Periodic homeomorphisms on S^3 and cubes with torus knotted holes

by

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Abstract. In this paper we obtain the classification of the orbit space of a free periodic group action on a cube with torus knotted hole. We also give a short geometric proof of the well known fact that a non-trivial torus knot cannot be the fixed point set of a piecewise linear periodic homeomorphism on S^3. While the Smith Conjecture has been solved, our geometric proof could conceivably lead to a purely geometric proof of the Smith Conjecture for all fibered knots.

In this paper we establish the following three results:

1. The classification of the orbit space of a free periodic group action on a cube with torus knotted hole.

2. A free periodic group action on S^3 is conjugate to a standard rotation if and only if it is invariant on a torus knot.

3. A non-trivial torus knot cannot be the fixed point set of a piecewise linear periodic homeomorphism on S^3.

Although the third result has been established by C. H. Giffen [4] and R. H. Fox [3], our geometric proof is extremely short in comparison to Giffen's and does not involve the special algebraic techniques used by Fox. While the Smith conjecture has evidently been solved, our geometric proof could conceivably lead to a purely geometric proof of the Smith conjecture for all fibered knots.

We shall only be concerned with the piecewise linear action of a given cyclic group Z_n on a triangulated 3-manifold M. For this reason all objects in this paper should be viewed as piecewise linear objects. In particular we shall assume without loss of generality that Z_n acts simplicially on M, for every $h \in Z_n$ the fixed point set of h is a subcomplex of M and that the natural projection $\varrho: M \to M/Z_n$ is simplicial and maps simplexes homeomorphically.

Boundary, *closure* and *interior* will be denoted by ∂, cl, and int, respectively. By a *fibering* of a 3-manifold we shall mean a decomposition of that manifold into simple closed curves in the sense of Seifert [8]. If m and n are relatively prime positive integers, then the standard *(m, n)-fibering* of S^3 is given by the transformation

$$x_1' = x_1 \cos(mt) + x_2 \sin(mt),$$
$$x_2' = -x_1 \sin(mt) + x_2 \cos(mt),$$
$$x_3' = x_3 \cos(nt) + x_4 \sin(nt),$$
$$x_4' = x_3 \sin(nt) + x_4 \cos(nt).$$

Thus the point $(x_1', x_2', x_3', x_4') \in S^3$ traverses a simple closed curve as t ranges from 0 to 2π. In [8], Seifert proved the following:

THEOREM 1. *Given a fibering of S^3, then there exists a pair of relatively prime positive integers m and n and a homeomorphism of S^3 onto itself which carries the fibering onto the standard (m, n)-fibering. Conversely, every pair of relatively prime positive integers m and n determine a fibering of S^3.*

We note that under the above transformation, the points $(x_1, x_2, 0, 0) \in S^3$ and $(0, 0, x_3, x_4) \in S^3$ trace the standard 1-spheres $x_1^2 + x_2^2 = 1$ and $x_3^2 + x_4^2 = 1$ in S^3. Furthermore, all other points trace curves which wind about these orthogonal 1-spheres. Thus we have:

THEOREM 2. *Every fibering of S^3 contains at least two unknotted simple closed curves and if a fiber is a non-trivial torus knot, then it is a regular fiber.*

We are now ready to establish the main results of this paper.

THEOREM 3. *Let K be a cube with torus knotted hole and let Z_n act freely on K. If $h \in Z_n$ is a generator, then there is an annulus $A \subset K$ which splits K into two solid tori N_1 and N_2 such that $h(N_i) = N_i$ and $h(A) = A$. In particular, the orbit space $K/\langle h \rangle$ is obtained by attaching the two solid tori $N_i/\langle h \rangle$ along their common boundary annulus $A/\langle h \rangle$.*

Proof. Let A_1 be an annulus in K such that K can be thought of as two solid tori, V_1 and V_2 sewn together along A_1. We can think of h^i as a self-homeomorphism of K for all powers of i. Thus by [2] or [5] we know that $h^i(A_1)$ is an annulus which is isotopic to A_1. Therefore we can find an embedding of $A_1 \times [-1, 1]$ into K with $\partial K \cap \partial(A_1 \times [-1, 1]) = \partial A_1 \times \times [-1, 1]$, $A_1 = A_1 \times \{0\}$, and $h^i(A_i) \subset A_1 \times [-1, 1]$ for all powers of i.

Let $B = A_1 \cup h(A_1) \cup \ldots \cup h^{n-1}(A_1)$. Clearly, $h(B) = B$. Hence we can find a regular neighbourhood of the polyhedron B, say $N(B)$, such that $N(B) \subset A_1 \times [-1, 1]$ and $h(N(B)) = N(B)$. Let $\{C_1, C_2, \ldots, C_m\}$ be the collection of components of $K - N(B)$. Exactly two of these components will contain a core of V_1 or a core of V_2. Call these components C_1 and C_2. Let $M = N(B) \cup C_3 \cup C_4 \cup \ldots \cup C_m$. Then M is a cube with handles and conceivably holes, which separates C_1 from C_2 since $A_1 \subset M$. Furthermore, $h(M) = M$.

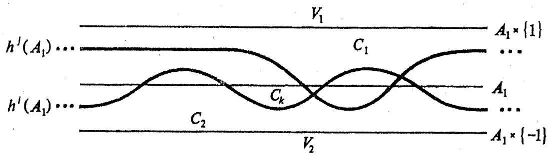

Fig. 1

Clearly, any hole in M is associated with a handle of either C_1 or C_2. Let D be a disk with $\partial D \subset \partial C_1 \cup \partial C_2$ which cuts that handle. Then $D \cup h(D) \cup \ldots \cup h^{n-1}(D) = D'$ is setwise fixed by h. As before we can find a regular neighborhood of D', $N(D')$, such that $h(N(D')) = N(D')$. Therefore $h(M \cup N(D')) = M \cup N(D') = M'$. If $\{C_1', C_2', \ldots, C_p'\}$ are the components of $K - M'$, we still have only two components which contain a core of V_1 or a core of V_2. Let C_1' and C_2' be these components, and let $M'' = M' \cup C_3' \cup \ldots \cup C_p'$. We still have that $h(M'') = M''$, and that M'' separates C_1' and C_2'. Continuing this process we obtain a cube with handles M''' with $h(M''') = M'''$. Also, there are two components of $K - M'''$, C_1''' and C_2''' which contain a core of V_1 or V_2. We redefine $M = M'''$, $C_1 = C_1'''$, and $C_2 = C_2'''$. Since M is a cube with handles, we know by Dehn's lemma and the loop theorem that the orbit space $M/\langle h \rangle$ is a cube with handles.

We can find a solid torus $V_1 \subset C_1$ with $\partial V_1 \cap \partial K$ an annulus and $\mathrm{cl}(K - V_1)$ a solid torus. If $h(C_1) = C_2$, then K can be thought of as $V_1 \cup h(V_1) \cup C$ where C is a solid torus which intersects both V_1 and $h(V_1)$ in an annulus in their respective boundaries. Since a core of V_1 and a core of $h(V_1)$ can be used as geometric generators of $\pi_1(K)$ we would have h_* an isomorphism of $\pi_1(K)$ to itself taking the generators a and b of $\pi_1(K)$ to a_1 and b_1 respectively. Since $\pi_1(K) \simeq \{a, b |\ a^x = b^y\}$ and $\pi_1(h(K)) \simeq \{a_1, b_1 |\ a_1^y = b_1^x\}$, this is impossible. Therefore $h(V_1) \subset C_1$ and hence $h(C_1) = C_1$ and $h(C_2) = C_2$. This means that the orbit spaces $C_1/\langle h \rangle$ and $C_2/\langle h \rangle$ are separated by $M/\langle h \rangle$ in $K/\langle h \rangle$.

We call a handle of $M/\langle h \rangle$ *inessential* if the handle contains a meridian disc D with $\partial D \subset \partial K/\langle h \rangle \cup \partial C_1/\langle h \rangle$ or $\partial D \subset \partial K/\langle h \rangle \cup \partial C_2/\langle h \rangle$. Let $\{D_1, \ldots, D_m\}$ be a complete set of meridian discs for inessential handles of $M/\langle h \rangle$ with $D_i \cap D_j = \emptyset$ for $i \neq j$. By considering the inverse image of these discs in K, we can find a set of meridian discs $\{D_1', \ldots, D_p'\}$ of M with $D_i' \cap D_j' = \emptyset$ for $i \neq j$.

We call a handle of M *inessential* if the handle contains a meridian disc D with $\partial D \subset \partial K \cup \partial C_1$ or $\partial D \subset \partial K \cup \partial C_2$. Since $h(C_1) = C_1$ and $h(C_2) = C_2$, we have that the orbit space of an inessential handle of M will be an inessential handle of $M/\langle h \rangle$. So $\{D_1', \ldots, D_p'\}$ will be a complete set of meridian discs for inessential handles of M.

Let the inverse image of D_i in $M/\langle h\rangle$ be $\{D'_{i1}, \ldots, D'_{in}\} \subset \{D'_1, \ldots, D'_p\}$. Let $N(D'_{i1})$ be a regular neighbourhood of D'_{i1} in M such that

$$h^i(N(D'_{i1})) \cap h^j(N(D'_{i1})) = \emptyset \quad \text{for} \quad 1 \leqslant i < j \leqslant n.$$

Then $C = C_1 \cup C_2 \cup N(D'_{i1}) \cup \ldots \cup h^{n-1}(N(D'_{i1}))$ has two components and $M' = \mathrm{cl}(K-C)$ has fewer inessential handles than M. Clearly $h(M') = M'$. Continuing in this fashion we find a cube with handles which contains no inessential handles; call it M''.

We note that M'' still separates C_1 from C_2. Therefore M'' contains an annulus which separates a core of V_1 from a core of V_2 in K. Hence M'' contains an essential handle that separates C_1 from C_2. But any other handle of M'' would have to be inessential. Thus M'' is itself a solid torus.

We can find a meridian disc D for M'' with the property that $h^i(D) \cap h^j(D) = \emptyset$ for $1 \leqslant i < j \leqslant n$. The iterates of D will split M'' into $n-1$ 3-cells. We find a disc D' contained in one of these 3-cells such that D' splits the 3-cell into two 3-cells with $\partial D' = \alpha \cup \beta \cup \gamma \cup h(\alpha)$, where α is an arc in D, β and γ are arcs in the two annuli $\partial M'' \cap \partial K$. We assumed here, without loss of generality, that in the 3-cell we chose the two discs D and $h(D)$ were adjacent. Clearly $A = D' \cup h(D') \cup \ldots \cup h^{n-1}(D')$ is an annulus with $h(A) = A$.

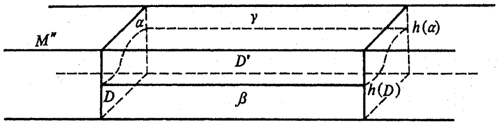

Fig. 2

Let N_1 and N_2 be the solid tori which when sewn together along A form K. Suppose that $h(N_1) = N_2$. This would mean that $h(C_1) = C_2$. As we saw before, this is impossible. Therefore, $h(N_1) = N_1$ and $h(N_2) = N_2$. This proves Theorem 3.

We consider the following lemmas.

Lemma 1. *Let T be a torus in S^3 and $D \subset S^3$ a disc such that $D \cap T = \partial D$. If ∂D is homotopically non-trivial on T, then there is a solid torus V in S^3 such that D is a meridian disc for V and $\partial V = T$.*

Lemma 2. *Let V be a solid torus in S^3, A an annulus in $\mathrm{cl}(S^3 - V)$ with $A \cap V = \partial A$, and let A_1 and A_2 denote the two annuli in ∂V with $\partial A_i = \partial A$ for $i = 1, 2$. If the components of ∂A are meridians for V, then there exist two solid tori V_1 and V_2 in S^3 such that $\partial V_i = A \cup A_i$, $V \subset V_i$ and the components of ∂A are meridians for V_i.*

Lemma 3. *Let V be a solid torus in S^3 and A an annulus in $\mathrm{cl}(S^3 - V)$ which splits $\mathrm{cl}(S^3 - V)$ into two solid tori N_1 and N_2. If V is knotted, then the components of ∂A cannot be meridians for V, N_1 or N_2.*

The proof of Lemma 1 is clear. Lemmas 2 and 3 can be inferred from [7].

Theorem 4. *Let Z_n act freely on S^3. If $h \in Z_n$ is a generator, then h is topologically equivalent to a standard rotation on S^3 if and only if there is a torus knot J in S^3 such that $h(J) = J$.*

Proof. If h is equivalent to a standard rotation, then such knots obviously exist. On the other hand, if J is a trivial (unknotted in S^3) torus knot with $h(J) = J$, then it follows from [6] that h is a standard rotation up to equivalence. Thus we shall assume that J is a non-trivial torus knot in S^3.

Since $h(J) = J$, there is a regular neighbourhood $N(J)$ of J such that $h(N(J)) = N(J)$. Let $K = \mathrm{cl}(S^3 - N(J))$. By Theorem 3, there is an annulus A in K which splits K into two solid tori N_1 and N_2 with $h(N_i) = N_i$ and $h(A) = A$. Let $p\colon S^3 \to S^3/\langle h\rangle$ denote the natural projection $p(x) = p(h^i(x))$, $1 \leqslant i \leqslant n$, and let J_1 and J_2 denote the components of $\partial p(A)$. Then $J_1 \cup J_2$ splits $\partial p(N_1)$ into two annuli A' and $p(A)$. Using a piecewise linear homeomorphism we may view A' as the standard annulus $\{(x, y) \in E^2 |\ 1 \leqslant x^2 + y^2 \leqslant 2\}$. A concentric contraction of the annulus which takes the outer boundary onto the inner boundary determines a fibering of A'. Similarly, we obtain a fibering of $p(A)$ and, hence, a fibering for $\partial p(N_i)$ such that J_1 and J_2 are two fibers and any other fiber of $\partial p(N_i)$ is parallel to J_i. Furthermore, since $h(p^{-1}(J_i)) = p^{-1}(J_i)$ and using standard general positioning arguments, we may choose this fibering such that:

1. for any fiber F, $h(p^{-1}(F)) = p^{-1}(F)$ is a simple closed curve;
2. there exists a pair of meridian discs $D_i \subset p(N_i)$ such that $\bigcup p^{-1}(D_i)$ is a disjoint union of meridian discs in N_i and
3. ∂D_i intersects a fiber exactly k_i points, where k_i is the winding number of J_1 on $p(N_i)$.

Cutting $p(N_i)$ open along D_i yields a solid cylinder C_i. In view of Lemma 3, $k_i > 0$. Thus the fibering of $\partial p(N_i)$ decomposes the "annular" boundary of C_i into line segments. Using a concentric contraction of this decomposition onto the axis of C_i decomposes C_i into such line segments. Reidentifying top and bottom of C_i gives us a fibering of $p(N_i)$ such that for each fiber F in $p(N_i)$, $p^{-1}(F)$ is a simple closed curve with $h(p^{-1}(F)) = p^{-1}(F)$. Since the fiberings of $p(N_1)$ and $p(N_2)$ agree on $p(A) = \partial p(N_1) \cap \partial p(N_2)$, we have a fibering of $K/\langle h\rangle$ and, hence, of $\partial p(N(J))$. As before, we may extend the fibering of $\partial p(N(J))$ to the core $p(J)$ of $p(N(J))$ such that every fiber lifts to a simple closed curve in S^3 which remains invariant under h.

We now have a fibering of S^3 with each fiber invariant under h.

According to Theorem 2 there must be at least two unknotted fibers. Since h is invariant on each of these, it follows from our earlier remark that h is equivalent to a standard rotation. This verifies Theorem 4.

THEOREM 5. *If h is a piecewise linear homeomorphism of period $n > 1$ on S^3 with fixed point set a simple closed curve J, then J cannot be a non-trivial torus knot.*

Proof. Suppose J is a non-trivial torus knot with $h(J) = J$. Then there is a regular neighborhood $N(J)$ of J such that $h(N(J)) = N(J)$. Using the argument given in the proof of Theorem 4, we may fiber S^3 such that J is a fiber and every fiber remains invariant under the action of h.

If F is a fiber in $\partial N(J)$, then we may choose a meridian disc D for $N(J)$ such that $D \cap J$ is a point, $h(D) = D$ and the number of points in $F \cap \partial D$ equals the winding number of F on $N(J)$. By our previous observation, this winding number must be greater than zero. Since the only fixed point of h on D is $D \cap J$ and $h(F \cap \partial D) = F \cap \partial D$, the winding number of F on $N(J)$ must be greater than 1. Thus J is an exceptional fiber and, according to Theorem 2, cannot be a non-trivial torus knot. But this contradicts our supposition and proves Theorem 5.

In view of Theorem 5 and the fact that every torus knot is a fibered knot, it seems feasible that a purely geometric proof of the Smith Conjecture for fibered knots can be developed.

References

[1] J. W. Alexander, *On the subdivision of 3-space by a polyhedron*, Proc. Nat. Acad. Sci. 10 (1924), p. 6.
[2] B. E. Clark, *Knot theories of the lens spaces*, University of Wyoming Ph. D. thesis (1976).
[3] R. H. Fox, *Two theorems about periodic transformations of the 3-sphere*, Michigan Math. J. 14 (1967), pp. 331–334.
[4] C. H. Giffen, *Fibered knots and periodic transformations*, Princeton Ph. D. thesis (1964).
[5] W. Jaco, and R. Myers, *An algebraic characterization of closed 3-manifolds*, preprint.
[6] G. X. Ritter, *Free Z_8 actions on S^3*, Trans. Amer. Math. Soc. 181 (1973), pp. 195–212.
[7] H. Schubert, *Knoten und Vollringe*, Acta Math. 90 (1953), pp. 131–286.
[8] H. Seifert, *Topologie dreidimensionaler gefaserter Raume*, Acta Math. 60 (1933), pp. 147–238.

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Accĕpté par la Rédaction le 6. 1. 1981

On the triangulation of smooth fibre bundles

by

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Abstract. In this paper we prove that if $p: (E, \partial E) \to B$ is a smooth fibre bundle, where $(E, \partial E)$ is a smooth compact manifold with corners in the boundary E, then p admits a piecewise differentiable (= P.D.) triangulation by a PL bundle, and moreover that any such triangulation of $p: \partial E \to B$ extends to one on the whole of E. This generalises the theorems of Putz, in the case where ∂E is smooth, and Lashof and Rothenberg, in the case of a vector bundle.

The main technical result is that if $\alpha: K \to M$ is a P.D. triangulation of a smooth manifold m-ad M by a PL manifold m-ad K, then the simplicial set $\mathrm{PL}(K) \backslash \mathrm{PD}(K, M)$ is contractible, where $\mathrm{PL}(K)$ is the simplicial group of PL automorphisms of K, and $\mathrm{PD}(K, M)$ is the simplicial set whose n-simplices are P.D. triangulations $\Delta^n \times K \to \Delta^n \times M$ commuting with projection onto Δ^n.

§0. Introduction. In its simplest form, the main theorem proved in this paper is that a smooth fibre bundle with compact fibre is triangulable by a PL bundle, and that, if the fibre is bounded, such a triangulation of the subfibre space determined by the fibre boundary can be extended to one of the whole bundle. In its most general form, we wish, in addition, for the fibre to have corners, and to be given a labelled collection of transversely intersecting submanifolds of codimension zero in the boundary, and a compatible collection of PL bundles triangulating the subfibre spaces corresponding to the labelled faces. The theorem then asserts the existence of a PL bundle triangulating the whole bundle and extending the given triangulations.

Our main theorem, in its greatest degree of generality, is a necessary ingredient of our paper [5], in which we prove that compact stratified sets in the sense of Thom are triangulable by simplicial complexes. A proof of the simplest form of our main theorem has been given by Putz [9]. Were this form sufficient for our application, this present paper would be unnecessary. However, Putz gives no consideration to the case where the fibre boundary has corners and since we definitely require the theorem in this degree of generality, and since it is also not clear how to modify Putz' somewhat adhoc argument to give the result, we are forced to give an independent treatment.

In fact, the published result which is closest to ours, and from which