} \to S^{n-1}$. Let $c_n \in H^n(X, S^{n-1}; \pi_{n-1}(S^{n-1})) \cong H^n(X, S^{n-1}; Z)$ be the obstruction to extending r to $X^{(n)}$; then it suffices to show $c_n = 0$. But by Proposition 13.1 of Chapter VI of [Hu],

$$c_n \in \operatorname{image}(\delta: H^{n-1}(S^{n-1}; Z) \to H^n(X, S^{n-1}; Z)).$$

Since $\partial = 0$, this image is zero.

2.8. LEMMA. *Suppose $m+n = N$, $X, Y \subset I^N$ are compacta, X is essential in the first m directions, Y is essential in the last n directions, and $X \cap Y = \emptyset$. Then there exist properly embedded polyhedra X_1, Y_1 such that $\dim X_1 = \dim X$, X_1 is essential in the first m directions, $\dim Y_1 = \dim Y$, Y_1 is essential in the last n directions, and $X_1 \cap Y_1 = \emptyset$.*

Proof. Choose a small $\varepsilon > 0$ and let $\varrho: I_\varepsilon^N \to I^N$ be a "radial" homeomorphism. Assume ϱ moves no point more than some preassigned distance $\delta > 0$. Let $\mathscr{U}$ be an open cover of X of fine mesh and of order $\dim X + 1$, and let $N(\mathscr{U})$ denote the polyhedron of the nerve of $\mathscr{U}$. There is a canonical map $f: X \to N(\mathscr{U})$ and then a PL map $g: N(\mathscr{U}) \to I^N$ so that $gf(X) \cap I_\varepsilon^N$ is essential in the first m directions of I_ε^N. Then $X_1 = \varrho(gf(X) \cap I_\varepsilon^N)$ is a polyhedron essential in the first m directions of I^N, and $\dim X_1 \leqslant \dim X$. The maps ϱ, g, f, are to be chosen so that $X_1 \cap Y = \emptyset$. Move Y off the faces $\bigcup_{i=1}^{m} (A_i \cup B_i)$; then change X_1 to be properly embedded.

When we use the term n-manifold, we assume the possibility of boundary. Also, by an N-manifold lying in I^N we mean a triangulated subcomplex of a triangulation of I^N in which it is assumed that all faces of I^N are subcomplexes. Let $X^{(n)}$ denote the n-skeleton of X. In our proofs, we shall implicitly assume that triangulations are fine enough to meet necessary criteria, leaving it to the reader to fill in routine details.

All homology and cohomology will use integer coefficients.

3. Essential compacta that must intersect. Assume $N \geqslant 2$ unless otherwise specified. The proofs of the next two theorems will use some lemmas that are stated and proved later in this section.

3.1. THEOREM. *Suppose $X, Y\subset I^N$ are compacta, X is essential in the first $N-1$ directions, and Y is essential in the last direction. Then $X\cap Y\neq\emptyset$.*

Proof. Assume $X\cap Y=\emptyset$. Use 2.4 to replace Y by an arc β so that $X\cap\beta=\emptyset$. Let α be a polyhedral neighborhood of X in I^N so that $\alpha\cap\beta=\emptyset$. By 3.4, $\alpha^{(N-1)}$ is essential in the first $N-1$ directions. By 3.6, $\alpha^{(N-1)}\cap\beta\neq\emptyset$, so we must conclude that $X\cap Y\neq\emptyset$.

3.2. THEOREM. *Suppose X, $Y\subset I^N$ are compacta, X is essential in the last two directions, and Y is essential in the first $n=N-2$ directions. Then $X\cap Y\neq\emptyset$.*

Proof. Assume $X\cap Y=\emptyset$, and using 2.8, assume X and Y are properly embedded polyhedra. We shall show that $X^{(2)}\cap Y^{(n)}\neq\emptyset$, contradicting the assumption that $X\cap Y=\emptyset$.

By 3.3, $X^{(2)}$ is essential in the last two directions, so if we can show that $Y^{(n)}$ is essential in the first n directions, then 3.6 will yield, $X^{(2)}\cap Y^{(n)}\neq\emptyset$. If $N=4$, then $n=2$, and we are finished; so, assume $N\geqslant 5$ and that $Y^{(n)}$ is not essential in the first n directions.

Let $f\colon Y^{(n)}\to I^n$ be the restriction of the projection map $I^N\to I^n$. Then f is as in 2.5, so f is not stable. Hence there exists a map $g\colon Y^{(n)}\to S^{n-1}$ such that $g=f$ on $f^{-1}(S^{n-1})$. Extend g to $G\colon I^N\to I^n$ so that on S^{N-1}, G is just the coordinate projection. We may assume G is PL, G is transverse at $0\in I^n$, and $0\notin G(Y^{(n)})$. The component S of $G^{-1}(0)$ containing S^1 is an orientable 2-manifold with boundary S^1, and $S\cap Y^{(n)}=\emptyset$.

Put S in general position with $Y^{(N-1)}$. If σ is an $(N-1)$-simplex of $Y^{(N-1)}$, then $S\cap\sigma\subset\operatorname{int}\sigma$ consists of a pairwise disjoint finite collection of unknotted, unlinked simple closed curves. By removing small open annular neighborhoods of these curves in S, and then sewing on disks which do not intersect $Y^{(N-1)}$ (apply this process to each σ), we create an orientable 2-manifold S_0 which does not intersect $Y^{(N-1)}$. Let S_0^* be the component of S_0 that contains S^1. Any properly embedded surface homeomorphic to S_0^* is ambient isotopic to S_0^* rel(S^{N-1}) by Unknotting Theorem 24, Chapter 8 of [Z] (or see page 177 of [R], but note that the last reference to ∂M there should be ∂Q). Then S_0^* can be written as the intersection of separators in the first n directions by 3.5, yet by 3.4, $Y^{(N-1)}$ is essential in the first n directions.

3.3. LEMMA. *Suppose $X\subset I^N$ is a polyhedron essential in the first two directions. Then $X^{(2)}$ is essential in the first two directions.*

Proof. Assume $X^{(2)}$ is not essential in the first two directions, and let $f\colon X\to I^2$ be the restriction of the coordinate projection. Using 2.5, there is a map $g\colon X^{(2)}\to S^1$ so that $g=f$ on $f^{-1}(S^1)\cap X^{(2)}$. Let G be defined as the map of $X^{(2)}\cup f^{-1}(S^1)$ to S^1 that equals g on $X^{(2)}$ and f on $f^{-1}(S^1)$. Since $\pi_k(S^1)=0$ for all $k\geqslant 2$, there is no obstruction to extending G to the entire polyhedron X. But because X is essential in the first two directions, this contradicts 2.5.

3.4. LEMMA. *Suppose $X\subset I^N$ is a polyhedron essential in the first $n<N$ directions. Then $X^{(N-1)}$ is essential in the first n directions.*

Proof. We do this by induction on the number of N-simplexes of X, the result being certainly true if there are none. We may assume $X\cap A_N=\emptyset=X\cap B_N$; thus there is an N-simplex σ of X having an $(N-1)$-face τ lying in the topological boundary of X. Let $X_0=X-(\operatorname{int}\sigma\cup\operatorname{int}\tau)$. Then X_0 is also essential in the first n directions. For suppose $S_1,\ldots,S_n$ are separators of $(A_1,B_1),\ldots,(A_n,B_n)$, and $\bigcap\{S_i|\ 1\leqslant i\leqslant n\}\cap X_0=\emptyset$. Isotop $\bigcap\{S_i|\ 1\leqslant i\leqslant n\}\cap\sigma$ off X through the face τ (with support in a small neighborhood of σ), thus removing any intersections with σ, but creating no new intersections with X. This contradicts essentiality of X. By induction, $X_0^{(N-1)}\subset X^{(N-1)}$ is essential in the first n directions.

3.5. LEMMA. *Let $N\geqslant 4$ and $k\geqslant 0$. Then there exist embedded $(N-1)$-cells $T_1,T_2,\ldots,T_{N-2}$ in I^N such that*

(1) *T_i is a separator of (A_i,B_i) for $1\leqslant i\leqslant N-2$,*

(2) *$\bigcap_{i=1}^{N-2}T_i$ is a properly embedded 2-disc with k 1-handles attached.*

Proof. Let $S_i=\{(x_1,x_2,\ldots,x_N)|\ x_i=0\}$ be the standard separators. Then $\bigcap_{i=1}^{N-3}S_i=\{(0,0,\ldots,0,x_{N-2},x_{N-1},x_N)\}=D^3$ is a 3-cell. We will embed I^{N-1} in I^N so that its image intersects D^3 in the desired surface.

Let $f\colon I^3\to I$ be a map such that

(a) f equals projection to the first coordinate outside $[-\varepsilon,\varepsilon]^3\subset I^3$ for some small ε,

(b) $f^{-1}(0)$ is a 2-disc with k 1-handles attached.

The embedding $g\colon I^{N-1}\to I^N$ is defined by

$$g(x_1,x_2,\ldots,x_{N-1})=\big(x_1,x_2,\ldots,x_{N-4},f(x_{N-3},x_{N-2},x_{N-1}),\tfrac{1}{2}(x_{N-3}),x_{N-2},x_{N-1}\big).$$

Then

$$g(I^{N-1})\cap D^3=\{(0,0,\ldots,0,y_{N-2},y_{N-1},y_N)|-\tfrac12\leqslant y_{N-2}\leqslant\tfrac12 \text{ and } f(2y_{N-2},y_{N-1},y_N)=0\}$$

is a properly embedded 2-disc with k 1-handles attached. It is easy to see that $T_{N-2}=g(I^{N-1})$ is a separator of A_{N-2} and B_{N-2}; in fact, it is properly isotopic in $I^N-(A_{N-2}\cup B_{N-2})$ to S_{N-2}. Letting $T_i=S_i$ for $1\leqslant i\leqslant N-3$, the lemma is proved.

3.6. LEMMA. *Let m and n be positive integers with $m+n=N$. Suppose X and Y are compacta in I^N such that X is essential in the first m directions, Y is essential in the last n directions, $\dim X=m$, $\dim Y=n$. Then $X\cap Y\neq\emptyset$.*

Proof. By 2.8 we may assume X and Y are properly embedded polyhedra. In $\partial I^N=\partial(I^m\times I^n)=S^{m-1}\times I^n\cup I^m\times S^{n-1}$, let $J_m=S^{m-1}\times I^n$ and $J_n=I^m\times S^{n-1}$. Let $\gamma_m=[I^m\times 0]$ generate $H_m(I^N,J_m)\cong \boldsymbol{Z}$ and let $\gamma_n=[0\times I^n]$ generate $H_n(I^N,J_n)\cong\boldsymbol{Z}$. Consider the bilinear intersection pairing $I(\_,\_)\colon H_m(I^N,J_m)\times H_n(I^N,J_n)\to H_0(I^N)\cong\boldsymbol{Z}$ (Lefschetz-dual to the cup product pairing $H^n(I^N,J_n)\times H^m(I^N,J_m)\to$

$H^N(I^N, \partial I^N)$). Since $I^m \times 0$ and $0 \times I^n$ intersect transversely in one point, $I(\gamma_m, \gamma_n) = \pm 1$. Applying Lemma 2.7 to X and Y, we obtain relative cycles $x \in H_m(X, S^{m-1})$ and $y \in H_n(Y, S^{n-1})$ so that $i_*(x) = \alpha\gamma_m$ and $i_*(y) = \beta\gamma_n$ with $\alpha, \beta \neq 0$. Because $I(i_*(x), i_*(y)) = I(\alpha\gamma_m, \beta\gamma_n) = \pm\alpha\beta \neq 0$, x and y must intersect. Therefore $X \cap Y \neq \emptyset$.

The following question was related to us by F. D. Ancel. Does there exist $n \geqslant 1$ and a compactum X of dimension n so that every map of X into R^{2n} has arbitrarily close embeddings? The answer is no, as shown by the next theorem.

For a compactum X, we use $C(X; R^n)$ to denote the space of continuous maps of X to R^n with the compact-open topology and $E(X; R^n) \subset C(X; R^n)$ to denote the subspace of embeddings.

3.7. THEOREM. *Let X be a compactum; then $\dim X < n$ if and only if $E(X; R^{2n})$ is dense in $C(X; R^{2n})$.*

Proof. If $\dim X < n$, then Theorem V2 of [H–W] shows $E(X; R^{2n})$ is dense in $C(X; R^{2n})$. Conversely, suppose $E(X; R^{2n})$ is dense in $C(X; R^{2n})$. We want to show that $\dim X < n$, so suppose $\dim X = n$ (If $\dim X > n$, then replace X by a closed subspace of dimension n). Let $f\colon X \to I^n$ be a stable map. Choose a map $g\colon I^n \to R^{2n}$ having the property that $g^{-1}(0)$ consists of two points a, b while $g^{-1}(x)$ is either singleton or empty for $x \neq 0$. Assume further that there are two disjoint n-cubes σ_a, σ_b contained in I^n so that $a \in \operatorname{Int}\sigma_a$, $b \in \operatorname{Int}\sigma_b$ and so that $g(\sigma_a) \cap I^{2n} = I^n \times 0$ while $g(\sigma_b) \cap I^{2n} = 0 \times I^n$. Let $\varepsilon = \frac{1}{2}$; then the following is true. Let $h\colon X \to R^{2n}$ be continuous and let $L = h^{-1}(I_\varepsilon^{2n}) \cap f^{-1}(\sigma_a)$, $M = h^{-1}(I_\varepsilon^{2n}) \cap f^{-1}(\sigma_b)$. If h is sufficiently close to $gf\colon X \to R^{2n}$, then $h(L)$ is essential in the first n directions of I_ε^{2n} and $h(M)$ is essential in the last n directions. Choosing h to be an embedding contradicts 3.6.

4. Disjoint essential compacta. If X and Y are properly embedded compacta in I^N with X essential in the first m directions and Y essential in the last $n = N-m$ directions, we have seen that X and Y must intersect if $m \in \{1, 2, N-2, N-1\}$. The following examples show that X and Y can be disjoint when $m \geqslant 3$.

4.1. EXAMPLE. Let X be the complex projective plane, obtained from S^2 by attaching a 4-cell D^4 using the Hopf map $h\colon S^3 \to S^2$. If N is sufficiently large, we can embed X in I^N so that $X \cap \partial I^N = S^2$ and $\operatorname{int}(D^4)$ is smoothly embedded in $\operatorname{int}(I^N)$. Let D^n be the standard convex n-cube ($n = N-3$) in I^N which is essential in the last n directions; then $D^n \cap X \subset \operatorname{int} D^4$. By a small ambient isotopy we move D^n into general position with $\operatorname{int} D^4$. Their intersection will be a disjoint union of a finite collection $\sigma_1, \ldots, \sigma_k$ of circles. Choose a small tubular neighborhood $M = \operatorname{int} D^4 \times D^{N-4}$ so that $M \cap D^n = \bigcup_{i=1}^{k} \sigma_i \times D^{N-4}$. There are disjoint 2-discs $E_1, E_2, \ldots, E_k$ in $\operatorname{int}(D^4)$ with $\partial E_i = \sigma_i$, $1 \leqslant i \leqslant k$. Form the manifold

$$V = (D^n - \bigcup_{i=1}^{k} \sigma_i \times \operatorname{int} D^{N-4}) \cup (\bigcup_{i=1}^{k} E_i \times \partial D^{N-4}).$$

Since a manifold cannot be retracted to its boundary, V is essential in the last n directions by Lemma 2.6, yet $X \cap V = \emptyset$. Since X is essential, V cannot equal, or even contain, an intersection of separators of the first 3 faces.

To produce similar examples essential in the first m directions for any $m \geqslant 3$, embed the k-fold suspension $\Sigma^k X$ in $I^k \times I^N = \Sigma^k I^N$ so that $\Sigma^k X \cap \partial(I^k \times I^N) = S^{k+2} = \Sigma^k S^2$, and $\Sigma^k X \cap (\{0\} \times I^N) = X$. Since the k-fold suspension of the Hopf map is essential, $\Sigma^k X$ is essential in the first $m = k+3$ directions. Taking the same V as above, $V \subset \{0\} \times I^N \subset I^{k+N}$, we have V essential in the last $N-3$ directions and $\Sigma^k X \cap V = \emptyset$.

In the above examples, the m-skeleton of $\Sigma^k X$ is inessential but the $(m+1)$-skeleton is essential. The following generalized construction produces examples, for m odd and arbitrarily large r, of disjoint essential compacta X and V such that the $(m+r)$-skeleton of X is inessential while V is an n-manifold.

4.2. LEMMA. *Suppose given $r \geqslant 0$ and $m \geqslant 3$. Then for sufficiently large N, there exists a subcomplex $X \subset I^N$ with the following properties:*

1. *$X \cap \partial I^N = S^{m-1}$.*
2. *The $(m+r)$-skeleton of X retracts to S^{m-1}.*
3. *X is essential in the first m directions.*

Proof. Choose $m_1 \geqslant m+r$ so that $\pi_{m_1}(S^{m-1}) \neq 0$ (this is always possible, see e.g. [Hu], Corollaries 9.3 and 13.3 of Chapter XI) and a nonzero element $\varphi \in \pi_{m_1}(S^{m-1})$. Form X by attaching an (m_1+1)-cell to S^{m-1} using φ. Choose N so large that X can be embedded in I^N with $X \cap \partial I^N = S^{m-1}$. Since X with any interior point of its (m_1+1)-cell removed (deformation) retracts to S^{m-1}, its $(m+r)$-skeleton retracts to S^{m-1}. But since φ is nonzero, X does not retract to S^{m-1}; hence X is essential by Lemma 2.6.

4.3. THEOREM. *Let m be odd, $m \geqslant 3$. Given $X \subset I^N$ satisfying the conclusion of Lemma 4.2, there exists an n-dimensional manifold $V \subset I^N$ ($n = N-m$) such that V is essential in the $\{m+1, m+2, \ldots, m+n\}$ directions but $V \cap X$ is empty.*

Proof. Let K_m be a $K(Z, m)$ CW complex obtained from S^m by attaching cells of dimension $m+2$ and higher.

LEMMA. *Let m be odd, $m \geqslant 3$. Given $q > 0$, there is a positive integer k and a cellular map $F\colon K_m \to K_m$ such that $F|S^m\colon S^m \to S^m \subset K_m$ is a map of degree k and $F(K_m^{(m+q)}) \subset S^m$.*

Note. Using the homotopy extension property, we may assume $F|S^m$ is any specified map of degree k.

Proof of the lemma. Let $k_0 = 1$ and let k_i be the order of $\pi_{m+i}(S^m)$ for $i > 0$. Since m is odd, each k_i is finite. Let $f_0\colon S^m \to S^m$ be the identity map.

Induction hypothesis (i): There are maps $f_0, f_1, \ldots, f_i$ from S^m to S^m such that

(1) degree $(f_j) = k_j$ for $0 \leqslant j \leqslant i$,

(2) $f_i \circ f_{i-1} \circ \ldots \circ f_2 \circ f_1 \circ f_0\colon S^m \to S^m$ extends to a map from $K_m^{(m+i+1)}$ to S^m.

For $i = 0$, $K_m^{(m+i+1)} = S^m$, so the hypothesis holds. Assume it holds for i. Choose $f_{i+1}\colon S^m \to S^m$ of degree k_{i+1}. Let $F_i\colon K_m^{(m+i+1)} \to S^m$ be the extension of $f_i \circ f_{i-1} \circ \dots \circ f_0$. For each $(m+i+2)$-cell D of K_m with attaching map $\varphi\colon \partial D \to K^{(m+i+1)}$, the composite $f_{i+1} \circ F_i \circ \varphi$ is null homotopic, since $\langle F_i \circ \varphi \rangle \in \pi_{m+i+1}(S^m)$ has order dividing k_{i+1}. Therefore $f_{i+1} \circ F_i$ extends to D. Proceeding cell by cell we extend $f_{i+1} \circ F_i$ over all of $K^{(m+i+2)}$. This completes the induction.

Let $k = \prod_{i=1}^{q-1} k_i$ and let $F = f_{q-1} \circ f_{q-2} \circ \dots \circ f_1 \circ f_0$. We have seen that F extends to a map from $K_m^{(m+q)}$ to $S^m \subset K_m$. Since the homotopy groups of K_m vanish above dimension m, we can then extend F to a cellular map $F\colon K_m \to K_m$. This completes the proof of the lemma.

Let γ be a generator of $H_m(I^N, J_m) \cong Z$. Since the $(m+r)$-skeleton of X retracts to S^{m-1}, $H_m(X, S^{m-1}) \to H_m(I^N, J_m)$ is the zero map by Lemma 2.7. Therefore in the exact homology sequence

$$H_m(X, S^{m-1}) \to H_m(I^N, J_m) \xrightarrow{i_*} H_m(I^N, X \cup J_m)$$

the second map is injective. Let γ_1 be the image of γ in $H_m(I^N, X \cup J_m)$; then γ_1 has infinite order. Therefore there is a homomorphism in $\mathrm{Hom}_Z\big(H_m(I^N, X \cup J_m), Z\big)$ which is nonzero on γ_1. In the commutative diagram

$$\begin{tikzcd}
H^m(I^N, X \cup J_m) \arrow[r] \arrow[d, "i^*"] & \mathrm{Hom}\big(H_m(I^N, X \cup J_m), Z\big) \arrow[r] \arrow[d, "\mathrm{Hom}(i_*)"] & 0 \\
H^m(I^N, J_m) \arrow[r] & \mathrm{Hom}\big(H_m(I^N, J_m), Z\big) \arrow[r] & 0
\end{tikzcd}$$

we pull this homomorphism back to a cohomology class $\{c_1\} \in H^m(I^N, X \cup J_m)$. Then if c is the restriction of c_1 to an element of $H^m(I^N, J_m)$, c has a nonzero value on γ.

Let $\varphi \in H^m(K_m, *)$ be an *m-characteristic* element (p. 425 [Sp]); that is, choose $\hat{\varphi}\colon H_m(K_m, *) \to Z$ to be an isomorphism. Then choose φ to be the preimage of $\hat{\varphi}$ under the isomorphism $H^m(K_m, *) \to \mathrm{Hom}\big(H_m(K_m, *), Z\big)$ of the universal coefficient theorem. Hence,

$$\pi_m(K_m, *) \xrightarrow{h} H_m(K_m, *) \xrightarrow{\hat{\varphi}} Z$$

is an isomorphism (where h is the Hurewicz isomorphism). By Theorem 8.1.10 of [Sp], the map

$$\Psi\colon [(X, A); (K_m, *)] \to H^m(X, A)$$

defined by $\Psi(\langle g \rangle) = g^*(\varphi)$ is a natural bijection when (X, A) is a relative CW complex. Therefore we have a commutative diagram

$$\begin{tikzcd}
H^m(I^N, X \cup J_m) \arrow[r, "i^*"] \arrow[d, "\Psi^{-1}"'] & H^m(I^N, J_m) \arrow[d, "\Psi^{-1}"] \\
{[(I^N, X \cup J_m); (K_m, *)]} \arrow[r, "i^{\#}"] & {[(I^N, J_m); (K_m, *)]}
\end{tikzcd}$$

If $\langle g \rangle \in [(I^N, J_m); (K_m, *)]$ corresponds to $\{c\} = i^*\{c_1\}$, then $\langle g \rangle$ is in the image of $i^{\#}$. Therefore we may assume $g(X \cup J_m) = *$. Changing g by a homotopy (all homotopies of g will be fixed on $X \cup J_m$) we may assume g is cellular; in particular, $g\big((I^N)^{(m-1)}\big) = *$. For any m-simplex Δ^m of $(I^N, X \cup J_m)$, define F_{Δ^m} by $g|\Delta^m\colon \Delta^m \to \Delta^m/\partial\Delta^m = S^m \xrightarrow{F_{\Delta^m}} S^m \subset K_m$. The condition $g^*(\varphi) = \{c\}$ implies that, after perhaps changing c by a coboundary, we have

$$\mathrm{degree}(F_{\Delta^m}) = c(\Delta^m).$$

Assume that I^N is oriented and g maps an open neighborhood U of $(I^N)^{(m)}$ to S^m. Then for any $p \in S^m$ with g transverse to p on U, $g^{-1}(p) \cap U$ consists of n-dimensional oriented manifold components ($n = N-m$) that intersect the m-skeleton transversely, and whose intersection numbers with the m-skeleton give the homomorphism $c\colon C_m(I^N, J_m) \to Z$.

Using the lemma, we obtain a map $F\colon (K_m, *) \to (K_m, *)$ so that

1. $F|S^m\colon S^m \to S^m$ has degree $k \neq 0$,
2. $F(K_m^{(N)}) \subset S^m \subset K_m$.

We may assume $F \circ g$ is transverse to a point $p \in S^m$ with $p \neq *$ and that $F^{-1}(p) \cap S^m = p_1 \cup p_2 \cup \dots \cup p_k$ are regular values of $g|U$. Then $(F \circ g)^{-1}(p) = g^{-1}(p_1) \cup g^{-1}(p_2) \cup \dots \cup g^{-1}(p_k)$ will be a properly embedded n-manifold V such that the intersection number of V with each m-simplex Δ^m is $k \cdot c(\Delta^m)$. Since $(F \circ g)(X \cup J_m) = *$, $V \cap (X \cup J_m) = \emptyset$, so $[V, \partial V] \in H_n(I^N, J_n)$, and the intersection number of $[V, \partial V]$ with γ is $k \cdot c(\gamma) \neq 0$. Therefore $[V, \partial V] \neq 0$ and V is essential in the last n directions.

4.4. Corollary. *V does not contain an intersection of separators of the $\{1, 2, \dots, m\}$ faces.*

5. Dimension drop. We have seen (4.2, 4.3) that there are many examples of subcomplexes X and Y of I^N with X essential in the first m directions, Y essential in the last $n = N-m$ directions, and $X \cap Y = \emptyset$. We noted that such a Y cannot be written as an intersection of separators S_i of (A_i, B_i), $1 \leqslant i \leqslant m$. Let W be a regular neighborhood of X, so that W is an N-dimensional submanifold of I^N. Suppose that $Z = \bigcap_{i=1}^{m} S_i$. Then W and Z must intersect, and Z must have dimension $\geqslant n$. We would expect that Z would intersect W in a set of dimension $\geqslant n$, but the following example shows that this needs not be true.

5.1. Example. Let X, N, D^n, σ_i, M, and E_i be as in Example 4.1. Now D^n is an intersection $\bigcap_{i=1}^{3} S_i$ where $S_i = \{(x_1, \dots, x_N) |\ x_i = 0\}$ is a separator of (A_i, B_i) for $1 \leqslant i \leqslant 3$. Consider the disjoint $(N-2)$-cubes $E_i \times D^{N-4} \subset M$. There is a quotient map $\eta\colon I^N \to I^N$ which collapses each $E_i \times \{x\} \subset E_i \times D^{N-4}$ to $0 \times \{x\}$, where 0 is the center of the 2-disc E_i. The image $\eta(X)$ is still essential and $\eta(M)$ is still a manifold neighborhood. Let $T_i = \eta(S_i)$. It is clear that T_i is still a separator of (A_i, B_i), and $\eta(D^n) = \bigcap_{i=1}^{3} T_i$ intersects $\eta(M)$ in the set $\bigcup_{i=1}^{k} \eta(E_i \times 0) \times D^{N-4}$. One of the dimensions of $D^n \cap M$ has been collapsed away and the dimension of the intersection is $N-4$ rather than the expected $N-3$. This is not an isolated example as is shown by the next result.

5.2. THEOREM. *Suppose $X, Y \subset I^N$ are compacta, X is essential in the first n directions, Y is essential in the remaining $N-n$ directions, and $X \cap Y = \emptyset$. Then there exists $k \in \{n, N-n\}$ and an N-manifold $M \subset I^N$ essential in the first k directions and separators $S_1, \ldots, S_k$ of $(A_1, B_1), \ldots, (A_k, B_k)$ respectively, such that*

$$\dim[M \cap \bigcap \{S_i| \ 1 \leqslant i \leqslant k\}] \leqslant N-k-1 .$$

Proof. By taking polyhedral neighborhoods, we may as well assume X and Y are N-manifolds. Using 3.6, we may assume that $X^{(n)}$ is not essential. Choose $k = n$ and $M = X$. Now choose separators $S_1^*, \ldots, S_k^*$ so that $M^{(k)} \cap S = \emptyset$, where $S = \bigcap \{S_i^*| \ 1 \leqslant i \leqslant k\}$. Let K denote the k-skeleton of I^N and L denote the dual $(N-k-1)$-skeleton.

We consider I^N as a subset of the nonsingular join of K and L; we will denote the segments of the join of K and L whose union makes up I^N by $[-1, 1]$ where $-1 \in K$, $1 \in L$. Let $T = M \cap S$. Then $T \cap K = \emptyset$, and we may assume that $T \cap [-1, 1] \subset [0, 1]$ for each segment $[-1, 1]$ of the join structure.

We now describe a monotone map f of I^N onto itself by requiring that for each segment $[-1, 1]$ of the join, f will carry $[0, 1]$ to $\{1\} \subset L$ and $[-1, 0]$ linearly and homeomorphically onto $[-1, 1]$. Note that for any subcomplex W, $f(W) = W$ and $f(I^N - W) = I^N - W$.

Let $S_i = f(S_i^*)$, $1 \leqslant i \leqslant k$. If α is a continuum from A_i to B_i that misses S_i, then $f^{-1}(\alpha)$ is a continuum from A_i to B_i missing S_i^*. Hence S_i is a separator of (A_i, B_i). It is routine to check that $M \cap \bigcap \{S_i| \ 1 \leqslant i \leqslant k\} \subset L$. This concludes the proof, since $\dim L = N-k-1$.

An *n-dimensional Cantor manifold* (see Definition VI 6 of [H–W]) is a compact n-dimensional space, $n \geqslant 1$, which cannot be separated by a subset of dimension $\leqslant n-2$. By Theorem VI 11 of [H–W], an irreducible compact separator of R^{n+1} is an n-dimensional Cantor manifold. One can show that an irreducible compact separator of I^{n+1} that separates two boundary points of I^{n+1} is an n-dimensional Cantor manifold (first pass to S^{n+1} by forming the quotient I^{n+1}/S^n). On the other hand, an arbitrary compact minimal separator of a compact $(n+1)$-manifold need not even be connected. Our next result can be construed as a generalization upon the concept of Cantor manifold.

5.3. THEOREM. *Let $M \subset I^N$ be an N-manifold which is essential in the first 2 directions. If S_1, S_2 are separators of (A_1, B_1), (A_2, B_2) respectively, then*

$$\dim(M \cap S_1 \cap S_2) \geqslant N-2.$$

Proof. Let $S = S_1 \cap S_2$ and suppose, on the contrary, that $\dim(M \cap S) \leqslant N-3$. Fix $2 < k \leqslant N$ and consider the coordinate $I_k = [-1, 1]$. There is a linear homeomorphism of I_k to $[-\frac{1}{2}, \frac{1}{2}]$ so that $-1 \to -\frac{1}{2}$ and $1 \to \frac{1}{2}$. This homeomorphism determines an embedding H of I^N into I^N which is fixed on each coordinate but the kth. It is not difficult to see that $H(M)$ is essential in the first 2 directions. We want to replace S_1, S_2 by separators T_1, T_2 so that $\dim(H(M) \cap T_1 \cap T_2) \leqslant N-3$.

We will write each T_i as the union of three sets, T_i^L, T_i^M, T_i^R. Let $T_i^M = H(S_i)$. A point $(x_1, \ldots, x_N)$ will be in T_i^L if $-1 \leqslant x_k \leqslant -\frac{1}{2}$ and $(x_1, \ldots, x_{k-1}, -\frac{1}{2}, x_{k+1}, \ldots, x_N) \in H(S_i)$; it will be in T_i^R if $\frac{1}{2} \leqslant x_k \leqslant 1$ and $(x_1, \ldots, x_{k-1}, -\frac{1}{2}, x_{k+1}, \ldots, x_N) \in H(S_i)$. It is left to the reader to show that for each i, T_i is a separator of (A_i, B_i). Noting that $T_i \cap H(M) \subset H(S_i)$, then $\dim(H(M) \cap T_1 \cap T_2) \leqslant N-3$. Hence there is no loss of generality in assuming that from the very beginning, $M \cap A_k = \emptyset = M \cap B_k$. Furthermore, if $M \cap A_i = \emptyset$, then $H(M) \cap A_i = \emptyset$, so the process can be repeated in such a way that $M \cap A_k = \emptyset = M \cap B_k$ for all $2 < k \leqslant N$, and we assume M has this property to begin with. Since $S_i \cap A_i = \emptyset = S_i \cap B_i$ for $i = 1, 2$, we have $M \cap S \subset \operatorname{int} I^N$.

Let $P \subset M$ be a copy of M obtained by pushing M inward along a collar neighborhood of $\partial M \cap \operatorname{int} I^N$. There is an ambient isotopy of I^N which carries P to M. By this fact, we may assume that there is a neighborhood U of M such that $\dim(U \cap S) \leqslant N-3$.

We are now going to alter $M^{(2)}$ in such a way as to produce a 2-complex K which is essential in the first 2 directions, but which does not intersect S. This contradiction of 3.3 will prove the result.

It is convenient here to make use of the dimension theory of [E2]. Since $\dim(U \cap S) \leqslant N-3$, then $\operatorname{dem}(U \cap S)$ in U is either $N-3$ or $N-2$. Hence we may assume that $S \cap M^{(1)} = \emptyset$.

For each 2-simplex σ of M such that $\sigma \cap S \neq \emptyset$, let $N_\sigma \subset U$ be a small neighborhood of $\operatorname{int}\sigma$, so that

(a) if τ is also a 2-simplex, then $\operatorname{int} N_\sigma \cap \operatorname{int} N_\tau = \emptyset$ and

(b) N_σ can be written in the form $\sigma \times D^{N-2}$ where D^{N-2} is a closed $(N-2)$-ball and $\sigma \times \{0\}$ identifies with σ.

Let $p_\sigma \colon \sigma \times D^{N-2} \to D^{N-2}$ be the natural projection. Now $\sigma \cap S \subset \operatorname{int}\sigma$, and $\dim(\operatorname{int} N_\sigma \cap S) \leqslant N-3$. Therefore we can find a map $q_\sigma \colon \sigma \times D^{N-2} \to D^{N-2}$ which agrees with p_σ near $\operatorname{bd}(\sigma \times D^{N-2})$ and such that $q_\sigma^{-1}(0) \cap S = \emptyset$. We may assume q_σ is PL and transverse at 0 so that $q_\sigma^{-1}(0)$ is an orientable 2-manifold whose only boundary component is $\operatorname{bd}\sigma$. Let K_σ be the component of $q_\sigma^{-1}(0)$ containing $\operatorname{bd}\sigma$. Now remove the part of $M^{(2)}$ that lies in $\operatorname{int} N_\sigma$ and replace it by $K_\sigma \cap \operatorname{int} N_\sigma$. Having done this for each σ, we arrive at our new 2-complex K, and $K \cap S = \emptyset$. It remains only to prove that K is essential in the first 2 directions.

Suppose K is not essential in the first 2 directions. Let $\pi \colon I^N \to I^2$ be the coordinate projection, and using 2.5, let $f \colon K \to S^1$ be an extension of the map $\pi|\pi^{-1}(S^1) \cap K$. Our goal now is to show that for each $\sigma \in M^{(2)}$, the map $f|\operatorname{bd}\sigma \colon \operatorname{bd}\sigma \to S^1$ is null homotopic. Before accomplishing that, let us see how that would be used to complete the proof.

We will construct a map $g\colon M^{(2)} \to S^1$ which agrees with π on $\pi^{-1}(S^1) \cap M^{(2)}$. By 2.5, this will show $M^{(2)}$ is not essential in the first 2 directions, a contradiction. If $\sigma \in M^{(2)}$ and $\sigma \subset S^{N-1}$, then $\sigma \subset \pi^{-1}(S^1)$ because $M \cap A_k = \emptyset = M \cap B_k$ for $2 < k \leqslant N$. The map g will equal f on such a σ. In fact g will also agree with f on $M^{(1)} \subset K$. Hence, if the above null homotopy property is true, g will extend to $M^{(2)}$ in such a way as to agree with π on $\pi^{-1}(S^1) \cap M^{(2)}$.

We only need to consider a 2-simplex σ such that $\sigma \cap S \neq \emptyset$. The exact cohomology sequence of the pair $(K_\sigma, \operatorname{bd} K_\sigma) = (K_\sigma, \operatorname{bd}\sigma)$ yields:

$$\begin{array}{ccccccc} H^1(K_\sigma) & \to & H^1(\operatorname{bd} K_\sigma) & \to & H^2(K_\sigma, \operatorname{bd} K_\sigma) & \to & H^2(K_\sigma) \\ & & \| & & \| & & \| \\ & & Z & & Z & & 0 \end{array}$$

The map $Z \to Z$, being a surjection, must in fact be an isomorphism. Hence the map $H^1(K_\sigma) \to H^1(\operatorname{bd} K_\sigma)$ is the zero map. On the other hand, if $f|\operatorname{bd} K_\sigma\colon \operatorname{bd} K_\sigma \to S^1$ has non-trivial degree, then by Hopf's Extension Theorem [H–W], the map $H^1(K_\sigma) \to H^1(\operatorname{bd} K_\sigma)$ cannot be the zero map. This concludes the proof.

For further references to the notion of Cantor manifold, see [A1] and [A2].

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