On the rank of a topological convexity

by

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Abstract. The rank of a convexity is an invariant which describes the degree of variation that convex sets are allowed to have. It relates to dimension, and finiteness of the rank has some strong consequences on the behaviour of the topological convexity structure. Rank can also be used to give some simple counterexamples concerning embeddability in finite products of trees, or in cubes.

0. Introduction

The present paper is partially of a preparatory nature. Our subject of investigation is a notion of rank of a (topological) convex structure, due originally to Jamison (cf. [J_2, p. 10]), and which appears to be closely related to the dimension of a certain convex hyperspace: see the main theorem in our subsequent paper [V_4].

The *rank* of a convex structure is defined as the supremum in $\{0, 1, \ldots, \infty\}$ of all $n < \infty$ such that there exists a free subset with n points. A set is called *free* if no one of its points is in the convex hull of the other ones. Some set-theoretic results and some relevant examples will be presented in section 2 below.

Section 3 is entirely concerned with the computation of the rank for two classes of topological convexities: the natural convexity on a *tree-like space*, and "*linear-like*" convexities, which are required to fullfill a condition due to Fuchssteiner (cf. [F, p. 152]). In the first case we found out that the rank equals the number of endpoints of (a suitable compactification of) the tree-like space, and in the second case that the convexity either has infinite rank, or the underlying space is tree-like with the natural convexity.

In section 4 we investigate some topological aspects of rank. Restricting to convexities with connected convex sets and with some separation property, we have obtained the following results:

(1) *the rank of an n-dimensional convexity is at least* $2n$;

(2) *in a convexity of finite rank, the interior of a dense convex subset is* (*weakly*) *dense, and every compact convex set is a polytope*;

(3) *in a convexity of finite rank on a separable metric space, every convex set is the hull of a countable subset.*

The results in (2) and (3) remain valid for convexities of *weakly infinite rank* (that is: the rank is infinite, but no infinite set is free), provided that the convexity is finite-dimensional. Note that in case the rank is finite, the latter condition also holds: see (1). Unfortunately, all known examples of convexities with weakly infinite rank are (weakly) infinite-dimensional. Also, it follows from our results in section 3 that no 1-dimensional or "linear-like" convexity can have weakly infinite rank. This leaves us with the rather intriguing question whether or not there exist finite-dimensional convexities with weakly infinite rank.

Weakly infinite rank is related to weak infinite dimensionality of convex hyperspaces by the main result of [V_4]. It is also related with *non-embeddability* in finite products of tree-like spaces, as will be shown in section 5. We also use the rank to give examples of

(1) *a two-dimensional convexity which cannot be embedded in a product of finitely many tree-like spaces;*

(2) *a sequence of two-dimensional convexities of increasing finite rank such that there is no upper bound for the number of tree-like factors needed for an embedding.*

Some open problems concerning embeddings in cubes or in finite products of tree-like spaces are also mentioned in section 5.

1. Preliminaries

1.1. Set-theoretic convexity. A *convexity* on a set X is a collection $\mathscr{C}$ of subsets of X such that (1) $\varphi, X \in \mathscr{C}$; (2) $\mathscr{C}$ is closed under formation of intersections; and (3) $\mathscr{C}$ is closed under formation of upward filtered unions. The members of $\mathscr{C}$ are called *convex sets*, and the pair $(X, \mathscr{C})$ is called a *convex structure*. It will be assumed throughout that singletons are convex.

Associated to a convexity $\mathscr{C}$ on X there is a *hull operator* $h = h_{\mathscr{C}}$ defined on $A \subset X$ as follows:

$$h(A) = \bigcap \{C \mid A \subset C \in \mathscr{C}\}.$$

The set $h(A)$ is called the *(convex) hull* of A, and the hull of a finite set is called a *polytope*. From the third axiom of convexity it follows that a set $C \subset X$ is convex iff $h(F) \subset C$ for each finite $F \subset C$. This property is called *domain finiteness*.

A collection $\mathscr{S}$ of subsets of X is said to *generate* the convexity $\mathscr{C}$ – and $\mathscr{S}$ is then said to be a *subbase* for $\mathscr{C}$ – if $\mathscr{C}$ is the smallest convexity including $\mathscr{S}$. According to [J_1, p. 8, 9] $\mathscr{S}$ is a subbase for $\mathscr{C}$ iff every nonempty polytope can be obtained as the intersection of a subfamily of $\mathscr{S}$.

Two set-theoretic *separation properties* will be of interest. A convexity $\mathscr{C}$ on X is said to be:

an S_3*-convexity*, if for each $C \in \mathscr{C}$ and for each $x \in X \setminus C$ there is a *half-space* (i.e. a convex set with a convex complement) including C and missing x;

an S_4*-convexity*, if for each pair of disjoint convex sets C, D there is a half-space including C and missing D.

See [J_1], [V_1] for further details.

1.2. Topological convexity. If, in addition to a convexity $\mathscr{C}$, the set X also carries a topology (which will henceforth be implicit in the set symbol X), then $(X, \mathscr{C})$ is called a *topological convex structure* provided all polytopes are closed. Equivalently, $\mathscr{C}$ has a subbase of closed sets. In the sequel, $\mathscr{C}^*$ will denote the collection of all (nonempty) closed convex sets of $(X, \mathscr{C})$.

The convex closure operator h^* of $(X, \mathscr{C})$ is defined as follows:

$$h^*(A) = \bigcap \{C \mid A \subset C \in \mathscr{C}^*\} \qquad (A \subset X).$$

In general, $h^*(A)$ need not be the closure of $h(A)$. However, if $(X, \mathscr{C})$ is *closure-stable*, that is: if the closure of each convex set is convex again, then $h^*(A) = \mathrm{Cl}\, h(A)$.

If $(X, \mathscr{C})$, $(X', \mathscr{C}')$ are (set-theoretic) convex structures, then a function $f\colon X \to X'$ is said to be *convexity preserving* (*C.P.*) *relative to* $\mathscr{C}$ *and* $\mathscr{C}'$ if $f^{-1}(C') \in \mathscr{C}$ for each $C' \in \mathscr{C}'$. A continuous function is called a *map*. This gives rise to a category of set-theoretic (resp. topological) convex structures, the morphisms of which are the C.P. functions (resp. C.P. maps). The notion of "*isomorphism*" should be understood in this setting.

A function $f\colon X \to [0, 1]$ is said to *separate* the sets $A, B \subset X$ if

$$A \subset f^{-1}(0), \quad B \subset f^{-1}(1).$$

Let the unit interval be equipped with the *linear convexity*, that is, the topological convexity generated by the sets of type $[0, t]$ or $[t, 1]$ $t \in [0, 1]$. A topological convexity $\mathscr{C}$ on X is said to be:

semi-regular, if for each $C \in \mathscr{C}^*$ and for each $x \in X \setminus C$ there is a C.P. map $X \to [0, 1]$ separating C and $\{x\}$;

regular, if for each $C \in \mathscr{C}^*$ and for each polytope $P \subset X \setminus C$ there is a C.P. map $X \to [0, 1]$ separating C and P.

See [V_1, 1.5] for an alternative description not involving C.P. maps.

As follows from observations in [J_1, p. 24, 26], a semi-regular convexity is S_3 and a regular convexity is S_4. See [V_1, 2.2] for a different proof.

It was shown in [V_1, 2.4] that a regular closure-stable convexity on a compact space is even *normal*, that is: every two disjoint convex closed sets can be separated with a CP map into the unit interval. On non-compact spaces, normal convexities are rather exceptional.

1.3. Convex dimension. Let $\mathscr{C}$ be a convexity on the set X. If Y is a subset of X, then the *trace* of $\mathscr{C}$ on Y is the convexity

$$\mathscr{C} \upharpoonright Y = \{C \cap Y \mid C \in \mathscr{C}\}.$$

Two subsets A_1, A_2 of X are said to be *screened* by the sets S_1, S_2 if

$$A_1 \subset S_1 \backslash S_2, \quad A_2 \subset S_2 \backslash S_1, \quad S_1 \cup S_2 = X.$$

Let $(X, \mathscr{C})$ now be a topological convex structure. The (*small inductive*) *dimension* of $(X, \mathscr{C})$ is the number $\operatorname{ind}(X, \mathscr{C}) \in \{-1, 0, 1, 2, \ldots, \infty\}$ satisfying the following rules ($n < \infty$)

(1) $\operatorname{ind}(X, \mathscr{C}) = -1$ iff $X = \emptyset$;

(2) $\operatorname{ind}(X, \mathscr{C}) \leqslant n+1$ iff for each $C \in \mathscr{C}^*$ and for each $x \in X \backslash C$ there exist $S_1, S_2 \in \mathscr{C}^*$ screening C and $\{x\}$, and such that $\operatorname{ind}(S_1 \cap S_2, \mathscr{C} \upharpoonright S_1 \cap S_2) \leqslant n$.

As convexities on non-compact spaces mostly fail to be normal, it makes little sense to consider a „large inductive” dimension function for convexities. Instead, one may consider various „reasonable” dimension functions (as described in [V_2, 3.1]) which are based on screening other types of pairs of convex closed disjoint sets. For semi-regular closure-stable convexities with connected convex sets all these dimension functions coincide with the above ind: see [V_2, 3.2].

To shorten lengthly expressions as in (2) above we will often omit reference to the convexity, writing X instead of $(X, \mathscr{C})$. We hereby agree that ind stands for *convex* dimension. Topological dimension will be considered only in our subsequent paper [V_4].

1.4. The continuity property. Let $(X, \mathscr{C})$ be a topological convex structure. A subset Y of X is said to be *in continuous position* relative to $\mathscr{C}$ provided that for each convex open set $O \subset X$ meeting Y,

$$\bar{O} \cap Y = \mathrm{Cl}_Y(O \cap Y).$$

The convex structure $(X, \mathscr{C})$ is said to have the *continuity property* if each of its convex sets is in continuous position relative to $\mathscr{C}$. As was observed in [V_1, 4.3] the continuity property is not inherited by convex subsets, equipped with the trace convexity.

In the sequel, the boundary $\bar{O} \backslash O$ of an open set will also be denoted by $\dot{O}$, and a set with more than one point will be called *nontrivial*. We also agree (for convenience) that the intersection of the empty family of subsets of a given set X equals X.

2. Rank of a convexity

2.1. Definitions. Let $(X, \mathscr{C})$ be a (set-theoretic) convex structure. A subset F of X is said to be *free*(1) (*relative to* $\mathscr{C}$) if for each $x \in F$,

$$x \notin h(F \backslash \{x\}).$$

(1) In [J_3] the term “independent” is being used.

We note that F need not be finite. However, it follows directly from domain finiteness (cf. 1.1) that a set is free iff each of its finite subsets is free.

From an informal point of view, each point of a free set F contributes *essentially* to the size and shape of the convex set $h(F)$. As every convex set can be approximated by polytopes (domain finiteness) the following seems to provide an adequate measure for the degree of variation that convex sets are allowed to have.

The *rank of a convex structure* $(X, \mathscr{C})$ is the number

$$d(X, \mathscr{C}) \in \{0, 1, \ldots, \infty\}$$

such that $d(X, \mathscr{C}) \leqslant n$ iff no finite set in X with more than n points is free, $n < \infty$. Equivalently, $d(X, \mathscr{C}) \geqslant n$ iff there exists a free set in X with exactly n points. This invariant has already been considered in [J_2].

The convex structure $(X, \mathscr{C})$ is said to have *finite rank* if $d(X, \mathscr{C}) < \infty$, and to have *infinite rank* otherwise. We want to distinguish the following cases. If $d(X, \mathscr{C}) = \infty$ but no infinite set is free in X, then $(X, \mathscr{C})$ is said to have *weakly infinite rank*. If there exists an infinite free collection in X, then $(X, \mathscr{C})$ is said to have *strongly infinite rank*.

If no confusion can arise we simply write $d(X)$ instead of $d(X, \mathscr{C})$. The term “number” will henceforth refer to a member of the set $\{0, 1, 2, \ldots, \infty\}$.

The following auxilliary invariant will be of use in determining the rank of a convexity in many concrete cases. Let $(X, \mathscr{C})$ be a convex structure. The *generating degree* of $(X, \mathscr{C})$ is the number $\operatorname{gen}(X, \mathscr{C})$ determined by the following rule: $\operatorname{gen}(X, \mathscr{C}) \leqslant n$ ($n < \infty$) iff there exists a subbase $\mathscr{S}$ for $\mathscr{C}$ and a decomposition

$$\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \ldots \cup \mathscr{S}_n$$

of $\mathscr{S}$ consisting of totally ordered subfamilies $\mathscr{S}_1, \ldots, \mathscr{S}_n$. In the latter case we also say that the *degree* of $\mathscr{S}$ is (at most) n.

The generating degree is a useful device to estimate the rank from above, as is shown in our next result:

2.2. Theorem. *Let $\mathscr{C}$ be a convexity on the nontrivial set X. Then $d(X, \mathscr{C}) \leqslant \operatorname{gen}(X, \mathscr{C})$.*

Note that a one-point space has rank 1, whereas the empty family is a subbase.

Proof. We may assume that $\operatorname{gen}(X, \mathscr{C}) = n < \infty$. Let $\mathscr{S}$ be a subbase for $\mathscr{C}$ of degree n. If $n = 0$ then $\mathscr{S} = \emptyset$, and $\mathscr{C} = \{\emptyset, X\}$. As singletons are assumed to be convex, X consists of only one point. Therefore, $n > 0$.

Suppose that $d(X, \mathscr{C}) \geqslant n+1$. Then there exists a free subset $F = \{x_1, \ldots, x_{n+1}\}$ of X with exactly $n+1$ points. As every nonempty polytope in X is the intersection of subbasic sets (cf. 1.1) there exist $S_1, \ldots, S_{n+1} \in \mathscr{S}$ with

$$h(F \backslash \{x_i\}) \subset S_i, \quad x_i \notin S_i \quad (i = 1, \ldots, n+1).$$

It follows that $S_1, \ldots, S_{n+1}$ are pairwise incomparable, contradiction. ∎

We note that $d(X, \mathscr{C})$ and $\mathrm{gen}(X, \mathscr{C})$ need not be equal in general (though, as a rule, they seem to be equal for "sufficiently nice" convexities): let $X = \{0, 1\}^2$ be equipped with the convexity $\mathscr{C}$ consisting of all subsets of X of type $A \times B$. Then clearly $d(X, \mathscr{C}) = 2$, but every subbase for $\mathscr{C}$ must contain the four sets

$$\{i\} \times \{0, 1\}, \quad \{0, 1\} \times \{i\}, \quad i = 0, 1,$$

which are mutually incomparable.

When dealing with more complicated convexities, it may be difficult to keep an eye on every possible subbase. The following result is therefore of practical interest. Recall that the *Helly number* h of a convexity $\mathscr{C}$ on X is the infimum of all $n < \infty$ such that for each finite collection $\mathscr{D} \subset \mathscr{C}$, $\bigcap \mathscr{D} \neq \emptyset$ whenever the members of $\mathscr{D}$ meet n by n (cf. [KW]).

2.3. THEOREM. *Let $\mathscr{C}$ be an S_3-convexity on X. Then there exists a subbase $\mathscr{H}$ for $\mathscr{C}$ such that*

(1) *all members of $\mathscr{H}$ are half-spaces of X;*

(2) *the degree of $\mathscr{H}$ equals* $\mathrm{gen}(X, \mathscr{C})$;

(3) *if h is the Helly number of $(X, \mathscr{C})$, then every nonempty half-space of X is the intersection of at most $h-1$ members of $\mathscr{H}$.*

Consequently, if $(X, \mathscr{C})$ is binary (that is, if $h \leqslant 2$), then $\mathrm{gen}(X, \mathscr{C})$ *equals the degree of the subbase, consisting of ALL nonempty half-spaces of X.*

Proof. We first recall an elegant and useful construction from [J_1, p. 16]. Let 2^X be the Cantor space obtained as the X-fold product of the discrete space $\{0, 1\}$. A set $A \subset X$ corresponds to a point in 2^X whose x-coordinate ($x \in X$) is 0 if $x \in A$ and 1 otherwise. Expressed directly in terms of the power set of X, 2^X has a base of open sets of type

$$I(F, G) = \{A \mid F \subset A, G \cap A = \emptyset\},$$

where F, G are finite subsets of X. Note that these basic sets are also closed. Jamison observed that *a convexity $\mathscr{C}$ is always a compact subset of 2^X* (cf. [J_1, I.6]). Hence, as the formation of complements is a continuous operation $2^X \to 2^X$, *the collection of all half-space of $(X, \mathscr{C})$ is also compact.*

Let $\mathscr{S}$ be any subbase for $\mathscr{C}$. Its closure $\bar{\mathscr{S}}$ in 2^X satisfies $\bar{\mathscr{S}} \subset \mathscr{C}$, and it is easy to see that $\mathscr{S}$ and $\bar{\mathscr{S}}$ have the same degree. Let $C \subset X$ be nonempty convex, and let $x \in X \setminus C$. For each nonempty finite $F \subset C$ we have $x \notin h(F)$, and as $\mathscr{S}$ is a subbase, $x \notin S \supset h(F)$ for some $S \in \mathscr{S}$. Therefore the set $\mathscr{S} \cap I(F, \{x\})$ is nonempty. Also, if F_1 and F_2 are subsets of X then

$$I(F_1, \{x\}) \cap I(F_2, \{x\}) = I(F_1 \cup F_2, \{x\}).$$

Hence

$$\{\bar{\mathscr{S}} \cap I(F, \{x\}) \mid F \subset C \text{ nonempty finite}\}$$

is a downward filtered collection of compact sets, and there exists an element T common to all these sets. Clearly, $C \subset T$, $x \notin T$, showing that *every nonempty convex set is the intersection of a subfamily of $\bar{\mathscr{S}}$.*

Let C be nonempty convex again, and let $x \in X \setminus C$. By the third axiom of convexity there exists a maximal convex set C' with the properties

$$C \subset C'; \ x \notin C'.$$

Then C' is a half-space since by the axiom S_3 there exists a half-space H of X with

$$C' \subset H; \quad x \notin H.$$

Also, C' is in $\bar{\mathscr{S}}$ since by the above argument there is a $T \in \bar{\mathscr{S}}$ with

$$C' \subset T;\ x \notin T.$$

This shows that $\bar{\mathscr{S}}$ includes the subbase $\mathscr{H}'$ of all half-spaces H of X such that for some $x \in X$, H is maximal with the property that $x \notin H$. Then $\mathscr{H} = \bar{\mathscr{H}'}$ consists of half-spaces of X, and the degree of $\mathscr{H}$ is not larger than the degree of $\mathscr{S}(\bar{\mathscr{S}})$. As $\mathscr{H}$ does not depend on $\mathscr{S}$, we find that $\mathrm{gen}(X, \mathscr{C})$ equals the degree of $\mathscr{H}$, thus establishing (1) and (2).

Proof of (3). First, note that by an argument given above, every nonempty convex set is the intersection of a subfamily of $\mathscr{H}$. Hence (3) is valid for $h = \infty$ (no cardinal distinction is made at infinity). Note that $X = \emptyset$ if $h = 0$, and X is a singleton if $h = 1$. In both cases, statement (3) is trivially fullfilled. So we may assume $2 \leqslant h < \infty$.

Let $H \subset X$ be a nonempty half-space. For each $x \in X \setminus H$ there is a half-space H_x of X maximal with the properties

$$H \subset K, \quad x \notin K.$$

The subcollection $\mathscr{K}$ of $\mathscr{H}$, defined by

$$\mathscr{K} = \{K\colon K \in \mathscr{H}, H \subset K\}$$

is easily seen to be compact in 2^X, and the above argument shows that

$$\bigcap \mathscr{K} \cap X \setminus H = \emptyset.$$

By [J_2, prop. (B)], there exist already h or less sets among the members of $\mathscr{K} \cup \{X \setminus H\}$ which have an empty intersection. Note that $\bigcap \mathscr{K} = H \neq \emptyset$. Hence there exist $K_1, \ldots, K_m \in \mathscr{K}$ with $m \leqslant h-1$ such that

$$H \subset \bigcap_{i=1}^{m} K_i \subset H,$$

proving (3). The final part of the theorem easily follows. ∎

By definition, a convexity $\mathscr{C}$ on a space X is a topological convexity iff it admits a subbase consisting of *closed* convex sets. It is then natural to ask whether or not $\mathrm{gen}(X, \mathscr{C})$ can be obtained as the degree of some subbase of closed convex sets. We only have an affirmative answer in case $(X, \mathscr{C})$ is semi-regular and binary: it follows from semi-regularity that $(X, \mathscr{C})$ is S_3 (cf. (1.2)) and that the collection of all closed nonempty half-spaces of X is a subbase for $\mathscr{C}$. Hence by Theorem 2.3, the generating degree of a semi-regular binary convexity equals the degree of its subbase of all nonempty closed half-spaces.

Let us consider some concrete examples now.

2.4. EXAMPLES.

(2.4; 1). Let $0 < n \leqslant \infty$, and let the *n-cube* I^n be equipped with the "cubical" convexity $\mathscr{C}_n$, that is, the topological convexity generated by the subbase $\mathscr{S}_n$ consisting of all sets of type

$$\pi_i^{-1}[0, t] \text{ or } \pi_i^{-1}[t, 1], \quad t \in [0, 1],\ i = 1, \ldots, n,$$

where π_i: $I^n \to I$ denotes the ith projection. Then $d(I^n, \mathscr{C}_n) = 2.n$ (we hereby agree that $2.\infty = \infty \pm 1 = \infty$), and in particular, $(I^\infty, \mathscr{C}_\infty)$ has strongly infinite rank.

Indeed, a (standard) free collection F may consist of all $x \in I^n$ with $\pi_i(x) = \frac{1}{2}$ for all but one i, in which case $\pi_i(x) \in \{0, 1\}$ (for $n = 1$, F equals the endpoint set of I. For $n > 1$, we note that the set of corner points of I^n is not free relative to $\mathscr{C}_n$). Hence $d(I^n, \mathscr{C}_n) \geqslant 2.n$, and $(I^\infty, \mathscr{C}_\infty)$ has strongly infinite rank. As the above subbase $\mathscr{S}_n$ is built up by $2n$ totally ordered collections, we find that

$$d(I^n, \mathscr{C}_n) \leqslant \mathrm{gen}(X, \mathscr{C}) \leqslant 2n,$$

whence both number are equal to $2n$ (as $(I^n, \mathscr{C}_n)$ is semi-regular and binary, this agrees with the above conclusion that $\mathrm{gen}(I^n, \mathscr{C}_n)$ is determined by a subbase of closed half-spaces).

(2.4; 2). Let the square I^2 now be equipped with the (trace of the) *linear convexity*. Then I^2 has strongly infinite rank since every circle in I^2 is a free set. A similar argument applies on any linearly convex set C in a vector space, such that C is not a point or a line (segment). See 3.2 below for a generalization to convexities with a property of Fuchssteiner.

(2.4; 3). With the notation of (2.4; 1) we put for each finite n

$$X_n = \{x \in I^\infty |\ 2^{-n} \leqslant \pi_k(x) \leqslant 2^{-n+1} \text{ if } k \leqslant n, \pi_k(x) = 2^{-k} \text{ otherwise}\}.$$

Note that X_n is an n-dimensional subcube of I^∞. Let $\underline{o}$ denote the origin. Then we put

$$X = \bigcup_{n=1}^{\infty} X_n \cup \{\underline{o}\}.$$

X is equipped with the trace $\mathscr{C}$ of the cubical convexity $\mathscr{C}_\infty$ of I^∞. This convex structure was considered previously in [V$_2$, 4.12] where it was observed that $\mathscr{C}$ is a normal binary convexity on the continuum X, and that if $x \in X_m$, $y \in X_n$, $z \in X_p$ with $m < n < p$, then $\pi_k(x) \geqslant \pi_k(y) \geqslant \pi_k(x)$ for all k. In particular, y is then in the convex hull of x and z. Therefore, a free subcollection of X can meet at most two of the cubes X_n, $n = 1, 2, \ldots$, and its intersection with X_n corresponds to a free subset of $(I^n, \mathscr{C}_n)$ under any isomorphism $X_n \approx I^n$. Hence no free set in X is infinite, whereas there exist free sets in $X_n \subset X$ with $2n$ points, $n < \infty$, showing that $(X, \mathscr{C})$ has weakly infinite rank.

(2.4; 4). *n-folded squares.* With the notation of (2.4; 1), let

$$X_n = \{x \in I^{n+2} |\ \exists i \leqslant n+1\colon \forall j \neq i, i+1\colon \pi_j(x) = 0\} \quad (n \geqslant 0).$$

X_n is equipped with the trace of the cubical convexity of I^{n+2}. This space can informally be described as the union of $n+1$ squares, each spanned between two successive axes. X_n is topologically equivalent to the square itself, and the axes numbered by $2, 3, \ldots, n+1$ can be regarded as n "foldings" in X_n.

Each X_n is a normal binary convex structure as can be seen from the external characterization theorem [vMW, 3.4]. Note that X_0 is the ordinary square convex structure, and that X_1 is isomorphic to X_0. X_2 is isomorphic to a subspace of X_0, obtained by cutting out a rectangle at one corner point. Hence for $n \leqslant 2$,

$$d(X_n) = \mathrm{gen}(X_n) = 4.$$

For $n > 2$ we have

$$d(X_n) = \mathrm{gen}(X_n) = n+2.$$

Indeed, a free collection in X_n is obtained by choosing a point different from the origin on each axis of I^{n+2}, showing that $n+2 \leqslant d(X_n)$. In view of 2.2 it suffices to show that $\mathrm{gen}(X_n) \leqslant n+2$. To this end, let

$$\mathscr{H}_n = \{H \cap X_n |\ H \subset I^{n+2} \text{ a closed halfspace}\}.$$

As $\mathrm{gen}(X_n) = n+2$ for $n = 2$, it follows from the remarks after 2.3 that the subbase of all closed half-spaces of X_n has degree $n+2$, whence the degree of $\mathscr{H}_n$ is also $n+2$ in case $n = 2$. Assume by induction that $\mathscr{H}_m$ has degree $m+2$ for $2 \leqslant m < n$, and suppose that $H_1, \ldots, H_{n+3}$ are closed half-spaces of I^{n+2} such that $H_1 \cap X_n, \ldots, H_{n+3} \cap X_n$ are pairwise incomparable. It is easy to see that a closed half-space of I^{n+2} is of type

$$\pi_i^{-1}[0, t] \text{ or } \pi_i^{-1}[t, 1] \quad (i = 1, \ldots, n+2, t \in [0, 1]).$$

As $H_1, \ldots, H_{n+3}$ must be incomparable in I^{n+2}, there exist $k \neq l$ in $\{1, \ldots, n+3\}$ with

$$H_k \cap H_l = \varnothing \quad \text{or} \quad H_k \cup H_l = I^{n+2}.$$

Hence H_k and H_l are "perpendicular" to the same ith axis. Reserving the counting of the axes if necessary, we may assume that $i \leqslant n$. If some H_m were perpendicular to the $(n+2)$nd axis, then clearly $H_m \cap X$ were comparable with $H_k \cap X$ or with $H_l \cap X$. This shows that the $(n+2)$nd axis is not essentially

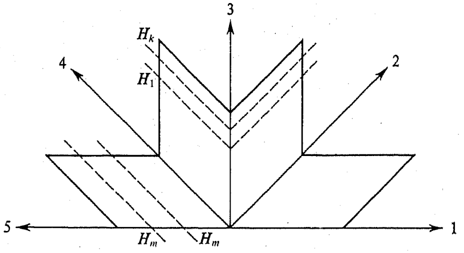

Fig. 1. X_3

involved. If $\pi\colon I^{n+2} \to I^{n+1}$ is the projection annihilating the $(n+2)$nd coordinate, then $\pi(X_n) = X_{n-1}$, and by inductive assumption two members of

$$\pi(H_1) \cap X_{n-1}, \ \ldots, \ \pi(H_{n+3}) \cap X_{n-1}$$

are comparable. As each H_m is orthogonal to one of the axes $1, 2, \ldots, n+1$, we find that

$$\pi^{-1}\big(\pi(H_m) \cap X_{n-1}\big) = H_m \cap X_n,$$

and hence that some two members of the original collection are comparable, a contradiction. As no $n+3$ members of $\mathscr{H}_n$ are pairwise incomparable, it follows from [D, 1.1] that $\mathscr{H}_n$ can be decomposed as a union of at most $n+2$ totally ordered families.

We conclude with two simple results which will be of use below.

2.5. THEOREM. *Let X, Y be convex structures, and let $f\colon X \to Y$ be a C.P. function onto. Then $d(X) \geqslant d(Y)$, and if X has weakly infinite rank, then Y has finite or weakly infinite rank.*

2.6. THEOREM. *Let X be a convex structure, $Y \subset X$. Then $d(X) \geqslant d(Y)$.*

The proofs are left to the reader.

3. Tree-like spaces and linear-like convexities

Recall that a *tree-like space* is a connected Hausdorff space in which every two distinct points can be separated by a third one. A non-cutpoint is usually called an *endpoint*. If x is a point of a tree-like space X then the *order of* x, $\operatorname{ord}(x)$, is defined to be the number of components of $X\backslash\{x\}$. Then x is called a *ramification point* if $\operatorname{ord}(x) > 2$. See [N, p. 143].

It was shown in [V_1, 2.10] that if X is a locally connected tree-like space, then the collection $\mathscr{C}$ of all connected subsets of X constitutes a normal binary convexity with compact polytopes. $\mathscr{C}$ is obviously closure stable, and $\operatorname{ind}(X, \mathscr{C}) \leqslant 1$ by [V_2, (2.6; 2)]. Conversely, if $(X, \mathscr{C})$ is a 1-dimensional semi-regular closure-stable convexity with connected convex sets, then X is a tree-like space. With some more efforts, it can even be shown that $\mathscr{C}$ then consists exactly

of *all* connected subsets of X. This correspondence between tree-like spaces and 1-dimensional convexities is illustrated by the proof of Theorem 3.1 below.

The above described convexity on a tree-like space will henceforth be referred to as the *natural* convexity.

3.1. THEOREM. *Let X be a locally connected nontrivial tree-like space, equipped with its natural convexity $\mathscr{C}$. Then the following numbers are equal:*

(a) *the rank of $(X, \mathscr{C})$;*

(b) *the sum of the number of endpoints of X and the number of free $\mathscr{C}^*$-ultrafilters on X;*

(c) *the generating degree of $(X, \mathscr{C})$;*

(d) $2+\Sigma(\operatorname{ord}(x)-2)$, *summation being taken over the ramification points of X.*

Proof that (a) = (b). We first consider the case where X is compact (then the condition of local connectivity is redundant by [W, Lemma 4]). Let E be the set of all endpoints of X. For each $x \in E$, $X\backslash\{x\}$ is a connected set including $E\backslash\{x\}$, whence $x \notin h(E\backslash\{x\})$ and E is free. It follows that (a) $\geqslant$ (b).

Let n be the number of elements of E. In order to show that (a) $\leqslant$ (b), we may assume that $n < \infty$. Suppose that there exists a free collection $F = \{y_1, \ldots, y_m\}$ in X with exactly m points, $n < m < \infty$. By semi-regularity and by the third axiom of convexity, we can find for each $i = 1, \ldots, m$ a maximal open half-space O_i of X with

(1) $$F\backslash\{y_i\} \subset O_i, \quad y_i \notin O_i.$$

Note that $y_i \in \bar{O}_i$, for otherwise another application of semi-regularity would provide us with a strictly larger open half-space with property (1). As $\operatorname{ind}(X, \mathscr{C}) = 1$, we find from [$V_2$, 2.7] that $\dot{O}_i = \bar{O}_i\backslash O_i$ is 0-dimensional, and hence that $\dot{O}_i = \{y_i\}$.

We now apply some techniques from binary convexities. If $(X, \mathscr{C})$ is a normal binary convex structure with compact X, and if $C \in \mathscr{C}^*$, then by [vMV_1, 3.4] there is a so-called *nearest point map* $p\colon X \to C$ with the following property. For each $x \in X$, $p(x)$ is the unique point in C with the property that

$$h\{x, p(x)\} \cap C = \{p(x)\}.$$

Returning to the original situation, take $C = h(F)$. Then

$$p^{-1}(y_i) = X\backslash O_i, \quad i = 1, \ldots, m.$$

Indeed, if $x \in O_i$ then

$$h(\{x\} \cup F\backslash\{y_i\}) \subset O_i,$$

and by the regularity of $(X, \mathscr{C})$ there is an open half-space O' of X with

$$h(\{x\} \cup F\backslash\{y_i\}) \subset O' \subset \bar{O}' \subset O_i.$$

Then $\dot{O}'$ separates each of x and $F\setminus\{y_i\}$ from y_i, and it follows from the connectedness of convex sets that

$$h\{x, y_i\} \cap \dot{O}' \neq \emptyset \neq h(F) \cap \dot{O}'.$$

Also, $h\{x, y_i\}$ meets $h(F)$, whence by binarity there is a point

$$u \in h\{x, y_i\} \cap h(F) \cap \dot{O}'.$$

Note that $u \neq y_i$, whence $p(x) \neq y_i$.

If, on the other hand, $x \notin O_i$, then

$$\emptyset \neq h\{x, y_i\} \cap h(F) \subset (X\setminus O_i) \cap \bar{O}_i = \{y_i\},$$

whence $p(x) = y_i$.

By [V_1, 3.5, 3.2] every nonempty closed half-space of X must include an endpoint of X. Hence we find an endpoint x_i in $p^{-1}(y_i)$ for each $i = 1, \ldots, m$, contradicting with our assumption on the total number of endpoints.

We now deal with the general case. Let X_w denote X with the *weak $\mathscr{C}$-topology*, that is, the topology generated by $\mathscr{C}^*$ (terminology of [V_1]). This passage from X to X_w does not effect most topo-convex properties of $(X, \mathscr{C})$: X and X_w have the same convex closed sets and hence they also have the same separation properties; convex sets are connected in X_w if they are in X, and $\operatorname{ind}(X, \mathscr{C}) = \operatorname{ind}(X_w, \mathscr{C})$ (see [V_2, (2.6; 4)]). In the present circumstances, we may conclude that X_w is a tree-like space with exactly the same endpoints as X (an endpoint is, in convex terms, a singleton half-space).

In general, if $\mathscr{S}$ is a closed subbase for a space Y, then the *superextension* $\lambda(Y, \mathscr{S})$ *of Y relative to $\mathscr{S}$* is the set of all *maximal* linked systems in $\mathscr{S}$, equipped with a Wallman-like topology generated by the closed subbase $\mathscr{S}^+ = \{S^+ \mid S \in \mathscr{S}\}$, where S^+ is the collection of all $\mathscr{M} \in \lambda(Y, \mathscr{S})$ with $S \in \mathscr{M}$. The interested reader is referred to [Ve] or [vM] for detailed information concerning superextensions. We will only need the following facts: $\lambda(Y, \mathscr{S})$ is a compact T_1-space, and if $\mathscr{S}$ is a normal subbase, then the convexity of $\lambda(X, \mathscr{S})$ generated by $\mathscr{S}^+$ is normal and binary, and its trace on the subspace Y (up to canonical embedding) is precisely the convexity generated by $\mathscr{S}$. We also note that normal binary convexities are closure-stable by [V_1, 2.9].

We now apply this construction on the space X_w with its closed subbase $\mathscr{C}^*$. As $\mathscr{C}$ is binary, we find that every maximal linked system $\mathscr{M} \subset \mathscr{C}^*$ is in fact an ultrafilter, whence $\lambda(X_w, \mathscr{C}^*)$ is a *compactification* of X_w. In particular, $\lambda(X_w, \mathscr{C}^*)$ is connected, and by the continuity of the nearest point map, it follows that all convex (closed) sets are connected. As polytopes are compact in X and hence in X_w, we find by domain-finiteness of the convexity on $\lambda(X_w, \mathscr{C}^*)$ that X_w is a dense *convex* subset of $\lambda(X_w, \mathscr{C}^*)$. In semi-regular closure-stable convexities with connected convex sets, convex dimension is not affected by the passage from a convex set to its closure (cf. [V_2, 2.9]). Hence

$$\operatorname{ind}\lambda(X_w, \mathscr{C}^*) = \operatorname{ind}(X_w, \mathscr{C}) = \operatorname{ind}(X, \mathscr{C}) = 1,$$

and in particular, $\lambda(X_w, \mathscr{C}^*)$ is a compact tree-like space. By [vMV_2, 1.7] such a space admits but *one* normal binary convexity, and hence the above convexity on $\lambda(X_w, \mathscr{C}^*)$ is the natural one for a tree-like space. Finally, the rank of a dense convex subspace is easily seen to be equal to the rank of the whole space by semi-regularity. Hence

$$d(X, \mathscr{C}) = d(X_w, \mathscr{C}) = d(\lambda(X_w, \mathscr{C}^*)) = \text{number of endpoints of } \lambda(X_w, \mathscr{C}^*).$$

One easily sees that the endpoints of $\lambda(X_w, \mathscr{C}^*)$ consist exactly of the end-points of X (X_w) and of the free ultrafilters of convex closed sets of X (X_w). This establishes the equality (a) = (b) in full generality.

Proof of (b) = (d). In case of compact tree-like spaces, this is well-known and easy to prove. If the tree-like space X is not compact, then we pass to $\lambda(X_w, \mathscr{C}^*)$ as above: no remainder point of $\lambda(X_w, \mathscr{C}^*)$ is a cutpoint (and *à fortiori*, a ramification point) whence the number (d) is not affected by the passage from X to $\lambda(X_w, \mathscr{C}^*)$, and the equality (b) = (d) follows from the compact case.

Proof of (b) = (c). We have (a) $\leqslant$ (c) by Theorem 2.2, and hence that (b) $\leqslant$ (c). In order to obtain that (c) $\leqslant$ (b), we will prove the following statements for $1 \leqslant n < \infty$:

$(I-n)$: if X is a locally connected tree-like space with $d(X, \mathscr{C}) \leqslant n$, then there are no $(n+1)$ incomparable half-spaces in X.

Note that by semi-regularity of $(X, \mathscr{C})$, the collection of all (closed) half-spaces forms a subbase for $\mathscr{C}$. By [D, 1.1], it follows from the conclusion of $(I-n)$ that $\operatorname{gen}(X, \mathscr{C}) \leqslant n$. $(I-1)$ being a triviality, we proceed by induction, assuming $(I-m)$ to hold for $m < n$, where $n > 1$. Let $d(X, \mathscr{C}) \leqslant n$, where X is a nontrivial locally connected tree-like space, and let $H_1, \ldots, H_{n+1}$ be half-spaces of X. It is clear that $H_1, \ldots, H_{n+1}$ are mutually incomparable iff there is a polytope P of X such that the relative half-spaces $P \cap H_1, \ldots, P \cap H_{n+1}$ are mutually incomparable. Note that P is compact. We may therefore assume that X is compact. As $d(X, \mathscr{C}) \leqslant n$, we find from (a) = (b) that the endpoint set E of X has at most n members. It is easily seen that $X = h(E)$. Pick a point $x \in E$ (which will be specified later). Then $C = h(E\setminus\{x\})$ is a compact tree-like space with endpoint set equal to $E\setminus\{x\}$ and hence with $d(C) \leqslant n-1$. Let $y = p(x)$, where $p\colon X \to C$ is the nearest point map (cf. above). Then

(1) $$\forall c \in C\colon\ y \in h\{c, x\};$$

(2) $$X = C \cup h\{x, y\};$$

(3) $h\{x, y\}$ is a totally ordered continuum and $\mathscr{C} \restriction h\{x, y\}$ is the natural convexity.

(1) is a direct consequence of binarity, and (2) follows from the fact that $C \cup h\{x, y\}$ is a connected set including E. (3) is well-known and easy to prove.

A proof of $(I-n)$ consists of the following analysis of possible situations. Assume first that there exist $i \neq j$ in $\{1, \ldots, n+1\}$ such that $C \subset H_i$ and $C \subset H_j$: then H_i and H_j are comparable by (2) and (3).

Note that if each H_i misses at most one endpoint of X, then two of them — say H_i and H_j — include $E \setminus \{u\}$ for some $u \in E$. By the above argument (with x equal to u), H_i and H_j are comparable. We henceforth assume that e.g. H_{n+1} mises at least *two* endpoints of X, one of them being chosen as our point x. Note that then $H_{n+1} \subset C$: let $u \in C$ be another endpoint of X with $u \notin H_{n+1}$. Then by (1),

$$y \in h\{u, x\} \subset X \setminus H_{n+1}, \tag{4}$$

and $H_{n+1} \subset C$ by (2).

Now let $F \subset \{1, \ldots, n+1\}$ be the set of all i with $H_i \cap C \neq \emptyset$. If $|F| < n$, then e.g. H_i and H_j $(i \neq j)$ miss C, and then H_i and H_j are comparable by (3). So assume $|F| \geqslant n$. By inductive hypothesis, there exist $k \neq l$ in F with

$$H_k \cap C \subset H_l \cap C. \tag{5}$$

Note that if $C \subset H_l$, then $l \neq n+1$ (since $y \in C \subset H_l$, $y \notin H_{n+1}$ by (4)), and the sets H_l, H_{n+1} are comparable. So assume $C \not\subset H_l$. If $y \notin H_l$, then take a point $c \in C \cap H_l$. We find that $x \notin H_l$, for otherwise, $y \in h\{c, x\} \subset H_l$ by (1). Hence $h\{x, y\} \subset X \setminus H_l$, and $H_l \subset C$ by (3). Note that $y \in C$, whence $y \notin H_k$. A similar argument then shows that also $H_k \subset C$. The sets H_k, H_l are then comparable by (5). If on the other hand $y \in H_l$, and if $x \notin H_l$, then take a point $c \in C \setminus H_l$. We find by (1) that

$$y \in h\{c, x\} \subset X \setminus H_l,$$

contradictory to our assumption. Hence $x \in H_l$, and $h\{y, x\} \subset H_l$. Now note that also $C \not\subset H_k$. Hence the above argument applies equally well on H_k: if $y \notin H_k$, then $H_k \subset C$, and the sets H_k, H_l are then comparable. If $y \in H_k$, then $h\{y, x\} \subset H_k$. Filling in the remaining possibility for H_k and H_l in (5), we obtain that

$$H_k = (H_k \cap C) \cup h\{y, x\} \subset (H_l \cap C) \cup h\{y, x\} = H_l,$$

completing the inductive proof of $(I-n)$. ∎

It is implicit in the proof of (a) = (b) that a tree-like convexity never has weakly infinite rank. Hence, for semi-regular closure-stable convexities with connected convex sets and with compact polytopes, the phenomenon of having weakly infinite rank occurs at earliest in dimension 2. We have not yet found such examples in finite dimensions [1].

[1] In a forthcoming paper of the author it will be shown that such examples do not exist for *binary* convexities. Our proof involves Theorem 4.7 below and some properties of other invariants.

The fact that for tree-like spaces (i.e. for 1-dimensional convexities) rank and generating degree coincide leads us to the question under which circumstances these invariants have to be equal. Some restriction seems necessary: see the example after Theorem 2.2. No counter-example is known in case the convexity is required to have connected convex sets and to have some separation property (the underlying set should be nontrivial).

We next discuss a class of convexities which share an important property with linear convexities on topological vector spaces. Let X be a topological convex structure. Then X has *Fuchssteiner's property* if for each convex open set $O \subset X$ and for each finite set $F \subset \bar{O}$,

$$h(F) \cap \dot{O} \subset h(F \setminus O). \tag{*}$$

If the convexity on X is regular and closure-stable, then it suffices to have (*) for *open half-spaces* of X, as the reader can easily verify. Note that if X is closure-stable, and if O is an open half-space, then $\bar{O} \setminus O$ is convex and (*) becomes

$$h(F) \cap \dot{O} = h(F \setminus O).$$

This gives us a fairly adequate picture of Fuchssteiner's property, namely that polytopes "sharply bend away" at their "corner" points.

It is clear that linear spaces satisfy Fuchssteiner's property with respect to linear convexity. We leave it to the reader to check that the natural convexity on a locally connected tree-like space also has Fuchssteiner's property. In general, the property is not inherited by convex subspaces.

The above condition (*) was first considered in [F] for the purpose of obtaining a Krein-Milman theorem. The continuity property (see 1.4) was originally designed for the same purpose. For convexities with connected convex sets, Fuchssteiner's condition implies the continuity property, as was observed in $[V_1, 4.3]$.

3.2. Theorem. *Let X a semi-regular closure-stable convexity structure with connected convex sets, and having Fuchssteiner's property. Let $C \subset X$ be convex. Then either C is a tree-like space, or C has strongly infinite rank.*

Note that even in the latter case, C may be a tree-like space. Also, the rank of C is never weakly infinite. Before starting with a proof, we present two auxilliary results.

3.3. Lemma. *Let X be a closure-stable convexity with the continuity property, and let $O_1, \ldots, O_n$ be convex open sets with $\bigcap_{i=1}^{n} O_i \neq \emptyset$. Then*

$$\mathrm{Cl}\Big(\bigcap_{i=1}^{n} O_i\Big) = \bigcap_{i=1}^{n} \bar{O}_i.$$

For $n = 1$, this is trivial. Next, let $n = 2$. As $O_1 \cap O_2 \neq \emptyset$, it follows from purely topological considerations on the openness of O_1 and O_2 that

$$\mathrm{Cl}(O_1 \cap O_2) = \mathrm{Cl}(O_1 \cap \bar{O}_2).$$

By the continuity property,

$$\mathrm{Cl}_X(O_1 \cap \bar{O}_2) = \mathrm{Cl}_{\bar{O}_2}(O_1 \cap \bar{O}_2) = \bar{O}_1 \cap \bar{O}_2.$$

Let $n \geqslant 2$ now be arbitrary. Then by the previous argument,

$$\mathrm{Cl}\Big(\bigcap_{i=1}^{n} O_i\Big) = \mathrm{Cl}\Big(\bigcap_{i=1}^{n-1} O_i\Big) \cap \bar{O}_i,$$

and the lemma follows from a straightforward inductive procedure. ∎

3.4. Lemma. *Let X be a semi-regular closure-stable convexity with connected convex sets, having the continuity property. Let $C \subset X$ be convex, and let V be a convex C-neighbourhood of $x \in C$. If x is a cutpoint of V, then x is also a cutpoint of C.*

Indeed, let $\mathscr{K}$ be the collection of all components of $V \setminus \{x\}$. For each $u \in C \setminus \{x\}$ we fix an open half-space $O(u)$ of X with

$$u \in O(u), \qquad x \in \dot{O}(u)$$

(see the "standard" argument at the beginning of 3.1). For each $K \in \mathscr{K}$ we put

$$P(K) = \{u \mid O(u) \cap K \neq \emptyset\}.$$

Note that by the continuity property

$$\overline{O(u)} \cap C = \mathrm{Cl}_C(O(u) \cap C).$$

Hence, as V is a C-neighbourhood of x, we find that $O(u)$ meets $V \setminus \{x\}$, and hence it meets some $K \in \mathscr{K}$, showing that

$$\bigcup_{K \in \mathscr{K}} P(K) = C \setminus \{x\}.$$

If $u \in P(K_1) \cap P(K_2)$, then $O(u)$ meets both K_1 and K_2. As $O(u) \cap V$ is convex and hence connected, we find that $K_1 = K_2$.

We finally show that $P(K)$ is open in C for each $K \in \mathscr{K}$. Let $u \in P(K)$, and let $v \in O(u) \cap C$. Then as $O(u) \cap O(v)$ meets C, we find by the continuity property and by Lemma 3.3 that

$$\text{(1)} \quad \mathrm{Cl}_C(O(u) \cup O(v) \cap C) = \mathrm{Cl}_X(O(u) \cap O(v)) \cap C = \overline{O(u)} \cap \overline{O(v)} \cap C.$$

Hence x is in the left hand set of the expression (1), and consequently

$$V \cap O(u) \cap O(v) \neq \emptyset.$$

The latter set is connected (being convex). Hence there is a $K' \in \mathscr{K}$ with

$$O(u) \cap O(v) \cap V \subset K'.$$

It follows that $K = K'$ and hence that $v \in P(K)$. This shows that $O(u) \cap C \subset P(K)$ for each $u \in P(K)$, and $P(K)$ is open in C. The point x being a cutpoint of V, we find that $\mathscr{K}$ has more than one member, and as $K \subset P(K)$ for each $K \in \mathscr{K}$, $\{P(K) \mid K \in \mathscr{K}\}$ is a proper decomposition of $C \setminus \{x\}$ into open sets. ∎

We now proceed with a proof of Theorem 3.2. First, note that if the rank of C is 0, then $C = \emptyset$, and if the rank of C is 1, then C is a singleton. So assume that the rank of C is at least two, and that it is either finite or weakly infinite. We will derive that C is a tree-like space.

Claim. *Let $D \subset C$ be a nontrivial convex subset. Then the space D has a cutpoint.*

Indeed, as D is nontrivial, there exists a proper, nonempty, and relatively open convex subset O of D. Note that the rank of the convexity subspace O is again finite or weakly infinite. In either case there exists a *maximal* finite free subset $F \subset O$, say:

$$F = \{y_1, \ldots, y_n\} \qquad (n > 1).$$

As $h(F) \subset O$ is closed in X, we find that $h(F)$ is a *proper* subset of O (connectedness of D). For each $i = 1, \ldots, n$ we fix an open half-space O_i of X with

$$F \setminus \{y_i\} \subset O_i, \qquad y_i \in \dot{O}_i.$$

Then $h(F) \subset \bar{O}_i$, and O_i meets $h(F)$ (since $n > 1$), whence by the continuity property of X, $O_i \cap h(F)$ is relatively dense in $h(F)$. Hence

$$\text{(2)} \qquad \bigcap_{i=1}^{n} O_i \cap h(F) \text{ is dense in } h(F).$$

Assume that

$$\bigcap_{i=1}^{n} O_i \cap O \setminus h(F) \neq \emptyset.$$

If y is in this set, then for each $i = 1, \ldots, n$

$$y_i \notin O_i \supset h(\{y\} \cup F \setminus \{y_i\}),$$

and as $y \notin h(F)$, we find that $F \cup \{y\} \subset O$ is a free set, larger than F. This contradicts with our maximality assumption on F, and we may conclude that

$$\text{(3)} \qquad \bigcap_{i=1}^{n} O_i \cap O \subset h(F).$$

It follows from (3), and from the fact that $O \setminus h(F) \neq \emptyset$, that $h(F) \setminus \bigcap_{i=1}^{n} O_i$ is a (nonempty) separator of the space O. Now $h(F) \subset \bar{O}_i$, and hence by Fuchssteiner's property

$$h(F) \setminus \bigcap_{i=1}^{n} O_i = \bigcup_{i=1}^{n} h(F) \cap \dot{O}_i = \bigcup_{i=1}^{n} h(F \setminus O_i) = F.$$

For each $i = 1, \ldots, n$ let $V_i \subset O$ be a convex D-neighbourhood of y_i with

$$V_i \cap F \setminus \{y_i\} = \emptyset.$$

Note that

$$h(F) \cup \bigcup_{i=1}^{n} V_i = \Big(\bigcap_{i=1}^{n} O_i \cap O\Big) \cup \bigcup_{i=1}^{n} V_i$$

is open in O, and hence it property includes $h(F)$. Then for some i, V_i meets $O \setminus h(F)$, whereas V_i meets $\bigcap_{i=1}^{n} O_i \cap O$ by (2) and (3). This shows that y_i is a cutpoint of V_i, and hence of D in view of Lemma 3.4.

Now assume that C is not a tree-like space. Then relative to the *weak topology*, C is not a tree-like space neither. We note that the assumptions on X remain valid with respect to the weak topology on (convex subspaces of) X, and in particular the above results apply equally well on this new topology, which has one additional feature, essential for the proof below: it is locally convex (and hence *locally connected*) by semi-regularity.

By assumption, there exists $u \neq v$ in C which cannot be separated by a third point, i.e. u and v are *conjugate* in the terminology of Whyburn, [Wh, p. 381]. Let $D \subset C$ be the corresponding *cyclic element*, that is:

$$D = \{y \mid y \text{ is conjugate to both } u \text{ and } v\}.$$

For each $x \in C \setminus \{u, v\}$ we let C_x denote the set consisting of x, together with the points of the component of $C \setminus \{x\}$ which contains u and v. Note that either $C_x = C$, or $\{x\}$ is a (convex) separator of C, in which case C_x is convex by [V_1, 5.4]. By definition,

$$D = \bigcap \{C_x \mid x \in C \setminus \{u, v\}\},$$

whence D is convex. It therefore admits a cutpoint. However, cyclic elements in a connected and locally connected T_2-space have no cutpoints by [Wh, Theorem 6.2]. ∎

4. Topological behaviour of rank

Let $(X_i, \mathscr{C}_i)$, $i \in I$ be a family of (set-theoretic) convexities. Its *product* is defined to be the convexity $(X, \mathscr{C})$, with $X = \prod_{i \in I} X_i$, and where $\mathscr{C}$ is generated by the subbase

$$\bigcup_{i \in I} \{\pi_i^{-1}(C_i) \mid C_i \in \mathscr{C}_i\},$$

where $\pi_i\colon X \to X_i$ denotes the ith projection. If there are but finitely many factors, then the members of $\mathscr{C}$ are of type

$$\prod_{i \in I} C_i \quad (C_i \in \mathscr{C}_i, i \in I),$$

(cf. [J_1, p. 20], [V_1, 1.8]). If each factor is topological, and if $X = \prod_{i \in I} X_i$ is equipped with the product topology, then $(X, \mathscr{C})$ is again a topological convex structure. If there are but finitely many factors, then $(X, \mathscr{C})$ is (semi-) regular provided each factor is.

4.1. THEOREM. *Let X_1, X_2 be semi-regular closure-stable convexities of which the underlying space is nontrivial and connected. If X is the product of X_1 and X_2, then*

$$d(X) = d(X_1) + d(X_2).$$

Informally, rank behaves *additively* under the formation of products.

Proof of 4.1. Let

$$F_1 = \{x_1^1, \ldots, x_m^1\}, \quad F_2 = \{x_1^2, \ldots, x_n^2\}$$

be free subsets of X_1 and X_2 respectively. A both X_1 and X_2 have more than one point, a standard maximality argument gives us open half-spaces O_i^1 of X_1 and O_j^2 of X_j, $i \leqslant m$, $j \leqslant n$, such that

$$x_i^1 \in \dot{O}_i^1; \quad F_1 \setminus \{x_i^1\} \subset O_i^1;$$
$$x_j^2 \in \dot{O}_j^2; \quad F_2 \setminus \{x_j^2\} \subset O_j^2.$$

Note that $\bigcap_{k \neq i} O_k^1$ is a neighbourhood of x_i^1, and hence that $\bigcap_{i=1}^{m} O_i^1 \neq \emptyset$. Similarly, $\bigcap_{j=1}^{n} O_j^2 \neq \emptyset$ (remark: the intersection of the "empty family" equals X by convention).

Then choose

$$y_k \in \dot{O}_k^1 \times X_2 \cap \Big(\bigcap_{i \neq k} O_i^1\Big) \times X_2 \cap X_1 \times \Big(\bigcap_{j=1}^{n} O_j^2\Big) \quad (k = 1, \ldots, m),$$

$$y_{n+l} \in X_1 \times \dot{O}_l^2 \cap \Big(\bigcap_{i=1}^{m} O_i^1\Big) \times X_2 \cap X_1 \times \Big(\bigcap_{j \neq l} O_j^2\Big) \quad (l = 1, \ldots, n).$$

We find that for $p \leqslant m$,

$$\{y_q \mid q \neq p\} \subset O_p^1 \times X_2$$

and for $m < p \leqslant m+n$ (say: $p = m+l$) that

$$\{y_q \mid q \neq p\} \subset X_1 \times O_l^2.$$

Hence the collection $\{y_p \mid p = 1, \ldots, m+n\}$ is free in X and $d(X) \geqslant d(X_1) + d(X_2)$.

The opposite inequality is a direct consequence of the fact that the (open) half-space of X are of type $O_1 \times X_2$ or $X_1 \times O_2$, where O_i is an (open) half-space of X_i, $i = 1, 2$. ∎

It is clear that 4.1 is not valid if one of the factors is empty or has only one point. Although the above argument can slightly be improved such that it also works for certain non-connected spaces, some restriction close to connectedness seems necessary: see the example in 2.2 (equipped with the discrete topology).

4.2. THEOREM. *Let X be a semi-regular closure-stable convexity with connected convex sets. Then* $2.\operatorname{ind}(X) \leqslant d(X)$.

Proof. The theorem is trivial for $\operatorname{ind} X = -1$. If $\operatorname{ind} X \leqslant n$, where $0 \leqslant n < \infty$, then by [V_2, 4.4] there exists a C.P. map

$$f: X \to [0, 1]^n$$

(the latter being equipped with the subcube convexity, c.f. (2.4; 1)) which is *onto*. Hence by (2.4; 1) and Theorem 2.5,

$$d(X) \geqslant d([0, 1]^n) = 2n. \blacksquare$$

We note that convex dimension behaves additively under the formation of products (cf. [V_2, 2.6]). Hence Theorems 4.1 and 4.2 are in good agreement. It is also clear form the examples in (2.4; 2) and (2.4; 4) that no other relationship between rank and dimension can be expected.

As a particular consequence of 4.2, a semi-regular closure-stable convexity with connected convex sets and of finite rank must be finite dimensional. One other application of rank is concerned with the following. It was shown in [V_2, 3.7] that if X is as in 4.2, and if X is finite-dimensional, then each dense half-space of X has nonempty interior. Also, an example was given, showing that this statement is false if "half-space" is replaced by "convex set". The convex structure in this example was a tree-like space with a dense (and hence infinite) collection of endpoints. By Theorem 3.1, this space has strongly infinite rank.

4.3. THEOREM. *Let X be a semi-regular closure-stable convexity with connected convex sets. If X has finite or weakly infinite rank, than the interior of a dense convex set meets every nonempty convex open set of X.*

Proof. Let $C \subset X$ be dense and convex, and let $O \subset X$ be a nonempty convex open set. Then $C \cap O$ is dense in O, and

$$(\operatorname{int} C) \cap O = \operatorname{int}_O(C \cap O).$$

As the rank of $C \cap O$ is also finite or weakly infinite, there exists a maximal free subset F of $C \cap O$ which is finite, say:

$$F = \{x_1, \ldots, x_n\}.$$

Let $O_i \subset X$ be a convex open set with

$$x_i \in \dot{O}_i, \quad h(F \setminus \{x_i\}) \subset O_i, \quad i = 1, \ldots, n.$$

Then $\bigcap_{i=2}^{n} O_i \cap O$ is a neighbourhood of $x_1 \in \bar{O}_1$, whence

$$\bigcap_{i=1}^{n} O_i \cap O \neq \emptyset.$$

Hence if $\operatorname{int}_O(C \cap O) = \emptyset$, then

$$\bigcap_{i=1}^{n} O_i \cap O \not\subset h(F) \subset C \cap O,$$

and by density of $C \cap O$ in O, there is a point

$$x_{n+1} \in \bigcap_{i=1}^{n} O_i \cap (O \setminus h(F)) \cap C.$$

Then $F \cup \{x_{n+1}\}$ is a free subcollection of $C \cap O$, a contradiction. $\blacksquare$

The following is a somewhat surprising result:

4.4. THEOREM. *Let X be a semi-regular closure-stable convexity with connected convex sets and with compact polytopes. If* $\operatorname{gen}(X) < \infty$, *then X has the weak topology, and a set $A \subset X$ is closed iff $A \cap P$ is closed in P for each polytope P of X. In particular, the topology of X is compactly generated.*

We note that the theorem applies for 1-dimensional X of finite *rank*: then X is tree-like and its rank equals its generating degree by Theorem 3.1. For convexity structures X as in the hypothesis of 4.4, there is no counterexample to the statement $d(X) = \operatorname{gen}(X)$. It were therefore a good testcase to try to "extend" the above theorem to convexities of finite rank.

Proof of 4.4. First note that

$$2.\operatorname{ind}(X) \leqslant d(X) \leqslant \operatorname{gen}(X) \tag{1}$$

by 4.2 and 2.2. Hence X is finite dimensional, say: $\operatorname{ind} X = n$. The theorem is obvious in dimensions -1, 0, and we proceed by induction, assuming the theorem to hold in dimensions $< n$, where $n > 0$. Let $A \subset X$ be such that $P \cap A$ is closed for each polytope P of X, and let $x \in X \setminus A$. We distinghuish between the following two cases.

Case 1. $x \notin \bigcap \{H \mid H \subset X \text{ a dense half-space}\}$ (we note that the latter collection is nonempty, and even dense in X by [V_1, 6.12]).

Let $H \subset X$ be a dense half-space with $x \notin H$, and let $F \subset H$ be a maximal subset with the property that $F \cup \{x\}$ is free in X. Note that F must be finite by (1), say:

$$F = \{x_1, \ldots, x_p\}.$$

For each $i = 1, \ldots, p$ there exists an open half-space O_i of X with

$$h(F \cup \{x\} \setminus \{x_i\}) \subset O_i, \quad x_i \notin O_i.$$

Then

$$\bigcap_{i=1}^{p} O_i \cap H \subset h(F \cup \{x\}),$$

for otherwise we can pick a point

$$y \in \bigcap_{i=1}^{p} O_i \cap H \setminus h(F \cup \{x\}),$$

and we obtain another free set $F \cup \{x, y\}$, with $F \cup \{y\} \subset H$, contradicting with the maximality of F. Hence

$$X \setminus h(F \cup \{x\}) \subset (X \setminus O_1) \cup \ldots \cup (X \setminus O_p) \cup \bar{H} \setminus H. \tag{2}$$

By [V_2, 3.4], ind $\bar{H} \setminus H < n$. By inductive assumption, $A \cap (\bar{H} \setminus H)$ is closed in $\bar{H} \setminus H$, and there exist convex relatively closed sets $C_1, \ldots, C_q \subset \bar{H} \setminus H$ such that

$$A \cap \bar{H} \setminus H \subset C_1 \cup \ldots \cup C_q; \quad x \notin C_1 \cup \ldots \cup C_q. \tag{3}$$

Note that for each $i = 1, \ldots, q$, $x \notin \bar{C}_i$ since $C_i \subset \bar{H} \setminus H$ is relatively closed, $x \in \bar{H} \setminus H$. On the other hand, $A \cap h(F \cup \{x\})$ is compact (compactness of polytopes). Hence for each $a \in A \cap h(F \cup \{x\})$ we can fix a closed convex set D_a with

$$a \in \operatorname{int} D_a, \quad x \notin D_a,$$

and a finite number of these sets, say: $D_{a_1}, \ldots, D_{a_r}$, suffices to cover $A \cap h(F \cup \{x\})$. Combining this with (2) and (3), we find

$$A \subset (X \setminus O_1) \cup \ldots \cup (X \setminus O_p) \cup \bar{C}_1 \cup \ldots \cup \bar{C}_q \cup D_{a_1} \cup \ldots \cup D_{a_r}$$

and x is not in the right hand set.

Case 2. $x \in \bigcap \{H \mid H \subset X \text{ a dense half-space}\}$.

Let $\mathcal{M}$ be the collection of all maximal half-spaces $H \subset X$ with the property that $x \notin H$. Note that (with the notation of 2.3),

$$\mathcal{M} \subset \mathcal{H}' \subset \mathcal{H},$$

and that the members of $\mathcal{M}$ are pairwise incomparable. By Theorem 2.3,

$$\text{degree of } \mathcal{H} = \operatorname{gen}(X) < \infty,$$

and hence the collection $\mathcal{M}$ is finite.

For each $M \in \mathcal{M}$ we find by assumption on x that there exists a point $x_M \in X \setminus \bar{M}$. Let

$$F' \subset \{x_M \mid M \in \mathcal{M}\}$$

be minimal with the property that

$$h(F') = h\{x_M \mid M \in \mathcal{M}\}.$$

Then F' is free in X, and it extends to a maximal free set $F \subset X$, which of necessity is finite again. We put

$$P = h(F), \quad F = \{x_1, \ldots, x_p\}.$$

F being free, we obtain open half-spaces O_i of X with

$$F \setminus \{x_i\} \subset O_i, \quad x_i \in \dot{O}_i \quad (i = 1, \ldots, p).$$

Again,

$$\bigcap_{i=1}^{p} O_i \subset h(F) \tag{4}$$

(otherwise F would extend to a larger free set). We claim that for each i, $x \in O_i$. For, suppose $x \notin O_i$. Then O_i extends to a maximal half-space M with $x \notin M$. Consequently, $M \in \mathcal{M}$, and

$$x_M \in P \setminus \bar{M}, \quad P = h(F) \subset \bar{O}_i \subset \bar{M},$$

a contradiction.

It follows from (4) that

$$X \setminus h(F) \subset X \setminus O_1 \cup \ldots \cup X \setminus O_p; \quad x \notin X \setminus O_1 \cup \ldots \cup X \setminus O_p. \tag{5}$$

On the other hand, $A \cap h(F)$ is compact and exactly as in case 1 we obtain convex closed sets $D_1, \ldots, D_r$ of X with

$$A \cap h(F) \subset D_1 \cup \ldots \cup D_r, \quad x \notin D_1 \cup \ldots \cup D_r. \tag{6}$$

Then (5) and (6) yield the desired result. ∎

Theorems 4.3 and 4.4 combine to obtain the following "Baire-type" theorem.

4.5. Corollary. *Let X be a semi-regular closure-stable convexity with connected convex sets and with compact polytopes, such that* $\operatorname{gen}(X) < \infty$. *Let $(C_i)_{i \in I}$ be a collection of convex sets, the closures of which cover X.*

(1) *If I is finite, then $\operatorname{int} C_i \neq \emptyset$ for some $i \in I$;*

(2) *If X is a Baire (topological) space, and if I is countable, then $\operatorname{int} C_i \neq \emptyset$ for some $i \in I$.*

Proof. We have that $\bigcup_{i \in I} \bar{C}_i = X$, and in either case we obtain an $i \in I$ with $\operatorname{int} \bar{C}_i \neq \emptyset$. As X is semi-regular and has the weak topology by 4.5, we obtain a convex open set O of X with

$$\emptyset \neq O \subset \operatorname{int} \bar{C}_i.$$

Then $C_i \cap O$ is a dense convex subset of O, and by Theorem 4.3,

$$\emptyset \neq \operatorname{int}_O(C_i \cap O) = \operatorname{int}(C_i \cap O) \subset \operatorname{int} C_i. \;∎$$

We note that 4.4 is used only to obtain neighbourhood bases of convex open sets at each point of X. Having the weak topology is not a necessity for this: by [V_3, 2.3], "uniformizable" convexities also have this property. In such circumstances, Theorem 4.4 can be avoided, and 4.5 works already for convexities of finite or weakly infinite *rank*.

We now pay some attention to "spanning" properties of convex sets.

4.6. THEOREM. *Let X be a semi-regular topological convexity of finite rank. Then every compact convex subset of X is a polytope.*

Proof. Let $C \subset X$ be compact convex, let d be the rank of X, and let $\varkappa$ be the cardinal number of C. We may assume that $\varkappa$ is infinite. We fix a 1-1 correspondence $\varkappa \to C$; $\alpha \to x_\alpha$ for notational convenience. By transfinite induction we will construct for each ordinal $\alpha \leqslant \varkappa$ a set $F_\alpha \subset C$ such that

(1) F_α has at most d points, and $x_\alpha \in h(F_\alpha)$ if $\alpha < \varkappa$;

(2) if $\beta < \alpha$, then $h(F_\beta) \subset h(F_\alpha)$.

Put $F_0 = \{x_0\}$ (we note that $d > 1$ since C is infinite). Having obtained F_β for $\beta < \alpha < \varkappa$, then F_α is constructed as follows. If α has a direct predecessor β, and if $F_\beta \cup \{x_\alpha\}$ has no more than d points, then put

$$F_\alpha = F_\beta \cup \{x_\alpha\}.$$

If $F_\beta \cup \{x_\alpha\}$ has more than d points, then there is an $x \in F_\beta \cup \{x_\alpha\}$ with

$$x \in h(F_\beta \cup \{x_\alpha\} \setminus \{x\}),$$

and we put

$$F_\alpha = F_\beta \cup \{x_\alpha\} \setminus \{x\}.$$

Note that $h(F_\alpha) = h(F_\beta \cup \{x_\alpha\})$, and hence that

$$x_\alpha \in h(F_\alpha) = h(F_\beta \cup \{x_\alpha\}) \supset h(F_\beta).$$

If α is a limit ordinal, then we use compactness of C to obtain a cluster point F'_α of the net $(F_\beta)_{\beta<\alpha}$ in the hyperspace of C. Clearly F'_α has no more than d points. Also,

$$\bigcup_{\beta<\alpha} h(F_\beta) \subset h(F'_\alpha).$$

Indeed, if $x \notin h(F'_\alpha)$ then there exists a convex open set $O \supset h(F'_\alpha)$ with $x \notin O$. Hence there is a cofinal collection $\Lambda \subset (\leftarrow, \alpha)$ with $F_\beta \subset O$ for all $\beta \in \Lambda$. Consequently,

$$x \notin O \supset \bigcup_{\beta\in\Lambda} h(F_\beta) = \bigcup_{\beta<\alpha} h(F_\beta)$$

by (2). Proceeding as above, we then obtain a set $F_\alpha \subset F'_\alpha \cup \{x_\alpha\}$ with at most d points, such that $h(F_\alpha) = h(F'_\alpha \cup \{x_\alpha\})$. If $\alpha = \varkappa$ (which is a limit ordinal since $\varkappa$ is infinite), we apply the second procedure and put $F_\varkappa = F'_\varkappa$. Having completed the induction, we note that

$$C \subset \bigcup_{\alpha<\varkappa} h(F_\alpha) \subset h(F_\varkappa) \subset C$$

by (1) and (2), establishing the theorem. ∎

The above argument fails to work for convexities having weakly infinite rank. Rather surprisingly, the theorem still holds for a "usual" class of convexities:

4.7. THEOREM. *Let X be a semi-regular, closure-stable convexity with connected convex sets, and of finite dimension. If X has weakly infinite rank, then every compact convex subset is a polytope.*

The proof of 4.7 requires the following simple result:

LEMMA. *Let $(Y, \leqslant)$ be a totally ordered set, and for each $y \in Y$ let $K(y)$ be a subset of $[y, \to)$ with $y \in K(y)$. If for each sequence*

$$y_0 < y_1 < \ldots < y_n < y_{n+1} < \ldots$$

in Y there exist $n < m$ with $y_m \in K(y_n)$, then there is a cofinal $Y' \subset Y$ such that

$$\forall\, y \in Y':\ K(y) \cap Y' \text{ includes an endsegment of } Y'.$$

Indeed: put $L(y) = [y, \to) \setminus K(y)$. Then $y \notin L(y)$ and if $y \in Y' \subset Y$ then $K(y) \cap Y'$ includes an endsegment of Y' iff $L(y) \cap Y'$ is not cofinal in Y'. Assume there is no cofinal $Y' \subset Y$ as required above. For each cofinal $Y' \subset Y$ we then obtain a point $y' \in Y'$ with $L(y') \cap Y'$ cofinal in Y', and hence also in Y. Applying this argument inductively, we obtain sequences $(Y_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, such that $y_n \in Y_n$, $Y_0 = Y$, and such that $Y_{n+1} = Y_n \cap L(y_n)$ is cofinal in Y_n and hence in Y. We then find that

$$y_0 < y_1 < \ldots < y_n < y_{n+1} \ldots,$$

and for all $n < m$, $y_m \notin K(y_n)$, contradictory to the assumption.

Let $C \subset X$ be compact convex. We first note that $\operatorname{ind} C \leqslant \operatorname{ind} X$, and that the theorem is obvious if $\operatorname{ind} X = -1$ or 0 (in fact, also for $n = 1$: then C is a compact tree-like space, its convexity must be the natural one by [vMV_2, 1.7], and the rank of C cannot be weakly infinite then, as was observed after 3.1. Hence C is of finite rank and 4.6 applies).

Assume the theorem to hold if X is of dimension $< n$, where $n \geqslant 1$ (or $n \geqslant 2$ if the reader wishes). Let $\operatorname{ind} X$ now be equal to n, and assume that C is not a polytope. Then by [J_1, I.12] there exists an increasing transfinite sequence $(C_\alpha)_{\alpha\in\varkappa}$ of convex sets of X with

(1) $\forall\, \alpha \in \varkappa$: C_α is properly included in C;

(2) $$\bigcup_{\alpha\in\varkappa} C_\alpha = C$$

(in the case of vector spaces this result on non-polytopes was discovered as late as 1972, cf. [K, Theorem 1], and it was extended soon afterwards by Jamison to abstract convexities).

Case I. *No C_α is dense in C.* Restricting to a cofinal subsequence if necessary, we may assume that for each $\alpha \in \varkappa$ there exists a point

$$x_\alpha \in C_\alpha \setminus \mathrm{Cl}\big(\bigcup_{\beta<\alpha} C_\beta\big).$$

By semi-regularity there exists an open half-space O_α of X with

$$\mathrm{Cl}\big(\bigcup_{\beta<\alpha} C_\alpha\big) \subset O_\alpha, \quad x_\alpha \notin O_\alpha. \tag{3}$$

Then put

$$K(\alpha) = \{\beta \geqslant \alpha \mid x_\beta \notin O_\alpha\}.$$

If $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \alpha_{n+1} < \ldots$ are in $\varkappa$, then $\alpha_m \in K(\alpha_n)$ for some $n < m$: if not, then $x_{\alpha_m} \in O_{\alpha_n}$ for all $n < m$. For $m < n$, we find by (3) that

$$x_{\alpha_m} \in C_{\alpha_m} \subset O_{\alpha_n},$$

and finally $x_{\alpha_n} \notin O_{\alpha_n}$. This shows that the infinite collection $\{x_{\alpha_n} \mid n = 0, 1, 2, \ldots\}$ is free, a contradiction.

We are now in a position to apply the lemma: there is a cofinal subset $\Lambda \subset \varkappa$ such that for each $\alpha \in \Lambda$ there is a $\beta \geqslant \alpha$ in Λ with the following property:

$$\text{if } \gamma \geqslant \beta \text{ is in } \Lambda, \text{ then } x_\gamma \notin O_\alpha.$$

This directly leads us to another cofinal set $\Omega \subset \Lambda$ such that for each $\alpha, \beta \in \Omega$,

$$\alpha < \beta \Rightarrow x_\beta \notin O_\alpha. \tag{4}$$

As Ω is cofinal in $\varkappa$, we have by (2) and (3) that

$$\bigcup_{\alpha\in\Omega} O_\alpha \supset C.$$

For each $\alpha \in \Omega$ we put

$$D_\alpha = h^*\{x_\beta \mid \beta \in \Omega, \beta > \alpha\} \text{ (notation of 1.2)}.$$

Then $D_\alpha \subset C \setminus O_\alpha$ by (4), and hence

$$\bigcap_{\alpha\in\Omega} D_\alpha \subset \bigcap_{\alpha\in\Omega} C \setminus O_\alpha = \emptyset.$$

However, $(D_\alpha)_{\alpha\in\Omega}$ is a decreasing sequence of closed subsets of the compact C, a contradiction. So we are lead to consider the remaining

Case II. *Some C_α is dense in C.* We may as well assume then that *all* C_α are dense in C by the monotony of the sequence $(C_\alpha)_{\alpha\in\varkappa}$. Restricting to a cofinal subset if necessary, we may also assume that for each $\alpha \in \varkappa$ there exists a point

$$x_\alpha \in C_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta.$$

X being a semi-regular convexity space, it follows from [J_1, p.26] that X is also an S_3-convexity. Hence for each $\alpha \in \varkappa$ there exists a half-space H_α of X with

$$x_\alpha \notin H_\alpha, \quad \bigcup_{\beta<\alpha} C_\beta \subset H_\alpha.$$

By the same argument as in case I, we obtain a cofinal subset $\Omega \subset \varkappa$ such that for each $\alpha < \beta$ in Ω, $x_\beta \notin H_\alpha$. Let μ be the first member of Ω. For each $\alpha > \mu$ in Ω we have

$$x_\alpha \in C_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta; \quad x_\alpha \in C \setminus H_\mu \subset \mathrm{Cl}(C \setminus H_\mu),$$

and hence the transfinite sequence of sets $(C_\alpha \cap \mathrm{Cl}(C \setminus H_\mu))_{\alpha\in\Omega, \alpha\geqslant\mu}$ is properly increasing. Also, its union equals $\mathrm{Cl}(C \setminus H_\mu)$, and by [$J_1$, I.12] again, the latter cannot be a polytope. However, $H_\mu \cap C$ is a dense half-space in C, whence by [V_2, Theorems 2.7 and 2.9],

$$\mathrm{ind}\,\mathrm{Cl}(C \setminus H_\mu) = \mathrm{ind}(C \setminus H_\mu) < \mathrm{ind}\, C \leqslant n,$$

contradictory to our inductive assumption. ∎

We do not know whether the restriction to finite-dimensional convexities is essential for the conclusion of 4.7. The situation is even more complicated by the fact that we have not found examples of (sufficiently nice) convexities of finite dimension and of weakly infinite rank.

The above results 4.6, 4.7 deal with compact convex sets. In the case of separable metrizable spaces some conclusions can be drawn concerning arbitrary convex sets as well:

4.8. Theorem. *Let X be a separable metrizable space equipped with a semi-regular closure-stable convexity all convex sets of which are connected. If X has finite rank, or, if X is finite-dimensional and of weakly infinite rank, then every convex set in X is the convex hull of a countable subset.*

We note that some countability condition on X is indispensible: let X be a totally ordered continuum with its natural convexity. If X is not first countable at an endpoint x, then $X \setminus \{x\}$ is a convex set which cannot be the hull of a countable set, in spite of the fact that this convexity space is 1-dimensional and of rank 2.

Polytopes of a convexity are often required to be compact (see, for instance, Theorem 4.4, and certain results in [V_2], [V_3], [V_4]). In these circumstances, it follows from 4.8 that every convex set in X is σ-compact. In many concrete situations this gives a simple way to see that a convexity does not have finite or weakly infinite rank.

Again, we do not know whether or not finite-dimensionality is an essential condition for the above thereom.

Proof of Theorem 4.8. With wither assumption on X we have ind $X < \infty$. We proceed by induction on $n = \operatorname{ind} X$, leaving the cases $n = -1, 0$ as trivialities: assume the theorem to hold in dimensions $< n$, and let $C \subset X$ be a convex set which is not the hull of a countable subset. By an inductive procedure we construct a sequence $(A_i)_{i=0}^{\gamma}$ of countable subsets of C such that

(1) $\quad h(A_i)$ is a nonempty half-space $\quad$ for $\quad i \geqslant 1$;

(2) $\quad h(A_i) \cap h(A_0 \cup \ldots \cup A_{i-1}) = \emptyset \quad$ for $\quad i \geqslant 1$.

Let $A_0 \subset C$ be a countable dense subset. Having constructed the sequence up to $i \geqslant 0$, we find that there exists a point

$$x \in C \setminus h(A_0 \cup \ldots \cup A_i)$$

by assumption on C. As X is S_3, there exists a half-space H of C such that

$$x \in H, \qquad H \cap h(A_0 \cup \ldots \cup A_i) = \emptyset.$$

Note that $\operatorname{int}_C H = \emptyset$. Hence by [$V_2$, 2.7], ind $H <$ ind X and by inductive assumption there exists a countable set $A_{i+1} \subset H$ with $h(A_{i+1}) = H$.

Having completed the induction, we fix a point $x_i \in h(A_i)$ for each $i \geqslant 1$. Then $\{x_i \mid i \geqslant 1\}$ is a free subset of C since for each $i \neq j$ we have by (2)

$$x_i \in h(A_i), \qquad x_j \notin h(A_i)$$

whence by the convexity of $C \setminus h(A_i)$,

$$x_i \notin C \setminus h(A_i) \supset h\{x_j \mid j \neq i\}.$$

The desired result follows from this contradiction. ∎

5. Embedding in cubes or products of tree-like spaces

Let X, Y be topological convex structures. An *embedding* of X in Y is a C.P. map $f\colon X \to Y$ with is an isomorphism between X and $f(X)$ (where the latter is equipped with the trace convexity).

5.1. Example. A two-dimensional, normal, closure-stable convexity with connected convex sets which cannot be embedded in a finite product of tree-like spaces.

Let X be the square, equipped with the (trace of the) linear convexity. The following geometrically obvious fact is used without proof: *if $C \subset X$ is a convex closed set then $X \setminus C$ has at most 4 components. If C_1, C_2 are disjoint convex closed sets each dividing X into at least three components, then $X \setminus C_1$ and $X \setminus C_2$ have exactly three components, and no convex closed $D \subset X \setminus C_1 \cup C_2$ divides the square into more than two components.*

This situation leads to the following consequence. Let $f\colon X \to T$ be a C.P. map onto the tree-like space T. Then T must be of the following type (Fig. 2): Indeed, if $x \in T$ is a ramification point, then $f^{-1}(x)$ is a convex closed set of X

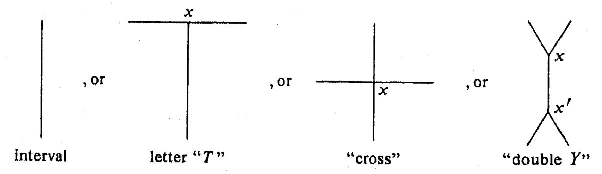

Fig. 2

such that $X \setminus f^{-1}(x)$ has at least 3 components. Hence, $\operatorname{ord}(x) \leqslant 4$ and if there is a second ramification point x', then

$$\operatorname{ord}(x) = \operatorname{ord}(x') = 3,$$

and there cannot exist a third ramification point. We note that C.P. maps of X onto each of the above trees can easily be constructed. The above argument is therefore sharp.

If $f\colon X \to \prod_{i=1}^{n} T_i$ is a C.P. map into a finite product of tree-like spaces, and if π_i denotes the ith projection, then $\pi_i \circ f$ may be assumed to be onto. As π_i is also C.P., we find from the above argument and from Theorem 3.1 that $d(T_i) \leqslant 4$. By the product Theorem 4.1, we find that $d(\prod_{i=1}^{n} T_i) \leqslant 4.n < \infty$. Hence by Theorem 2.6, f cannot be an embedding. ∎

Even in case the rank is finite, it is not possible in general to embed a (sufficiently nice) n-dimensional convexity into a product of $m = m(n) < \infty$ many tree-like spaces (the number of factors cannot depend exclusively on the dimension n):

5.2. Example. There exists a sequence $(X_p)_{p=2}^{\gamma}$ of 2-dimensional, normal, closure-stable convexities with connected convex sets and of finite rank, such that X_p is not embeddable in a product of less than $p/2$ many tree-like spaces.

Proof. Let X_p be the unit square again. In $\boldsymbol{R}^2$ we fix p distinct directions ($p \geqslant 2$), and we let the convexity of X_p be generated by the sets of type $H \cap X_p$, where H is a closed half-space of the vector space $\boldsymbol{R}^2$ with bounding hyperplane parallel to one of the chosen directions.

As $p \geqslant 2$, it follows that singletons are convex. As the linear convexity on $\boldsymbol{R}^2$ has Helly number 3, the same holds for X_p (except for $p = 2$, where the Helly number will be 2). The collection $\mathscr{D}_p$ of all nonempty intersections of subbasic sets is obviously compact in the hyperspace of X_p (such sets are exactly the ones determined by certain linear inequalities). Hence by [J_1, III.2], $\mathscr{D}_p$ is exactly the collection of *all* convex closed sets in X_p. If $D, D' \in \mathscr{D}_p$ are disjoint, and as the Helly number of X_p is at most 3, we find (at most) three subbasic sets H_1, H_2, H_3 each including either D or D', and such that

$$H_1 \cap H_2 \cap H_3 = \emptyset,$$

say: $D \subset H_1$, $D' \subset H_2 \cap H_3$. The linear map, determined by H_1, then separates D from D' and it is obviously C.P.

This shows that X_p is a normal convex structure. As its collection of closed convex sets is compact, it follows from [J_1, III.2] that X_p is also closure-stable. Finally,

$$2p \leqslant d(X_p) \leqslant \operatorname{gen}(X_p) \leqslant 2p$$

as one can easily check (as for the first inequality, construct a $2p$-gon in X_p with edges parallel to the selected directions, and pick a point on the inside of each boundary segment. The resulting collection is free). If

$$f\colon X_p \to \prod_{i=1}^{k} T_i$$

is a C.P. map into a product of k tree-like spaces and (notation of 5.1) with $\pi_i \circ f$ onto, we find again that the rank of T_i is at most 4 (the argument of 5.1 is now applied on a more restrictive collection of linearly convex sets). Hence

$$d\Big(\prod_{i=1}^{k} T_i\Big) \leqslant 4k,$$

and if f has to be an embedding, we must have $2p \leqslant 4k$. ∎

We note that an embedding of X_p in a p-cube can easily been constructed.

Our interest in finite-dimensional convexities of weakly infinite rank is partially motivated by the following result.

5.3. Theorem. *Let X a convex structure of weakly infinite rank, and with a connected underlying space. Then X is not embeddable in a product of finitely many trees.*

Proof. Let $f\colon X \to \prod_{i=1}^{k} T_i$ be a C.P. map into a product of $k < \infty$ tree-like spaces. X being connected, we find that $\pi_i \circ f(X)$ is also a tree, and hence we may assume $\pi_i \circ f$ to be onto for each i. By Theorem 2.5, each T_i can have at most weakly infinite rank, whence by a remark following 3.1, $d(T_i) < \infty$. Hence by 4.1,

$$d\Big(\prod_{i=1}^{k} T_i\Big) < \infty,$$

and by Theorem 2.6, f cannot be an embedding. ∎

From the above results one might draw the rather superficial conclusion that tree-like spaces and their finite products are too "simple" to include complicated substructures. This is not true. In fact, measuring "complication" in terms of rank, there exist tree-like spaces of the largest possible "degree of complication", namely strongly infinite rank. The leitmotive in the above arguments is rather that certain special circumstances force the tree-like factors to stay far below their maximal complexity.

As a general conclusion it appears that finite-dimensional convexities need not be embeddable in products of finitely many 1-dimensional convex structures and if it is possible to do so, then the number of factors may depend not only on the dimension, but also on the rank.

In a sense, the situation is less complicated if one restricts attention to cubes (with cubical convexity). In view of Theorem 2.6, a convexity of infinite rank cannot be embedded in a finite-dimensional cube. From various concrete examples we got the impression that the following problem might have a positive solution. Let X be a semi-regular closure-stable convexity with connected convex sets. If the rank d of X is finite, it is then possible to embed X in an n-cube, where n is in between $d/2$ and d? Of course, X should be at least separable and metrizable for this.

We note that if X is a convex structure of finite rank and with the usual properties, and if $f\colon X \to \prod_{i=1}^{k} T_i$ is an embedding in a finite product of trees (in particular, in a cube), then each factor T_i may be assumed to have finite rank. Consequently, $\prod_{i=1}^{k} T_i$ has the weak topology by Theorem 4.4, and then every subspace also has the weak topology. Hence, for an affirmative answer to the above embedding problems, one must obtain an extension of 4.4 to convexities of finite rank.

Let us end with the mentioning of a weaker "embedding" problem. Let X be a (sufficiently nice) convex structure of finite dimension. Does there exist a *topological embedding* f of X into a finite product of tree-like spaces *such that f is C.P.*? We note that a C.P. homeomorphism need not be a convexity isomorphism, unless the convexities in consideration are *binary*: see for instance [vMV_2, 1.5]. In this case the weaker embedding problem reduces to the original one, and for binary convexities we know of no counterexample.

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Hereditarily indecomposable continua with trivial shape

by

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The result presented in this paper, in a slightly weaker form, was discovered by the second author. The proof was complicated. Later the first author found simpler proof and it was decided to write a joint paper with the simpler proof.

Morton Brown [1] has proved that the limit X of an inverse sequence of n-spheres S^n, $n \geqslant 2$, is not hereditarily indecomposable provided each bonding map is essential, i.e. not homotopic to a constant. In this situation we have: (1) X is the limit of an inverse sequence of locally connected unicoherent continua, and (2) $\check{H}^n(X) \neq 0$. We shall prove more: any h.i. continuum satisfying (1) must be acyclic (even tree-like).

A space X is said to be *contractible relatively another space Y* provided any mapping from X into Y is nullhomotopic. If a mapping f is nullhomotopic, we write $f \simeq 0$.

THEOREM. *If an hereditarily indecomposable continuum X is the limit of an inverse sequence of locally connected and unicoherent continua, then X is tree-like.*

Proof. Let $X = \varprojlim \{X_n, g_{nm}\}$, where X_n's are locally connected and unicoherent continua, and let $f\colon X \to Y$ be a mapping into a 1-dimensional polyhedron. We shall show that f is nullhomotopic. Since $Y \in \mathrm{ANR}$, there are an index $n \geqslant 1$ and a mapping $f_n\colon X_n \to Y$ such that $f \simeq f_n \circ g_n$, where g_n is the projection from X into X_n. Hence it suffices to show that $f_n \simeq 0$. By the Whyburn factorization theorem there exists a continuum Z, a monotone surjection $k\colon X_n \to Z$ and a 0-dimensional map $l\colon Z \to Y$ such that $f_n = l \circ k$. It follows that Z is a locally connected and unicoherent continuum. Since l is 0-dimensional and $\dim Y = 1$, by the Hurewicz theorem [4, p. 114, Th. 1] we infer that Z is a curve. It follows that Z is a dendrite [4, p. 442, Cor. 8], hence an absolute retract. This proves that $f_n \simeq 0$ because k (and also l) is nullhomotopic. Thus we have proved that X is contractible relatively any graph. By [3, Cor. 4]