already a model A^* where $A = \bigoplus_{p\in P}\bigoplus_{n\leqslant N} Z(p^n)^{\alpha_{p,n}} \oplus \bigoplus_{p\in P} Z(p^\omega)^{\gamma_p}$ is determined by the finite tupel of numbers $\alpha_{p,n}$ and γ_p, it is possible to enumerate recursively all L^t-sentences which are consitent with T^t_{pf}. This implies decidability.

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On the span of weakly-chainable continua

by

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Abstract. A continuum is weakly chainable provided it is the continuous image of the pseudo-arc. It is an open problem to classify weakly chainable atriodic tree-like continua. In particular, the following problem, due to Mohler, is open: Suppose X is a weakly-chainable atriodic tree-like continuum, is X chainable? In this paper we will give a necessary condition for weak-chainability of certain continua by proving the following theorem: Suppose X is a weakly-chainable (atriodic) tree-like continuum such that every proper subcontinuum is chainable, then the span of X is zero. This answers a question of Ingram. We will also investigate some related problems.

1. Introduction and preliminaries. By a *mapping* we mean a continuous function and by a continuum a compact, connected metric space. A *tree* is a finite, connected and simply connected graph. A continuum is *tree-like* (*arc-like*) if it is an inverse limit of trees (arcs, respectively). A continuum X is *atriodic*, provided for every pair Y_1, Y_2 ($Y_2 \subset Y_1$) of subcontinua of X, $Y_1 \setminus Y_2$ has at most two components.

Let (X, d) be a connected metric space. For $i = 1, 2$ let $\pi_i\colon X\times X \to X$ be the ith coordinate projection. We define the *surjective span* $\sigma^*(X)$ (respectively the *surjective semi-span* $\sigma_0^*(X)$) (see [6], [7]), of X to be the least upper bound of the set of real numbers $\alpha \geqslant 0$ with the following property: there exists a connected set $C_\alpha \subset X\times X$ such that $d(x, y) \geqslant \alpha$ for $(x, y)\in C_\alpha$ and $\pi_1 C_\alpha) = X = \pi_2(C_\alpha)$ [resp. $\pi_1(C_\alpha) = X$]. The *span* $\sigma(X)$ [resp. *semi-span* $\sigma_0(X)$] of X is defined by

$$\sigma(X) = \sup\{\sigma^*(A)|\ A \subset X, A \neq \emptyset \text{ connected}\}$$
$$(\text{resp. } \sigma_0(X) = \sup\{\sigma_0^*(A)|\ A \subset X,\ A \neq \emptyset \text{ connected}\}.$$

It is known that the (semi-) span of a chainable continuum is zero. It is an open question of Lelek whether a continuum of span zero is chainable. It follows from [8] that such a continuum is atriodic tree-like.

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Let C be the class of all mappings of the closed unit interval $I = [0, 1]$ onto itself and let $f, g \in C$. We say that f and g have an *ε-uniformization* (see [11]) provided there exist $a, b \in C$ such that $fa \underset{\varepsilon}{=} gb$, where the meaning of $f \underset{\varepsilon}{=} g$ is: $|f(t)-g(t)| \leqslant \varepsilon$ for every $t \in I$. It was proved in [11] that for every $\varepsilon > 0$ every pair of mappings $f, g \in C$ admit an ε-uniformization.

2. The span of continua. Let $X = \varprojlim(X_n, f_m^n)$ and $Y = \varprojlim(Y_n, g_m^n)$ be two inverse sequences of continua. Let $\{\varphi_n\colon X_n \to Y_n\}$ be a sequence of mappings and let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\lim \varepsilon_n = 0$ and for each triple k, n, m $(k < n < m)$, the following diagram is ε_n commutative:

$$\begin{array}{ccccc} & & X_n & \xleftarrow{f_n^m} & X_m \\ (1) & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_m} \\ Y_k & \leftarrow & Y_n & \leftarrow & Y_m \end{array}$$

Define $\varphi\colon X \to Y$ by $\varphi(x_1, x_2, x_3, \ldots) = (y_1, y_2, y_3 \ldots)$ where

$$y_k = \lim_{m\to\infty} g_k^m \varphi_m(x_m).$$

It was proved by Mioduszewski [12] that the map φ is well defined and continuous. The map φ is said to be weakly induced by the sequence $\{\varphi_n\colon X_n \to Y_n\}$ with respect to the sequence $\{\varepsilon_n\}$. Conversely he has also shown that for every map $\varphi\colon X \to Y$ there exist infinite subsequences $\{n_k\}$, $\{m_k\}$ and mappings $\{\varphi_k | X_{n_k} \to Y_{m_k}\}$ satisfying (1). Similar diagrams (see [12] for details) exist between homeomorphic continua.

Let X and Y be continua. A mapping $f\colon X \to Y$ is called *confluent* (resp. *weakly confluent*) provided for every continuum $K \subset Y$ and every component (resp. some component) C of $f^{-1}(K)$, $f(C) = K$. A continuum X is in class W provided every mapping of any continuum onto X is weakly confluent. It is known [3] that all atriodic tree-like continua are in class W. It is an easy observation that every tree-like continuum, such that every proper subcontinuum is arc-like, is atriodic and hence in class W. A mapping $f\colon X \to Y$ is said to be irreducible provided no proper subcontinuum of X is mapped onto Y.

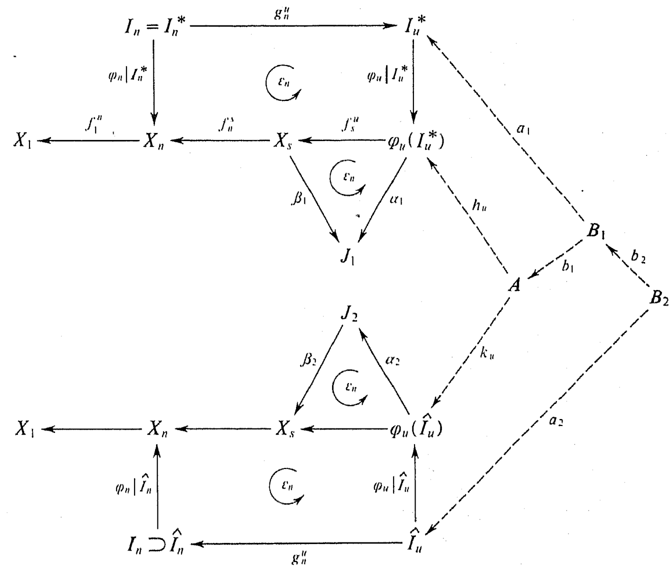
2.1. THEOREM. *Let X be a weakly-chainable atriodic tree-like continuum such that every proper subcontinuum is arc-like. Then the (semi-) span of X is zero.*

Proof. Let $X = \varprojlim(X_P, f_m^P)$ and let the pseudo-arc $P = \varprojlim(I_P, g_m^P)$ where the X_P's are trees and the I_P's are arcs. Suppose that the map $\varphi\colon P \to X$ is weakly induced by the sequence $\{\varphi_P\colon I_P \to X_P\}$ with respect to the sequence $\{\varepsilon_P\}$. We may assume that φ is irreducible.

Suppose that the semi-span of X is positive. Hence there exists a number $\eta > 0$, a continuum C and mappings $h, k\colon C \to X$ such that $h(C) = X$ and $d(h(c), k(c)) > \eta$ for every $c \in C$. We may assume that $k(C) \neq X$ or k is irreducible. Moreover, we may assume that $d(f_1 \circ h(c), f_1 \circ k(c)) > \eta$ for every $c \in C$. Here $f_P\colon X \to X_P$ denotes the natural projection. Choose $n > 0$ such that $10\varepsilon_n < \eta$.

Let $P^* = \varprojlim(I_P^*, g_m^P | I_P^*)$ be a proper-subcontinuum of P such that I_P^* is a subcontinuum of I_P and $g_n(P^*) = I_n$. Put $Y_1 = \varphi(P^*)$, then Y_1 is a proper subcontinuum of X. Since X is in class W, there exists a proper subcontinuum C^* of C such that $h(C^*) = Y_1$. Put $k(C^*) = Y_2$, then Y_2 is also a proper subcontinuum of X. Note that $Y_i = \varprojlim(f_P(Y_i), f_m^P | f_P(Y_i))$ $(i = 1, 2)$. Since X is in class W, there exists a continuum $\hat{P} = \varprojlim(\hat{I}_P, g_m^P | \hat{I}_P) \subset P$ such that $\varphi(\hat{P}) = Y_2$.

(2)



(3)

Since the map φ is almost induced and the continua Y_i are chainable, it follows that there exist integers s, u $(n < s < u)$, arcs J_i and mappings $\alpha_1\colon \varphi_u(I_u^*) \to J_1$, $\alpha_2\colon \varphi_u(\hat{I}_u) \to J_2$ and $\beta_i\colon J_i \to X_s$ $(i = 1, 2)$ such that diagrams (2) and (3) are $2\varepsilon_n$-commutative.

Since $\varphi_u(I_u^*)$ and $\varphi_u(\hat{I}_u)$ are locally connected continua (approximating Y_1 and Y_2), there exist an arc A (approximating C^*), mappings $h_u\colon A \to \varphi_u(I_u^*)$ and $k_u\colon A \to \varphi_u(I_u)$ (approximating $f_u \circ h$ and $f_u \circ k$, resp.) such that

$$(4) \qquad d(f_1^u \circ h_u(t), f_1^u \circ k_u(t)) > \eta \qquad \forall t \in A.$$

Let $\delta > 0$ be such that $x, y \in J_i$ with $d(x, y) < \delta$ implies

$$d(f_1^s \circ \beta_i(x), f_1^s \circ \beta_i(y)) < \varepsilon_n$$

for $i = 1, 2$.

Consider the mappings $\alpha_1 \circ \varphi_u | I_u^*$: $I_u^* \to J_1$ and $\alpha_1 \circ h_u$: $A \to J_1$. It follows from [11] that these mappings have a δ-uniformization. Hence there exists an arc B_1 and mappings a_1: $B_1 \to I_u^*$, b_1: $B_1 \to A$ such that

(5) $$\alpha_1 \varphi_u a_1 \underset{\delta}{=} \alpha_1 h_u b_1.$$

Similarly the mappings $\alpha_2 k_u b_1$: $B_1 \to J_2$ and $\alpha_2 \circ \varphi_u$: $\hat{I}_u \to J_2$ admit a δ-uniformization. Hence there exist an arc B_2 and mappings a_2: $B_2 \to \hat{I}_u$, b_2: $B_2 \to B_1$ such that

$$\alpha_2 \varphi_u a_2 \underset{\delta}{=} \alpha_2 k_u b_1 b_2.$$

By (5)

$$\alpha_1 \varphi_u a_1 b_2 \underset{\delta}{=} \alpha_1 h_u b_1 b_2.$$

Hence

$$f_1^s \beta_1 \alpha_1 \varphi_u a_1 b_2 \underset{\varepsilon_n}{=} f_1^s \beta_1 \alpha_1 h_u b_1 b_2 \quad \text{and} \quad f_1^s \beta_2 \alpha_2 \varphi_u a_2 \underset{\varepsilon_n}{=} f_1^s \beta_2 \alpha_2 k_u b_1 b_2.$$

It follows from (2) and (3) that

$$f_1^s \beta_1 \alpha_1 \underset{\varepsilon_n}{=} f_1^u \quad \text{and} \quad f_1^s \beta_2 \alpha_2 \underset{\varepsilon_n}{=} f_1^u.$$

Hence

$$f_1^u \varphi_u a_1 b_2 \underset{3\varepsilon_n}{=} f_1^u h_u b_1 b_2 \quad \text{and} \quad f_1^u \varphi_u a_2 \underset{3\varepsilon_n}{=} f_1^u k_u b_1 b_2.$$

Also $f_1^u \varphi_u \underset{\varepsilon_n}{=} f_1^n \varphi_n g_n^u$ and hence

(6) $$f_1^n \varphi_n g_n^u a_1 b_2 \underset{4\varepsilon_n}{=} f_1^u h_u b_1 b_2 \quad \text{and} \quad f_1^n \varphi_n g_n^u a_2 \underset{4\varepsilon_n}{=} f_1^u k_u b_1 b_2.$$

Since the map $g_n^u a_1 b_2$: $B_2 \to I_n^* = I_n$ is onto and $g_n^u a_2$: $B_2 \to \hat{I}_n \subset I_n$, there exists a $t_0 \in B_2$ such that $g_n^u a_1 b_2(t_0) = g_n^u a_2(t_0)$. Hence by (6) $d(f_1^u h_u b_1 b_2(t_0),$ $f_1^u k_u b_1 b_2(t_0)) < 8\varepsilon_n < \eta$. This contradicts (4) and the proof is complete.

2.2. Remarks. In [4] Ingram constructed an uncountable planar collection of atriodic tree-like continua of positive span. These continua satisfy the additional condition that all proper subcontinua are arcs. He proved [5] that at most countably many members of this collection are weakly chainable and raised the question whether no member of this collection is weakly chainable. The above theorem answers this question in the affirmative. Earlier it had been announced by T. Moebus

that one particular member of this collection is not weakly chainable. The authors were not able to solve the following problem:

2.3. Problem. Suppose X is a weakly-chainable atriodic tree-like continuum. Is the span of X zero? What if X is also hereditarily indecomposable? Does X have the fixed point property?

Problem 2.3 would have an affirmative answer if the following problem, due to Lee Mohler, has an affirmative answer.

2.4. Problem. Suppose X is a weakly-chainable atriodic tree-like continuum, is X arc-like?

3. Fixed point properties. It is known (see [1], [13], [14]) that there are atriodic tree-like continua without the fixed-point property. In addition these continua satisfy the property that every proper subcontinuum is an arc. It is an open problem to characterize the atriodic tree-like continua with the fixed-point property. It follows immediately from Theorem 2.1 that weakly-chainable atriodic tree-like continua such that all proper subcontinua are arc-like have the fixed-point property.

It is an open problem (cf. University of Houston problem book, problems 84 and 86) whether the confluent image of an arc-like continuum is arc-like or has span zero. It is known (cf [10]) that such an image is an atriodic tree-like continuum. The following problem is also open.

3.1. Problem. Suppose f: $X \twoheadrightarrow Y$ is a confluent map of an arclike continuum X onto a continuum Y. Does Y have the fixed-point property?

The following theorem gives a partial solution. By a ray we mean a one-to-one continuous image of $[0, \infty)$ or $(-\infty, \infty)$. An upper semi-continuous set valued function G: $X \to Y$ is called *refluent* [2] provided for every continuum $K \subset X$ and each component C of $G(K)$, we have $G(x) \cap C \neq \emptyset$ for every $x \in K$.

3.2. Theorem. *Suppose f: $X \to Y$ is a confluent map of an arc-like continuum X onto a continuum Y such that X contains a dense ray. Then Y has the fixed-point property.*

Proof. Suppose g: $Y \to Y$ is a fixed-point free map. Consider the set valued function $G = f^{-1} gf$: $X \to X$. It follows easily that G is a fixed-point free set valued refluent function. Let $\varepsilon > 0$ such that $d(x, G(x)) > \varepsilon$ for all $x \in X$. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be an ε-chain irreducibly covering X.

For $A, B \subset X$, we say that $A < B$ provided $\max\{i \mid A \cap U_i \neq \emptyset\}$ $< \min\{i \mid B \cap U_i \neq \emptyset\}$. Let φ: $[0, \infty) \to X$ be a one-to-one continuous function such that $\varphi([0, \infty)) \cap U_n \neq \emptyset$ and $\varphi(0) \in U_1$. Then $\varphi(0) < G\varphi(0)$. Let $t_1 = \sup\{t' \in [0, \infty) \mid \varphi(t') < G\varphi(t')$ for all $t' \in [0, t]\}$, then $t_1 < \infty$. Suppose $\varphi(t_1) \in U_q$. Choose $\delta > 0$ such that $t_1 - \delta > 0$ and $\varphi([t_1 - \delta, t_1 + \delta])$ $= K \subset U_q$. By the definition of t_1, there exist $x, y \in K$ and $z \in G(y)$ such that x $< G(x)$ and $z < y$. Let C be the component of $G(K)$ containing the point z. Since $z \in C$ and $G(x) \cap C \neq \emptyset$, C intersects elements U_p and U_r where $p < q$

$< r$. Hence $C \cap U_q \neq \emptyset$. Let $u \in K$ such that $G(u) \cap U_q \neq \emptyset$, then $d(u, G(u)) < \varepsilon$. This contradiction completes the proof.

Problem 3.1 would have an affirmative answer if the following problem, due to Maćkowiak (see [9]), has an affirmative answer.

3.3. PROBLEM. Do arc-like continua have the fixed-point property for upper semi-continuous refluent set valued functions?

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Metrizability of certain quotient spaces

by

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Abstract. The metrizability of certain sequential spaces can be characterized by whether or not they contain two canonical subspaces.

Introduction. Let us begin with the following well known examples. These example will play an important role in this paper. Let α be an infinite cardinal number. Let S_α be the space obtained from the topological sum of α convergent sequences by identifying all the limit points. S_ω is especially called sequential fan. We also need another canonical example S_2. That is, $S_2 = (N \times N) \cup \cup N \cup \{0\}$, N is the set of integers, with each point of $N \times N$ an isolated point. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n);\, m \geqslant m_0\}$. And U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$.

We recall some basic definitions. Let X be a space and Let $\mathfrak{A}$ be a cover (not necessarily closed or open) of X. Then X has the *weak topology* with respect to $\mathfrak{A}$, if $F \subset X$ is closed in X whenever $F \cap A$ is closed in A for each $A \in \mathfrak{A}$. Of course we can replace "closed" by "open". A space X is *sequential* (resp. a *k-space*), if X has the weak topology with respect to the cover consisting of all compact metric subsets (resp. compact subsets). As is well known, a sequential space (resp. k-space) is characterized as a quotient image of a metric space [5] (resp. locally compact space [2]). A space X is a *k_ω-space* [14], if it has the weak topology with respect to a countable cover consisting of compact subsets of X. A space X is *Fréchet* (resp. *strongly Fréchet* [21], E. Michael [15] calls it countably bi-sequential) if whenever $x \in \bar{A}$ (resp. $x \in \bar{A}_n$ with $A_{n+1} \subset A_n$), there exist $x_n \in A$ (resp. $x_n \in A_n$) such that $x_n \to x$. We shall remark that S_ω is a Fréchet k_ω-space which is not strongly Fréchet, and that S_2 is a non-Fréchet, k_ω-space.

Now, S_ω (resp. S_2) is helpful in analyzing the gap of Fréchet spaces and strongly Fréchet spaces [22; 16 (b)] (resp. gap of sequential spaces and Fréchet spaces [6; Proposition 7.3]). A. V. Arhangel'skiĭ and S. P. Franklin [1] introduced the sequential order $\sigma(X)$ of a space X. For a hereditarily normal sequential space X, V. Kannan [11] gave a characterization of $\sigma(X)$ by whether or not X contains spaces S_n defined inductively, and showed that such a space X