# On a relationship between countable functionals and projective trees*

by

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**Abstract.** The countable functionals of the type $(0 \to k) \to 0$ define a class of $k$-trees whose nodes are labeled by countable functionals of pure type $k$. The paper discusses some elementary definability properties of the countable functionals leading to the notion of partially wellfounded tree and identifies the supremum of the lengths of the above mentioned $k$-trees as the projective ordinal $\pi_k$. The countable functionals are presented in terms of convergence spaces with countable bases of finite functionals.

The theory of the countable functionals has many aspects, e.g. topological, proof theoretic, recursion theoretic ones. Because each countable functional can be coded by a function, and the set of codes of countable functionals of pure type $n+1$ is complete in $\underset{\sim}{\Pi}^1_n$, the theory is also part of descriptive set theory (Hyland [5], Normann [13]). The purpose of the present paper is to show that also the projective ordinals are well-known inhabitants in the land of the countable functionals. With a type $\sigma$ the type $\sigma^+ = (0 \to \sigma) \to 0$ is associated, and a countable functional $f$ of type $\sigma^+$ is envisaged as a tree $T_f$ of objects of type $\sigma$ as follows: If $s = (u_0, \ldots, u_{n-1})$ is a finite sequence of countable functionals of type $\sigma$ and $\bar{s}$ the corresponding infinite sequence given by $\bar{s}(i) = u_i$ if $i < n$, $=$ zerofunctional $0^\sigma$ otherwise, then $s \in T_f \Leftrightarrow f(\bar{s}) > n$. Scarpellini [15] exhibits Spector's bar recursion in his model as recursion on these wellfounded trees. Therefore we call the length $|T_f|$ of $T_f$ the *Spector ordinal of $f$* and denote the supremum of $|T_f|$ for countable $f$ of type $n^+$ by $\gamma_n$. The main result of the paper is the theorem of D. Normann, saying that $\gamma_n$ is the projective ordinal $\underset{\sim}{\pi}^1_n$, i.e. the supremum of the lengths of the $\underset{\sim}{\Pi}^1_n$ prewellorderings in the Baire space. But before this we give a brief introduction into the theory of the countable functionals from the point of view of limit-spaces, following Scarpellini [15]. Other treatments like the Kleene–Kreisel definition of Ct via associates [8], [9], Ershov's embedding in his general theory of partially continuous functionals [1], which is related to Scott's work on lattices, Hyland's filter spaces [4], [6], or Troelstra's ECF model [18] may be in some respect superior to the pure limit-space approach, but the

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latter makes it easy to emphasize the role of the finite functionals. These can be readily identified with a countable dense subset of Lim, the space of the continuous functionals. The finite functionals are codable as numbers, and so the set of the convergent sequences of (coded) finite functionals constitutes a set coff of descriptions for Lim. Each countable functional $f$ has a standard description obtained by restricting $f$ to $f|n$ of lenght $n$; in Normann's terminology the sequence $\lambda n.f|n$ is (almost the same as) the trace of $f$, which is left by $f$ in the finite functionals. Traces or coffs can often play a similar role in Lim as Kleene's associates do in Ct, but there are differences: the evaluation expressed in coffs is not recursive; the function which maps a functional to its trace is continuous, but the reverse function which maps a coff to the represented functional is not. In section 2 Coff $(\varrho\to\sigma)$ is characterized as the set of those $\alpha$, which applied (explained in the text) to any $\beta\in\mathrm{Coff}(\varrho)$ gives a value $\alpha(\beta)\in\mathrm{Coff}(\sigma)$, and the completeness of $\mathrm{Coff}(n+1)$ is proved. There we need a lemma which says that each $\underset{\sim}{\Pi}^1_n$-set $A$ is of the form $\{\alpha\in\mathscr{R}\colon \forall g\in\mathrm{Lim}(n)\exists i\ R(\alpha, g, i)\}$ with a countable predicate $R$. In analogy to the tree representation of $\underset{\sim}{\Pi}^1_n$-sets the elements of $A$ can be considered as partially wellfounded trees. Although there is no obvious definition of length for partially wellfounded trees the axiom of projective determinacy allows comparison of such trees by a projectively defined prewellordering: $\tilde{\alpha}\leqslant\alpha$ iff there is a "monotone" mapping of the tree belonging to $\tilde{\alpha}$ into the tree belonging to $\alpha$. This discussion is continued in section 3 with the concept of Spector trees.

**§ 1. Limit continuous functionals.** There are various ways to introduce the countable functionals topologically. We mentioned already Ershov's partially continuous functionals and Hyland's filter spaces. In this paper we will use Scarpellini's model of limit continous functionals. Scarpellini [15] is inspired by the chapter on $\mathscr{L}^*$-spaces in Kuratowski [12], where the notion of a convergent sequence is basic. If $\to$ denotes a binary relation between sequences $F\colon\ \omega\to X$ of a set $X$ and elements $f$ of $X$, then $(X, \to)$ must satisfy three conditions in order to be a $\mathscr{L}^*$-space:

(i) if $F$ is eventually constant $f$, then $F\to f$,

(ii) all subsequences $G$ of $F\to f$ converge against $f$,

and the Urysohn-condition,

(iii) if $F$ does not converge against $f$, there is a subsequence $G$ of $F$ such that no subsequence $H$ of $G$ converges against $f$.

A $\mathscr{L}^*$-space is discrete, if the eventually constant sequences are the only convergent ones. The discrete space $\omega$ of the natural numbers will also be denoted by Lim (0). If $(X, \to)$ is a $\mathscr{L}^*$-space, let $L(X)$ denote the set of maps $F\colon\ \omega\cup\{\infty\}\to X$ s.t. $\lambda n\in\omega.\ F(n)\to F(\infty)$. A map $f$ between two $\mathscr{L}^*$-spaces $(X, L(X))$ and $(Y, L(Y))$ is continuous iff it transports all convergent sequences $G\in L(X)$ into convergent sequences $f\circ G\in L(Y)$. The set $Z$ of the continuous maps is endowed with a $\mathscr{L}^*$-structure: $F\in L(Z)$ iff for all $G\in L(X)$ $\lambda i\cdot F(i)\,(G(i))\in L(Y)$. To



emphasize the homomorphic aspect of this definition we often write $F(G)$ for $\lambda i\cdot F(i)\,(G(i))$. With these function spaces the category of $\mathscr{L}^*$-spaces is cartesian closed. If $X$ has elements of type $\varrho$ and $Y$ of type $\sigma$, then the elements of $Z$ are of type $\varrho\to\sigma$. Starting from 0, the type of the natural numbers, and disregarding the product types one obtains all finite types. For technical reasons the definition of Lim $(\varrho\to\sigma)$, the space of the continuous functionals of type $\varrho\to\sigma$, is slightly modified: let $(\mathrm{Lim}\,(\varrho),\ L(\varrho))$ and $(\mathrm{Lim}\,(\sigma),\ L(\sigma))$ be given, $\tau=\varrho\to\sigma$. We define first $L^*(\tau)$ as the set of $F\colon\ \omega\cup\{\infty\}\to\mathrm{Lim}\,(\varrho)\to\mathrm{Lim}\,(\sigma)$ s.t. $F(G)\in L(\sigma)$ for all $G\in L(\varrho)$. Then Lim $(\tau)$ consists of all $f\colon\ \mathrm{Lim}(\varrho)\to\mathrm{Lim}\,(\sigma)$ with $\lambda i\cdot f\in L^*(\tau)$ and $L(\tau)$ of all $F\in L^*(\tau)$ with range $(F)\subseteq\mathrm{Lim}\,(\tau)$. We write $F\underset{\tau}{\to}f$ iff $F'\in L^*(\tau)$, where $F'\colon\ \omega\cup\{\infty\}\to\mathrm{Lim}\,(\varrho)\to\mathrm{Lim}\,(\sigma)$ is defined by $F'(n)=F(n)$ and $F'(\infty)=f$, and say "$F$ converges against $f$ in Lim $(\tau)$", i.e. $F\to f$ in Lim $(\tau)$, iff $F\underset{\tau}{\to}f$ and $\forall n\ F(n)\in\mathrm{Lim}\,(\tau)$ and $f\in\mathrm{Lim}\,(\tau)$. In this notation $f\in\mathrm{Lim}\,(\varrho\to\sigma)$ iff $\lambda n\cdot g_n\to g$ in Lim $(\varrho)$ implies $\lambda n\cdot f(g_n)\to f(g)$ in Lim $(\sigma)$. The set of all strictly increasing $j\colon\ \omega\to\omega$ is denoted by mon. The subsequences of $F\colon\ \omega\to X$ are then given as the $F\circ j$ with $j\in$ mon. We leave it as an exercise to prove that $L^*(\tau)$ is closed against taking subsequences. The pure types are denoted by natural numbers and defined by $n+1=n\to 0$. Lim(0) is the discrete space $\omega$, Lim (1) is identified with the Baire space $\mathscr{R}$ of the irrationals, Lim (2) is the set of all continuous (in the usual topological sense) maps between $\mathscr{R}$ and $\omega$.

Often one wants to prove $f_n\to f$ in Lim $(\tau)$ by considering approximating sequences $F_n\to f_n$. Then Lemma 1.1 can be helpful.

LEMMA 1.1. *Let* $\tau=\varrho\to\sigma$ *and* $f_n\to f\colon$ Lim $(\varrho)\to$ Lim $(\sigma)$, $F_n\colon\ \omega\to$ Lim $(\varrho)\to$ Lim $(\sigma)$ *with* $F_n\to f_n$ *for all* $n$. *Then* $f_n\underset{\tau}{\to}f$, *if for all* $j\in$ mon $\lambda n\cdot F_n(j(n))\underset{\tau}{\to}f$.

Proof by induction on types. The corresponding statement for $\tau=0$ is true. Let $G\to g$ in Lim $(\varrho)$. Then $\lambda i\cdot F_n(i)\,(G(n))\to f_n(G(n))$ in Lim $(\sigma)$ for each $n$ and $\lambda n\cdot F_n(j(n))\,(G(n))\to f(g)$ in Lim $(\sigma)$ for each $j\in$ mon. With ind hyp for $\sigma$ we obtain $\lambda n\cdot f_n(G(n))\to f(g)$ in Lim $(\sigma)$.

As a corollary we have

LEMMA 1.2. *Let* $\tau=\varrho\to\sigma$ *and* $f\colon$ Lim $(\varrho)\to$ Lim $(\tau)$. *If* $F\underset{\tau}{\to}f$ *for a sequence* $F\colon\ \omega\to$ Lim $(\varrho)\to$ Lim $(\sigma)$ *then* $f$ *is continuous, i.e.* $f\in$ Lim $(\tau)$.

Proof. Let $g_n\to g$ in Lim $(\varrho)$. Then $\lambda i\cdot F(i)\,(g_n)\to f(g_n)$ for each $n$ and $\lambda n\cdot F(j(n))\,(g_n)\to f(g)$ for each $j\in$ mon. With Lemma 1.1 $f(g_n)\to f(g)$ in Lim $(\sigma)$.

In the remainder of section 1 we derive some well-known topological properties of the countable functionals in this limit-space setting. First Kreisel's question on the continuity of moduli of continuity. If $f$ is a continuous functional of type $\sigma^+=(0\to\sigma)\to 0$ then a function $\mu(f)\colon$ Lim $(0\to\sigma)\to\omega$ is a modulus of continuity (moc) for $f$ iff for all $g\in$ Lim $(0\to\sigma)$ $\mu(f)\,(g)$ gives a number $m$ s.t. $(\mu)\ \forall\tilde{g}\in$ Lim $(0\to\sigma)$ $\big(\forall i\leqslant m(g(i)=\tilde{g}(i))\Rightarrow f(g)=f(\tilde{g})\big)$. $\mu(f)$ is called the *minimal moc for* $f$, denoted $\mu_{\min}(f)$, if the $m$ in $(\mu)$ is minimal. Each continuous $f$ has a minimal moc, because otherwise there would be $g\in$ Lim $(0\to\sigma)$, and for each $m$ $g_m\in$ Lim $(0\to\sigma)$ with $\forall i\leqslant m\big(g(i)=g_m(i)\big)$ and $f(g)\neq f(g_m)$, contradicting the continuity of $f$. If $f$ is continuous of type 2 then each modulus $\mu(f)$: $\mathscr{R}\to\omega$ is continuous too. But there is no continuous $\mu$: $\mathrm{Lim}\,(2)\to\mathrm{Lim}\,(2)$ s.t. for all $f\in\mathrm{Lim}\,(2)$ $\mu(f)$ is a moc for $f$, an old result of Kreisel, see [11], p. 154. In a letter (11. 11. 1975) Kreisel asked wether every functional of type $2^+$ has a continuous modulus of continuity. Independently Howard, Hyland and the author gave a negative answer. The simple counter example is contained in

LEMMA 1.3. *Let* $f\in\mathrm{Lim}\,(2^+)$ *be defined by* $f(g)=g(0)\,(\lambda n\cdot gn0^1)$ *for* $g\in\mathrm{Lim}\,(0\to 2)$. *Then $f$ has no continuous modulus of continuity.*

Proof. Assume that $\mu(f)$ is a continuous moc for $f$. Let $k\in\omega$ be fixed, $h:=0^{0\to 2}$, $Hnm\alpha:=0$ if $\alpha(k+1)\leqslant n$, $=1$ otherwise; $Gnm:=Hnm$ if $m\leqslant k$, $=\lambda\alpha\cdot n+1$ otherwise. Because of $f(Hn)=0$ and $f(Gn)=1$ is $\mu(f)\,(Hn)>k$. Because of $H\to h$ and the presupposed continuity of $\mu(f)$ is $\mu(f)(h)>k$. But $k$ was arbitrary.

Convergence in Lim (2) can be reduced to pointwise convergence with the help of the minimal modulus of continuity.

LEMMA 1.4. *Let* $f_n, f\in\mathrm{Lim}\,(2)$. *Then* $f_n\to f$ *is equivalent to*

$$\text{(A)}\qquad \forall\alpha\,\exists m\,\exists n_0\,\forall n\geqslant n_0\;\big(\mu_{\min}(f_n)\,(\alpha)\leqslant m\wedge f_n(\alpha)=f(\alpha)\big).$$

*It follows that for each convergent* $F$: $\omega\to\mathrm{Lim}\,(2)$ *and each* $\alpha\in\mathscr{R}$ *the set* $\{\mu_{\min}(F(n))\,(\alpha)\colon n\in\omega\}$ *is bounded in* $\omega$.

Proof. Let $f_n\to f$ and assume not (A). By considering subsequences we can assume w.l.o.g. the existence of an $\alpha$ s.t. $\mu_{\min}(f_n)(\alpha)>n$ and $f_n(\alpha)=f(\alpha)$ for all $n$. Then for each $n$ there are $\alpha_n$ with $\forall i\leqslant n\;\alpha_n(i)=\alpha(i)$ and $f_n(\alpha_n)\neq f_n(\alpha)$. Therefore $\alpha_n\to\alpha$ and $f_n(\alpha_n)=f(\alpha)=f_n(\alpha)$ for large $n$, contradiction. The other direction is equally simple.

LEMMA 1.5. *For each* $f\in\mathrm{Lim}\,(2)$ *and* $m\in\omega$ *there is* $F\colon\omega\to\mathrm{Lim}\,(2)$ *with* $F\to f$ *and* $\forall n\,\mu_{\min}(F(n))(0^1)\geqslant m$.

Proof. Define $F(n)\,(\alpha):=f(\alpha)$, if $\alpha(m)\leqslant n$; $:=f(0^1)+1$ otherwise. $F\to f$ is clear. Let $\alpha=0^1$ and assume $\mu_{\min}(F(n))\,(\alpha)<m$ for some $n$. Define $\beta(i)=n+1$, if $i=m$, $=0$ otherwise. Then $\forall i<m\;\alpha(i)=\beta(i)$, but $F(n)\,(\alpha)=f(\alpha)$ and $F(n)\,(\beta)=f(\alpha)+1$, contradiction.

As a corollary we obtain the mentioned result of Kreisel:

Choose $f_n, f$ in Lim (2) with $f_n\to f$ s.t. $f_n\neq f$ for all $n$. Lemma 1.5 gives for each $n$ a sequence $\lambda i\cdot f_{ni}\to f_n$ with $\mu_{\min}(f_{ni})\,(0^1)\geqslant n$. If $\mu$ would be a continuous moc for Lim (2) then $\mu(f_n)\,(0^1)\geqslant n$ and $\mu(f)\,(0^1)\geqslant n$ for all $n$, which is absurd. Now set $A:=\{f_{ni}\colon n, i\in\omega\}$ with the above $f_{ni}, f_n, f$. Then $A$ contains no sequence converging against $f$. The argument uses Lemma 1.4: Let $G\colon\omega\to A$ and assume $G\to f$. Then only finitely many of the $f_{ni}$ of the range$(G)$ can have the same subscript $n$, because otherwise there would be a subsequence converging against $f_n\neq f$. Therefore $\{\mu_{\min}(G(k))\,(0^1)\colon k\in\omega\}$ is unbounded, contradicting Lemma 1.4. The natural closure operator cl $(B)$ for



subsets $B$ of $\mathrm{Lim}\,(\tau)$ is defined by: $f\in\mathrm{cl}\,(B)\Leftrightarrow$ there is $F\colon\omega\to B$ with $F\to f$ in $\mathrm{Lim}\,(\tau)$. For the $A$ as above we have shown $f\in\mathrm{cl}(\mathrm{cl}(A))\backslash\mathrm{cl}(A)$, and

LEMMA 1.6. *The natural closure operator for* Lim (2) *is not idempotent.*

Nevertheless each $\mathrm{Lim}\,(\tau)$ is separable in the sense that there is a countable set $\mathrm{Fin}(\tau)$ with $\mathrm{cl}(\mathrm{Fin}(\tau))=\mathrm{Lim}(\tau)$. The reason is that a continuous functional disposes only of countable information, and can be approximated by functionals containing only a finite amount of information: each natural number $n$ is identified with $\{i<n\}$. $\mathrm{Fin}(\tau, n)$ will denote the set of the finite functionals (ff) of type $\tau$ and length $n$. To avoid the empty set $\mathrm{Fin}(0, n)$ is defined as $n+1=\{i\leqslant n\}$. $\mathrm{Fin}(\varrho\to\sigma, n)$ is just the set of all functions from $\mathrm{Fin}(\varrho, n)$ into $\mathrm{Fin}(\sigma, n)$. The set $\mathrm{Fin}(\tau)$ of the ffs of type $\tau$ is the union of the $\mathrm{Fin}(\tau, n)$. The restriction $i|n$ of a natural number $i$ to $n$ is $i$ if $i\leqslant n$, $n$ otherwise. We define inductively two families of mappings, the sections $s_n^\tau$: $\mathrm{Fin}(\tau, n)\to\mathrm{Lim}(\tau)$ and the retractions $r_n^\tau$: $\mathrm{Lim}\,(\tau)\to\mathrm{Fin}\,(\tau, n)$: for $\tau=0$ we set $s_n^0(i):=i$ for $i\leqslant n$, $r_n^0(i):=i|n$ for $i\in\omega$. Let $\tau=\varrho\to\sigma$. For $f\in\mathrm{Fin}\,(\tau, n)$, $g\in\mathrm{Lim}\,(\varrho)$ we define

$$s_n^\tau(f)\,(g)=s_n^\sigma\big(f(r_n^\varrho(g))\big).$$

For $f\in\mathrm{Lim}(\tau)$, $g\in\mathrm{Fin}(\varrho, n)$ we define analogously

$$r_n^\tau(f)\,(g)=r_n^\sigma\big(f(s_n^\varrho(g))\big).$$

With the usual induction on types one sees easily that all $s_n^\tau$, $r_n^\tau$ are well-defined, and that $r_n^\tau\circ s_n^\tau$ is the identity on $\mathrm{Fin}\,(\tau, n)$. We abbreviate $s_n^\tau\circ r_n^\tau(f)$ as $f|n$, "$f$ restricted to $n$". For $\tau=\varrho\to\sigma$ and $f\in\mathrm{Lim}\,(\tau)$ $f|n$ has the direct definition $(f|n)\,(g)=f(g|n)|n$ for $g\in\mathrm{Lim}\,(\varrho)$. Obviously $f|n|\,n=f|n$. Often $\mathrm{Fin}(\tau, n)$ is identified with its image $s_n^\tau(\mathrm{Fin}\,(\tau, n))$ and $r_n^\tau(f)$ with $f|n$. The restriction operator gets on well with the convergence:

LEMMA 1.7. (i) *If* $\lambda n\cdot f_n\underset{\tau}{\to}f$ *and* $j\in\mathrm{mon}$ *then* $\lambda n\cdot f_n|j(n\underset{\tau}{\to}f$.

(ii) *If $f$ is continuous then* $\lambda n\cdot f|n\to f$.

Proof of (i) for $j=$ identity by induction on $\tau$ is straightforward. For $j\in\mathrm{mon}$ define an increasing function $k\colon\omega\to\omega$ with $k\circ j=\mathrm{id}$. Then $\lambda n\cdot f_{k(n)}\to f$, $\lambda n\cdot f_{k(n)}|n\to f$, $\lambda n\cdot f_{k(j(n))}|j(n)\to f$.

(ii) is immediate from (i).

With Lemma 1.1 we obtain the following converse of (i):

If $\lambda n\cdot f_n|j(n)\to f$ for all $j\in\mathrm{mon}$ then $\lambda n\cdot f_n\to f$.

The hypothesis can not be weakened to $j=\mathrm{id}$ as the following example shows: Define $F\colon\omega\to\mathrm{Lim}\,(2)$ by $F(n)\,(\alpha)=0$ if $\alpha(n)=\alpha(n+1)$, $=\alpha(n+1)$ otherwise. Because of $(\alpha|k)\,(n+1)=\alpha(n+1\,|k)|k=(\alpha|k)\,(n)$ for $k\leqslant n$ is $(F(n)|k)\,(\alpha)=0$ for $k\leqslant n$, therefore $\lambda n\cdot F(n)|n\to 0^2$ and $\lambda n\cdot F(n)|k\to 0^2$ for each $k$, but $F(n)$ (identity) $=n+1\neq 0$.

**§ 2. Convergent sequences of finite functionals.** A finite functional (abbr.: ff) is a finite object and as such can be coded by a number. Let $\mathrm{code}_\tau$: $\mathrm{Fin}\,(\tau)\to\omega$ and decode$_\tau$: $\omega \to \mathrm{Fin}(\tau)$ be two functions with $\mathrm{decode}_\tau \circ \mathrm{ocode}_\tau =$ identity on $\mathrm{Fin}(\tau)$ s.t. the relevant operations on ffs are primitive recursive in the codes. For example the length of a ff should be primitive recursively extractable from the code. The application between two ffs $f$ of type $\tau = \varrho \to \sigma$ and $g$ of type $\varrho$ is defined as

$$f(g) = r^{\sigma}_{\max}\big(s^{\tau}_{n}(f)\,(s^{\varrho}_{m}(g))\big),$$

where $f \in \mathrm{Fin}(\tau, n)$, $g \in \mathrm{Fin}(\varrho, m)$, max = maximum of $n$ and $m$. This reduces to ordinary application if $n = m = \max$. Application has also to be primitive recursive in the codes. Each continuous functional $f$ of type $\tau$ defines a sequence $\lambda n{\cdot}f|n$ of ffs describing $f$ completely. Adapting a denomination of Normann, we call the function $\lambda n{\cdot}\mathrm{code}_\tau(r^{\tau}_{n}(f))$ the finite functional trace of $f$ and denote it by $f^r$ or $r(f)$.

Remark. Let As$(k)$ denote the set of Kleene's associates for pure type $k$ and Ct$(k)$ the notion of the thereby defined functionals. Then $\mathrm{Ct}(k) = \mathrm{Lim}(k)$ (Hyland [4]). Let $B^k_s$ be the set of continuous functionals which have an associate $\alpha$ beginning with $s$, that is $\bar{\alpha}(n) = s$ for $n = \mathrm{lh}(s)$. Define $\varphi^k_i = s^k_n(f)$, if $f = \mathrm{decode}(i) \in \mathrm{Fin}(k, n)$. Then the family $(\varphi^k_i\colon i \in \omega)$ has the following properties:

(i) $\varphi^k_i \in \mathrm{Ct}(k)$.

(ii) If $B^k_s$ is not empty, we may find primitive recursively in $k$, $s$ an $i$ with $\varphi^k_i \in B^k_s$.

(iii) The relation $\{(i, s)\colon \varphi^k_i \in B^k_s\}$ is primitive recursive.

(The proof has to make use of the theory of associates and is outside the scope of this paper.)

Depending on a familiy $(\varphi^{k-1}_i)$ satisfying (i)–(iii) Normann [14] defines the trace $h_f$ of a functional $f \in \mathrm{Ct}(k)$ for $k \geqslant 2$ to be

$$h_f(i) = f(\varphi^{k-1}_i).$$

$h_f$ is recursive in $f^r$ and conversely: Let $g = \varphi^{k-1}_i$ of length $n$. Then $(r_n f)(g) = r_n\big(f_n(s_n g)\big) = r_n\big(h_f(\mathrm{code}(g))\big)$ and $f^r(n) = \mathrm{code}(r_n(f))$ can be composed of the $r_n(f)(g)$ for $g \in \mathrm{Fin}(\varrho, n)$. Conversely let $g = \mathrm{decode}(i) \in \mathrm{Fin}(\varrho, m)$ and choose $n \geqslant m$ so large that $f(s_m g) < n$. Then

$$\begin{aligned} h_f(i) &= f(\varphi_i) = f\big(s_m(g)\big) = r_n\big(f(s_m g)\big) = r_n\big(f(s_n r_n s_m g)\big) = (r_n f)(r_n s_m g) \\ &= \mathrm{code}\,(r_n f)\,\big(\mathrm{code}\,(r_n s_m g)\big) = f^r(n)|\mathrm{code}\,(r_n s_m)\,(\mathrm{decode}(i)). \end{aligned}$$

Each function $\alpha$ can be seen as a sequence $\lambda n{\cdot}\mathrm{decode}_\tau(\alpha(n))$ of ffs of type $\tau$. We call $\alpha$ a convergent sequence of ffs (coff) of type $\tau$ iff the corresponding sequence of ffs converges in $\mathrm{Lim}(\tau)$. We denote by $\mathrm{Coff}(\tau)$ the set of all coffs of type $\tau$ and by $s$ the map: $\mathrm{Coff}(\tau) \to \mathrm{Lim}(\tau)$, $\alpha \mapsto s(\alpha) =$ limit of $\lambda n{\cdot}\mathrm{decode}_\tau(\alpha(n))$. If $\mathrm{Coff}(\tau)$ is considered as a subspace of $\mathscr{R}$, then $s$ is not continuous for $\tau \geqslant 2$ (define the sequence $F\colon \omega \to \mathrm{Lim}(2)$ as at the end of § 1: $F(n)(\alpha) = 0$ if $\alpha(n) = \alpha(n+1)$, $= \alpha(n+1)$ ow., $\alpha_n(i) = \mathrm{code}(F(n)|i)$, $\alpha(i) = $ code of the zerofunctional in $\mathrm{Fin}(2, i)$. Then $\alpha_n, \alpha \in \mathrm{Coff}(\tau)$ and for $n \geqslant i$ $\alpha_n(i) = \alpha(i)$, so $\alpha_n \to \alpha$, but $s(\alpha_n) = F(n) \not\to 0^2 = s(\alpha)$. The trace function $r\colon \mathrm{Lim}(\tau) \to \mathrm{Coff}(\tau)$ is continuous and one-one. This contrasts to the properties of Kleene's associates: the function which maps an associate to the represented functional is continuous, but there is no continuous function selecting an associate for a given functional (because otherwise every modulus of continuity would be continuous). Observe also that for each $\varrho \neq 0$ there is a continuous injective map from $\mathrm{Lim}(\varrho \to \sigma)$ into $\mathrm{Lim}(\varrho)$, whereas there is no continuous surjective map of $\mathrm{Lim}(\varrho)$ onto $\mathrm{Lim}(\varrho \to \sigma)$ (assume $f$ is such a map and define $h$ by $h(g) = \mathrm{vary}\big(f(g)(g)\big)$ for a map $\mathrm{vary} \in \mathrm{Lim}(\sigma \to \sigma)$. Then $h$ would be a $f(g_0)$ and $h(g_0) = \mathrm{vary}\big(h(g_0)\big)$).



We search now for a simple inductive characterization of $\mathrm{Coff}(\tau)$. As a bridge we use the following equivalence relation: two convergent sequences are equivalent iff they have the same limit. This relation together with the corresponding variant Limv of Lim has the following inductive formulation. Limv(0) is $\omega$ and $\tilde{F} \underset{0}{\sim} F$ iff $\tilde{F}, F\colon \omega \to \omega$ are two convergent sequences with the same number as limit. Let $\tau = \varrho \to \sigma$. We define first $\tilde{F} \underset{\tau}{\sim} F$ for sequences $\tilde{F}, F\colon \omega \to \mathrm{Limv}(\varrho) \to \mathrm{Limv}(\sigma)$ by

$$\forall \tilde{G}, G\colon \omega \to \mathrm{Lim}(\varrho)\,\big(\tilde{G} \underset{\varrho}{\sim} G \Rightarrow \tilde{F}(\tilde{G}) \underset{\sigma}{\sim} F(G)\big).$$

$\mathrm{Limv}(\tau)$ consists of all $f$ with $\lambda i{\cdot}f \underset{\tau}{\sim} \lambda i{\cdot}f$. The following lemma has a straightforward proof by induction on types.

LEMMA 2.1. (i) $\tilde{F} \underset{\tau}{\sim} F \Rightarrow F \sim F$,

(ii) $\underset{\tau}{\sim}$ *is an equivalence relation on* $\{F\colon F \underset{\tau}{\sim} F\}$.

LEMMA 2.2. $\mathrm{Lim}(\tau) = \mathrm{Limv}(\tau)$ *and* $\tilde{F} \underset{\tau}{\sim} F \Leftrightarrow \exists f \in \mathrm{Lim}(\tau)\, \tilde{F} \to f \leftarrow F$.

Proof. Let $\tau = \varrho \to \sigma$ and $\tilde{F} \sim F$. With ind hyp we get for every $g \in \mathrm{Lim}(\varrho)$ a $f(g) \in \mathrm{Lim}(\sigma)$ with $\lambda n{\cdot}\tilde{F}(n)(g) \to f(g) \leftarrow \lambda n{\cdot}F(n)(g)$. We have to show that $\tilde{F} \to f$ and $F \to f$. If $G\colon \omega \to \mathrm{Lim}(\varrho)$, $g \in \mathrm{Lim}(\varrho)$ and $G \to g$, then because of $\lambda n{\cdot}g \to g$, $G \sim \lambda n{\cdot}g$ and therefore $\tilde{F}(G) \sim F(\lambda n{\cdot}g) \sim \lambda n{\cdot}f(g)$. This implies $\tilde{F}(G) \to f(g)$. By symmetry $F \to f$. The other direction is simple.

As a corollary we have

$$\alpha \underset{\tau}{\sim} \alpha \Leftrightarrow \alpha \in \mathrm{Coff}(\tau).$$

Lemma 2.2 shows that if two equivalent sequences $F$ and $G$ are merged into a sequence $H$ then $H$ is equivalent to $F$ and $G$ too. For two sequences $F, G\colon \omega \to X$ we define the merged sequence $H = F * G$ by $H(2n) = F(n)$ and $H(2n+1) = G(n)$, and the subsequences $(F)_{\mathrm{even}}$, $(F)_{\mathrm{odd}}$ by $(F)_{\mathrm{even}}(n) = F(2n)$, $(F)_{\mathrm{odd}}(n) = F(2n+1)$. The following lemma should be clear.

LEMMA 2.3. (i) $\tilde{F} \sim F \Rightarrow (\tilde{F})_{\mathrm{even}} \sim (F)_{\mathrm{even}} \wedge (\tilde{F})_{\mathrm{odd}} \sim (F)_{\mathrm{odd}}$.

(ii) $\tilde{F} \sim F \Rightarrow \tilde{F} \sim (F)_{\mathrm{even}} \sim (F)_{\mathrm{odd}}$.

(iii) $F$ *converges* $\Leftrightarrow F \sim F \Leftrightarrow (F)_{\mathrm{even}} \sim (F)_{\mathrm{odd}} \Leftrightarrow F * F$ *converges.*

LEMMA 2.4. *Let* $\tau = \varrho \to \sigma$.

(i) $\tilde{\alpha} \underset{\tau}{\sim} \alpha \Leftrightarrow \forall \tilde{\beta}, \beta\ (\tilde{\beta} \underset{\varrho}{\sim} \beta \Rightarrow \tilde{\alpha}(\tilde{\beta}) \underset{\sigma}{\sim} \alpha(\beta))$.

(ii) $\alpha \in \mathrm{Coff}(\tau) \Leftrightarrow \forall \beta \in \mathrm{Coff}(\varrho)\ \alpha(\beta) \in \mathrm{Coff}(\sigma)$,

*where $\alpha(\beta)(i) = \alpha(i)$ of type $\tau$ applied to $\beta(i)$ of type $\varrho$ as codes of ffs as defined above.*

Proof of (i). ($\Rightarrow$) is trivial. ($\Leftarrow$): Let $\tilde{G} \underset{\varrho}{\sim} G$ and $\tilde{\alpha}(i) \in \mathrm{Fin}(\tau, k(i))$, $\alpha(i) \in \mathrm{Fin}(\tau, l(i))$, $\tilde{\beta}(i) = r^{\varrho}_{k(i)}(\tilde{G}(i))$, $\beta(i) = r^{\varrho}_{l(i)}(G(i))$. Then $\tilde{\alpha}(i)(\tilde{G}(i)) = s_{k(i)}(\tilde{\alpha}(i)(\tilde{\beta}(i)))$ and $\alpha(i)(G(i)) = s_{l(i)}(\alpha(i)(\beta(i)))$. W.l.o.g. we assume $k, l \in \mathrm{mon}$. Then $\tilde{\beta} \sim \beta$ and $\tilde{\alpha}(\tilde{\beta}) \sim \alpha(\beta)$, then $\tilde{\alpha}(\tilde{G}) \sim \alpha(G)$.

Proof of (ii). ($\Rightarrow$): $\beta \in \mathrm{Coff}(\varrho) \Rightarrow \beta \sim \beta \Rightarrow \alpha(\beta) \sim \alpha(\beta) \Rightarrow \alpha(\beta) \in \mathrm{Coff}(\sigma)$. ($\Leftarrow$) $\tilde{\beta} \underset{\varrho}{\sim} \beta \Rightarrow \tilde{\beta} * \beta \in \mathrm{Coff}(\varrho) \Rightarrow \alpha(\tilde{\beta} * \beta) \in \mathrm{Coff}(\sigma)$. Therefore $(\alpha)_{\mathrm{even}} \sim (\alpha)_{\mathrm{odd}}$, then $\alpha \in \mathrm{Coff}(\tau)$.

With Lemma 2.4 it is easily seen that Coff(0) is $\Sigma^0_2$, Coff(1) is $\Pi^0_3$, Coff(2) is $\Pi^1_1$, and $\mathrm{Coff}(k+1)$ is $\Pi^1_k$ for $k \geqslant 1$. We say that a relation $P \subseteq \mathrm{Lim}(\tau_1) \times \ldots \times \mathrm{Lim}(\tau_n)$ is countable iff its characteristic function $\chi_P$, defined by $\chi_P(g_1)(g_2)\ldots(g_n) = 0 \Leftrightarrow P(g_1, \ldots, g_n)$ is continuous.

LEMMA 2.5. *Let $P$ be a countable predicate on $\mathscr{R} \times \mathrm{Lim}(k) \times \omega$ and $A \subset \mathscr{R}$ be given by*

$$\alpha \in A \Leftrightarrow \forall g \in \mathrm{Lim}(k)\ \exists i P(\alpha, g, i).$$

*Then $A$ is continuously reducible to $\mathrm{Coff}(k+1)$, i.e. there is a continuous map $\varphi\colon \mathscr{R} \to \mathscr{R}$ with $\alpha \in A \Leftrightarrow \varphi(\alpha) \in \mathrm{Coff}\ (k+1)$.*

Proof. For $\alpha \in \mathscr{R}$ and $g \in \mathrm{Fin}(k, n)$ define $\alpha^*(n) \in \mathrm{Fin}(k+1, n)$ by $\alpha^*(n)(g) =$ least $p \leqslant n$ with $P(\alpha, g, p)$, if there is such a $p$, $= n$ otherwise, and $\varphi(\alpha)(n) = \mathrm{code}_{k+1}(\alpha^*(n))$.

We have to show

(i) $\alpha \mapsto \alpha^*$ is continuous,

(ii) $\alpha \in A \Rightarrow \alpha^* \in \mathrm{Coff}(k+1)$,

(iii) $\alpha \notin A \Rightarrow \alpha^* \notin \mathrm{Coff}(k+1)$.

ad (i): Fix $n$ and let $\alpha_i \to \alpha$. For large $i$ and $g \in \mathrm{Fin}(k, n)$ is least $p \leqslant n$. $P(\alpha_i, g, p) =$ least $p \leqslant n \cdot P(\alpha, g, p)$. Then $\alpha^*_i(n) = \alpha^*(n)$ for large $i$, because there are only finitely many $g \in \mathrm{Fin}(k, n)$.

ad (ii): Let $\beta \in \mathrm{Coff}(k)$. We show that $\alpha^*(\beta)$ is eventually constant. Let $g \in \mathrm{Lim}(k)$ with $s(\beta) = g$. Because of $\alpha \in A$ there is a minimal $p$ with $P(\alpha, g, p)$. Because $\beta \to g$ this $p$ is also minimal w.r. to $\beta(n)$, that is $\alpha^*(n)(\beta(n)) = p$ for large $n$.

ad (iii): If $\alpha \notin A$ we have $g \in \mathrm{Lim}(k)$ with $\forall i \leqslant n \neg P(\alpha, g, i)$ for all $n$. Because $P$ is countable there is a $j \in \mathrm{mon}$ with $\forall i \leqslant n \neg P(\alpha, g|j(n), i)$. Put $\beta(n) := \mathrm{code}(g|j(n))$. Then $\beta \in \mathrm{Coff}(k)$ and $(\alpha^* \circ j)(\beta)(n) = \alpha^*(j(n))(\beta(n)) \geqslant n$. Therefore $(\alpha^* \circ j)(\beta)$ is unbounded and $\alpha^* \circ j$, $\alpha^*$ not a coff.

Kreisel observed in [9] that the quantifier-free axiom of choice is valid in Ct (see also Troelstra [18] for a proof related to his model ECF).



LEMMA 2.6. *Let $P$ be countable. Then*

$$\forall f \in \mathrm{Lim}(\varrho)\ \exists g \in \mathrm{Lim}(\sigma)\ P(f, g, h) \Rightarrow \exists G \in \mathrm{Lim}(\varrho \to \sigma)\ \forall f \in \mathrm{Lim}(\varrho)\ P(f, Gf, h).$$

Proof. Assume the hypothesis with fixed $h$ and define $G(f) = g$, if $g \in \mathrm{Fin}(\sigma)$ and $P(f, g, h)$ and $\mathrm{code}_\sigma(g)$ is minimal with this property.

LEMMA 2.7. *If $P$ is a $\Pi^1_k$-predicate ($k \geqslant 1$) then there is a countable $R$ with*

$$P(\alpha) \Leftrightarrow \forall g \in \mathrm{Lim}(k)\ \exists i R(\alpha, g, i).$$

Proof by induction on $k$. Let $k > 1$ and $P(\alpha) \Leftrightarrow \forall \beta\ Q(\beta, \alpha)$ with $Q \in \Sigma^1_{k-1}$. With ind hyp we have a countable $R_0$ with

$$\begin{aligned}
P(\alpha) &\Leftrightarrow \forall \beta\ \exists g \in \mathrm{Lim}(k-1)\ \forall i R_0(\beta, \alpha, g, i) \\
&\Leftrightarrow \forall \beta \forall f \in \mathrm{Lim}(k)\ \exists g \in \mathrm{Lim}(k-1) R_0(\beta, \alpha, g, f(g)) \\
&\Leftrightarrow \forall f \in \mathrm{Lim}(k)\ \exists g \in \mathrm{Lim}(k-1) R_1(\alpha, g, f(g)) \\
&\Leftrightarrow \forall f \in \mathrm{Lim}(k)\ \exists g \in \mathrm{Fin}(k-1) R_1(\alpha, g, f(g)) \\
&\Leftrightarrow \forall f \in \mathrm{Lim}(k)\ \exists i R_1(\alpha, \mathrm{decode}(i), f(\mathrm{decode}(i))) \\
&\Leftrightarrow \forall f \in \mathrm{Lim}(k)\ \exists i R_2(\alpha, f, i)
\end{aligned}$$

with appropriate countable $R_1$, $R_2$.

COROLLARY 2.8. $\mathrm{Coff}(k+1)$ *is complete in* $\Pi^1_k$ *for* $k \geqslant 1$.

GAMES AND TREES.

We discuss briefly the theorem of Martin and Moschovakis that $\Pi^1_n$ and $\Sigma^1_{n+1}$ for odd $n$ have the prewellordering property assuming projective determinacy (PD). The knowledge of the already classical proof (e.g. Hinman [2]) is presupposed. A prewellordering (pwo) is a binary relation on a set which is reflexive, transitive, connected, and wellfounded. We define the proper field $F$ of a pwo $\preccurlyeq$ by: $\alpha \notin F \Leftrightarrow \forall \alpha (\tilde{\alpha} \preccurlyeq \alpha)$. Let $\preccurlyeq_i$ ($i = 0, 1$) be pwo's on the Baire space with proper fields $F_i$. The union $\preccurlyeq$ of $\preccurlyeq_0$ and $\preccurlyeq_1$ is then defined by:

$$(\tilde{\alpha}, \tilde{\beta}) \preccurlyeq (\alpha, \beta) \Leftrightarrow \tilde{\alpha} \preccurlyeq_0 \alpha \wedge (\tilde{\alpha} \notin F_0 \wedge \alpha \notin F_0 \Rightarrow \tilde{\beta} \preccurlyeq_1 \beta).$$

It is easily verified that $\preccurlyeq$ is a pwo and the proper field $F$ of $\preccurlyeq$ is given by

$$(\alpha, \beta) \in F \Leftrightarrow \alpha \in F_0 \vee \beta \in F_1.$$

Now let $\tau = \varrho \to \sigma$ and suppose that pwo's $\preccurlyeq_\varrho$ with proper field $\neg\mathrm{Coff}(\varrho)$ (the complement of $\mathrm{Coff}(\sigma)$) and $\preccurlyeq_\sigma$ with proper field $\mathrm{Coff}(\sigma)$ are given. If $\preccurlyeq$ denotes the union of $\preccurlyeq_\sigma$ and $\preccurlyeq_\varrho$ and $\alpha(\beta)$ again application on $\mathrm{Coff}(\tau) \times \mathrm{Coff}(\varrho)$ then we have

$$(\alpha(\beta), \beta) \in \text{proper field } (\preccurlyeq) \Leftrightarrow (\beta \in \mathrm{Coff}(\varrho) \Rightarrow \alpha(\beta) \in \mathrm{Coff}(\sigma)).$$

The missing universal quantifier is provided by playing the following game.

Define $G(\tilde{\alpha}, \alpha) \subseteq \mathscr{R} \times \mathscr{R}$ by

$$(\tilde{\beta}, \beta) \in G(\tilde{\alpha}, \alpha) \Leftrightarrow (\tilde{\alpha}(\tilde{\beta}), \tilde{\beta}) \preccurlyeq (\alpha(\beta), \beta)$$

and $\tilde{\alpha} \preccurlyeq_{\tau} \alpha$ by:

*II* has a winning strategy in the game $G(\tilde{\alpha}, \alpha)$, where *I* plays $\tilde{\beta}$ and *II* plays $\beta$, and *II* wins iff $(\tilde{\beta}, \beta) \notin G(\tilde{\alpha}, \alpha)$. With the well-known arguments $\preccurlyeq_{\tau}$ is seen to be a pwo with proper field $\mathrm{Coff}(\tau)$.

$\mathrm{Coff}(0)$ is endowed with the trivial pwo: $\tilde{\alpha} \preccurlyeq_0 \alpha \Leftrightarrow (\alpha \in \mathrm{Coff}(0) \Rightarrow \tilde{\alpha} \in \mathrm{Coff}(0))$. Assume that there are pwo's $\preccurlyeq^n_{\Sigma} \in \Sigma^1_{n-1}$ and $\preccurlyeq^n_{\Pi} \in \Pi^1_{n-1}$ with proper field $\neg\mathrm{Coff}(n)$, which give $\Sigma^1_{n-1}$ the pwo property.

Define games $G_{\Sigma}$ and $G_{\Pi}$ by

$$(\tilde{\beta}, \beta) \notin G_{\Sigma}(\tilde{\alpha}, \alpha) \Leftrightarrow \tilde{\alpha}(\tilde{\beta}) \preccurlyeq_0 \alpha(\beta) \wedge (\tilde{\alpha}(\tilde{\beta}) \notin \mathrm{Coff}(0) \wedge \alpha(\beta) \notin \mathrm{Coff}(0) \Rightarrow \tilde{\beta} \preccurlyeq^n_{\Pi} \beta).$$

and $G_{\Pi}$ analogously with $\Pi$ and $\Sigma$ exchanged and prewellorderings $\preccurlyeq_{\Sigma}$, $\preccurlyeq_{\Pi}$ by

$\tilde{\alpha} \preccurlyeq_{\Sigma} \alpha \Leftrightarrow$ *II* has winning strategy in $G_{\Sigma}(\tilde{\alpha}, \alpha)$,

$\tilde{\alpha} \preccurlyeq_{\Pi} \alpha \Leftrightarrow$ *I* has no winning strategy in $G_{\Pi}(\tilde{\alpha}, \alpha)$.

Then: $\tilde{\alpha} \in \mathrm{Coff}(n+1) \vee \alpha \in \mathrm{Coff}(n+1) \Rightarrow (\tilde{\alpha} \preccurlyeq_{n+1} \alpha \Leftrightarrow \tilde{\alpha} \preccurlyeq_{\Sigma} \alpha \Leftrightarrow \tilde{\alpha} \preccurlyeq_{\Pi} \alpha)$; $\preccurlyeq_{\Sigma}$, $\preccurlyeq_{\Pi}$ and $\preccurlyeq_{n+1}$ with proper field $\mathrm{Coff}(n+1)$ give $\Pi^1_n$ the pwo property.

Lemma 2.7 motivates the notion of a partially wellfounded tree. Assume that a set $A \subseteq \mathscr{R}$ has the following description:

(V) $\alpha \in A \Leftrightarrow \forall \beta \in V\, \exists i\, Q(\alpha, \bar{\beta}(i))$, with $Q$ "simple".

(For example you can find a $\Pi^1_{n-1}$-complete $V$ s.t. for each $\Pi^1_n$-set $A$ there is a recursive $Q$ satisfying (V)). Like in $\Pi^1_1$ try to see $\alpha \in A$ as a tree:

$$s \in T_{\alpha} :\Leftrightarrow s \in \mathrm{Seq} \wedge \forall i \leqslant \mathrm{lh}(s)\, \neg Q(\alpha, s|i).$$

Define: $T$ is a $V$-founded tree $\Leftrightarrow T$ is a tree and $\forall \beta \in V \exists i \bar{\beta}(i) \notin T$.

Then: $\alpha \in A \Leftrightarrow T_{\alpha}$ is a $V$-founded tree.

Let $T$, $T'$ be $V$-founded trees. A function $\sigma\colon \omega \to \omega$ is $V$-monotone from $T$ into $T'$ iff

(i) $\sigma(<>) = <>$,

(ii) $\forall s \in \mathrm{Seq}\; \forall u\, \exists v\, (\sigma(s * \hat{u}) = \sigma(s) * \hat{v})$,

(iii) $\forall \beta \in V\, [\sigma(\beta) \in V \wedge \forall i\, (\bar{\beta}(i) \in T \Rightarrow \overline{\sigma(\beta)}\,(i) \in T')]$,

where $\sigma(\beta)\,(i) = (\sigma(\bar{\beta}(i+1)))_i$.

We define a relation on $V$-founded trees by

$$T \preccurlyeq T' \Leftrightarrow \text{there is } V\text{-monotone } \sigma \text{ from } T' \text{ into } T, \text{ and } \tilde{\alpha} \preccurlyeq \alpha \Leftrightarrow T_{\tilde{\alpha}} \preccurlyeq T_{\alpha}.$$

The relation $\preccurlyeq$ is reflexive and transitive.

If we define the game $G_V(\tilde{\alpha}, \alpha)$ by

$$(\tilde{\beta}, \beta) \notin G_V(\tilde{\alpha}, \alpha) \Leftrightarrow \big(\tilde{\beta} \in V \Rightarrow (\beta \in V \wedge \mu j \cdot \bar{\tilde{\beta}}(j) \notin T_{\tilde{\alpha}} \leqslant \mu j \cdot \bar{\beta}(j) \notin T_{\alpha})\big).$$

then $T_{\tilde{\alpha}} \preccurlyeq T_{\alpha} \Leftrightarrow$ *II* has winning strategy in $G_V(\tilde{\alpha}, \alpha)$.



If all these games $G_V(\tilde{\alpha}, \alpha)$ are determinate, then $\preccurlyeq$ is connected and wellfounded too. Of course these games can also be used to prove the zigzag picture of pwo properties.

**§ 3. Trees of higher types.** To explain the idea we describe the concept of Spector trees first in the classical theory of the Baire space $\mathscr{R}$.

Its topology is induced by the Baire neighborhoods $N(s)$ consisting of the $\alpha$'s with $\bar{\alpha}(\mathrm{lh}(s)) = s$. From a lecture of Kechris [7] we learned the following convenient way of associating a function $\varepsilon$ to $f \in \mathrm{Lim}(2)$ which describes $f$ completely. Say that $\varepsilon$ is an associate of the open set $A \subseteq \mathscr{R}$ iff $A = \bigcup_i N(\varepsilon(i))$. The graph $G_f$ of a partial $f\colon \mathscr{R} \to \omega$ is the set $\{\hat{u} * \alpha \mid f(\alpha) \simeq u\}$ where $(\hat{u} * \alpha)(0) = u$ and $(\hat{u} * \alpha)(i+1) = \alpha(i)$. Similarly the concatenation $s * \alpha$ of a finite sequence $s \in \mathrm{Seq}$ and an infinite sequence $\alpha$. Then $f$ is partially continuous on its open domain iff $G_f$ is open and we call $\varepsilon$ an associate of $f$ iff $\varepsilon$ is an associate of $G_f$. In this case we write $\{\varepsilon\}$ for $f$. The set $\mathrm{Tot} := \{\varepsilon \mid \{\varepsilon\} \text{ is total}\}$ plays the same role as $\mathrm{Coff}(2)$, in particular it is $\Pi^1_1$ and $\Pi^1_1$-complete:

An arbitrary $\Pi^1_1$-set $A$ is of the form $\forall \beta\, \exists i\, R(\alpha, \beta, i)$ with recursive $R$. Define the partial recursive selection functional Sel by

$$\mathrm{Sel}(\alpha, \beta) :\simeq \text{least } i \text{ with } R(\alpha, \beta, i).$$

With the parameter theorem find a primitive recursive functional $f_A\colon \mathscr{R} \to \mathscr{R}$ with

$$\{f_A(\alpha)\}\,(\beta) \simeq \mathrm{Sel}(\alpha, \beta).$$

Then $f_A$ reduces $A$ to Tot.

By using an idea of Spector we assign to each continuous functional $f$ of type 2 a countable ordinal $|f|$ which measures the complexity of $f$. Let $0^1$ be the constant zero function and say that $f$ has no predecessors if $f(0^1) = 0$ and that otherwise $f$ has the predecessors $f * \hat{u}$ for $u \in \omega$ where $f * \hat{u}$ is defined by

$$(f * \hat{u})\,(\alpha) :\simeq f(\hat{u} * \alpha) \dot{-} 1.$$

So we define $|f| := 0$ if $f(0^1) = 0$ and

$$|f| := \sup_u{}^{+} |f * \hat{u}|$$

otherwise. It is useful to iterate the process of getting predecessors:

$$f * s := \begin{cases} f & \text{if } \mathrm{lh}(s) = 0, \\ (f * r) * \hat{u} & \text{if } s = r * \hat{u}. \end{cases}$$

Let $\beta \upharpoonright i := \bar{\beta}(i) * 0^1$. Then the sequence $(\beta \upharpoonright i)$ converges against $\beta$ and the continuous $f$ satisfies the Spector condition

$$\forall \beta\, \exists i\, f(\beta \upharpoonright i) \leqslant i.$$

Because of the equivalence

$$f(\beta \upharpoonright i) \leqslant i \Leftrightarrow (f * \bar{\beta}(i))(0^1) = 0$$

$|f|$ is a welldefined ordinal. Observe that the argument goes through equally well for the set of partially continuous functionals $f$ which are defined for the countably many $\beta \upharpoonright i$ and satisfy the Spector condition. Let Fun $\supseteq$ Tot denote the corresponding set of associates and $|\varepsilon| := |\{\varepsilon\}|$ for $\varepsilon \in$ Fun. The same $f_A$ as above shows that also Fun is $\Pi^1_1$-complete.

The Spector tree $T_\varepsilon \subseteq$ Seq is defined for $\varepsilon \in$ Fun by

$$\bar{\alpha}(n) \in T_\varepsilon \;:\Leftrightarrow\; \forall i \leqslant n \; \{\varepsilon\}(\alpha \upharpoonright i) > i.$$

If $\varepsilon \in$ Fun then $T_\varepsilon$ is a wellfounded tree and the canonical length $|T_\varepsilon|$ of $T_\varepsilon$ is equal to $|\varepsilon|$. Shoenfield's Lemma 2 on page 182 in his book [17] says especially for $\tilde{\varepsilon}, \varepsilon \in$ Fun:

$$|\tilde{\varepsilon}| \leqslant |\varepsilon| \text{ iff there is a monotone mapping from } T_{\tilde{\varepsilon}} \text{ to } T_\varepsilon.$$

The right side is a $\Sigma^1_1$ condition and so the norm $|\cdot|$: Fun $\to \aleph_1$ for the $\Pi^1_1$-complete Fun gives $\Pi^1_1$ the prewellordering property. That this norm is equally well suited for recursion — and proof theoretical purposes is shown by the following table where for a set $M \subseteq \mathrm{Lim}(2)$ $|M|$ is the supremum of all $|f|$ with $f \in M$.

| $M$ | $\lvert M\rvert$ |
|---|---|
| continuous | $\aleph_1$ |
| recursive | $\omega_1$ |
| definable in PR+ | $\varphi\varepsilon_{\Omega_1+1}0$ |
| bar recursion of type 0 | the Bachmann-Howard-ordinal |
| definable in PR | $\varepsilon_0$ |

where PR denotes the set of the Hilbert-Gödel primitive-recursive functionals of finite types.

(The first two lines are classical results, the third line is due to Howard [3] (difficult direction) and the author [19] (simple direction), the last line is connected with the names of Tait, Schwichtenberg (see [16]) and Howard which also studied the fine structure of the hierarchy.)

The generalization is now straightforward. Let the variables $Y$, $c$, $u$ range over continuous functionals of types $\sigma^+ := (0 \to \sigma) \to 0$, $0 \to \sigma$, $\sigma$ resp. Let $c \upharpoonright n$ denote the sequence with $(c \upharpoonright n)(i) = c(i)$ for $i < n$ and $0^\sigma$ otherwise. Then each continuous $Y$ satisfies the Spector condition

$$\forall c \, \exists i \; Y(c \upharpoonright i) \leqslant i.$$

As above the corresponding Spector tree $T_Y \subseteq \mathrm{Seq}_\sigma := \{(u_0, \ldots, u_{n-1}) \mid \forall i < n \; u_i \in \mathrm{Lim}(\sigma)\}$ is wellfounded with the length $|T_Y|$. Let 0 denotes the zero functional of type $0 \to \sigma$ and $Y * \hat{u}$ be defined by

$$(Y * \hat{u})(c) := Y(\hat{u} * c) \dot{-} 1.$$

In case $Y0 > 0$ the $Y * \hat{u}$ are the predecessors of $Y$ and

$$|Y| = \sup\{|Y * \hat{u}| + 1 \mid u \in \mathrm{Lim}(\sigma)\},$$

in case $Y0 = 0$ $|Y| = 0$. Then $|Y| = |T_Y|$. To get back the $Y$ from the predecessors $Y * \hat{u}$ is possible only for "normed" $Y$ and is solved by the following supremum operation $\sup_\sigma$ of type $(\sigma \to \sigma^+) \to \sigma^+$:

$$\sup zc = z(c0)(c^+) + 1$$

where $z$ is of type $\sigma \to \sigma^+$ and $(\hat{u} * c)^+ = c$. Then

$$zu = (\sup z) * \hat{u}.$$

In analogy to the Kreisel–Troelstra set $K$ from [10] we define inductively a set $\mathscr{K}^\sigma \subseteq \mathrm{Lim}(\sigma^+)$ by

1. $0^{\sigma^+} \in \mathscr{K}^\sigma$,
2. $\forall u \in \mathrm{Lim}(\sigma) \; zu \in \mathscr{K}^\sigma \Rightarrow \sup z \in \mathscr{K}^\sigma$.

Then for all $Y \in \mathscr{K}^\sigma$:

$$Y = \sup(\lambda u \cdot Y * \hat{u}).$$

Recursion on the wellfounded trees $T_Y$ is Spector's bar recursion. The bar recursion operator $B^*_{\sigma\tau}$ of type $\sigma^+ \to \tau \to ((\sigma \to \tau) \to \tau) \to \tau$ is defined by

$$Y0 = 0 \Rightarrow B^* YGH = G,$$
$$Y0 > 0 \Rightarrow B^* YGH = H(\lambda u \cdot B^*(Y * \hat{u}) GH).$$

With induction on $|Y|$ one shows that $B^*$ is continuous which is essentially Scarpellini's argument from [15].

The functional $C := \lambda Y \cdot B^* Y0 \sup$ is a retraction from $\mathrm{Lim}(\sigma^+)$ onto $\mathscr{K}^\sigma$ with $T_Y = T_{CY}$ and $|Y| = |CY|$. So, if we want, we can restrict ourselves to $\mathscr{K}$.

For $\tilde{T}$, $T \subseteq \mathrm{Seq}_\sigma$ $F: \tilde{T} \to T$ is monotone iff $F$ transports branches in $\tilde{T}$ into branches in $T$. Then

$$|\tilde{Y}| \leqslant |Y| \Leftrightarrow \exists F: \tilde{T} \to T \text{ monotone}.$$

The argument is given by playing the following game $\bar{G}(\tilde{Y}, Y)$:

$$(\tilde{c}, c) \notin \bar{G}(\tilde{Y}, Y) \;:\Leftrightarrow\; \text{least } i \cdot \tilde{Y}(\tilde{c} \upharpoonright i) \leqslant \text{least } j \cdot Y(c \upharpoonright j) \leqslant j.$$

$I$ plays $\tilde{c}$, $II$ plays $c$ and $I$ wins iff $(\tilde{c}, c) \in \bar{G}(\tilde{Y}, Y)$. This is an open game in the usual sense and so by open game determinancy either $I$ or $II$ has a winning strategy (ws). If $II$ has a ws then there is a monotone $F$: $T_{\tilde{Y}} \to T_Y$; if $I$ has a ws then there is a $u \in \mathrm{Lim}(\sigma)$ and a monotone $F$: $T_Y \to T_{Y * \hat{u}}$.

For $M \subseteq \mathrm{Lim}(\sigma^+)$ $|M|$ is the supremum of all $|Y|$ with $Y \in M$. The function sp: $\omega \to \{\text{types}\}$ is defined by $\mathrm{sp}(0) = 0$ and $\mathrm{sp}(n+1) = \mathrm{sp}(n)^+$ and

$$\underset{\sim}{\gamma}_n := |\mathrm{Lim}(\mathrm{sp}(n+1))|.$$

$\underset{\sim}{\gamma}_0 = |\mathrm{Lim}(2)| = \aleph_1$ and by going down to the associates it is clear that $\underset{\sim}{\gamma}_n \leqslant \underset{\sim}{\pi}_n$, where $\underset{\sim}{\pi}_n$ is the supremum of the $\underset{\sim}{\Pi}_n^1$ prewellorderings of the Baire space.

The proof of the following unpublished result of D. Normann is included here with his permission.

THEOREM 3.1 (Normann). $\underset{\sim}{\gamma}_n = \underset{\sim}{\pi}_n$.

Proof. Let $T$ be a $\underset{\sim}{\Pi}_n^1$-tree over the Baire space. We construct a continuous functional $Y$ of type $(n+1)^+$ with $|Y| \geqslant$ lenght of $T$. To this end choose a countable predicate $R$ with

$$\alpha \in T \Leftrightarrow \forall g \in \mathrm{Lim}(n)\, \exists p\, R(\alpha, g, p)$$

and define $u_\alpha$, $u_\alpha^*$ by

$$u_\alpha(g) \simeq \text{least } p \cdot R(\alpha, g, p) \quad \text{and} \quad u_\alpha^* = \langle \alpha, u_\alpha \rangle,$$

where $\langle\ \rangle$ here and in the following is an appropriate coding functional. Define a new wellfounded tree $T^*$ by

$$T^* = \{\langle u^*_{\langle \alpha_0 \rangle}, \ldots, u^*_{\langle \alpha_0, \ldots, \alpha_n \rangle}\rangle : \langle \alpha_0, \ldots, \alpha_n \rangle \in T\}.$$

Obviously the length of $T$ is less than or equal to the length of $T^*$. The crucial observation is that for an arbitrary (total) continuous functional $v$ of type $n+1$ if we know $v \notin T^*$ we can continuously verify this fact. Because of the encoding of the parameter this boils down to finding a continuous $g$ with $u(g) \neq u_\alpha(g)$ under the proviso $u \neq u_\alpha$ for a fixed $\alpha$.

We have to consider two cases. If $u_\alpha$ is total we shall find eventually a finite functional $g$ with $u(g) \neq u_\alpha(g)$. If $u_\alpha$ is not defined or equivalently $\neg R(\alpha, g, p)$ for all $p$, then for $k := u(g)$ and large $i$ we have $\neg R(\alpha, g|i, k)$ and $u(g|i) = k$, so again $u(g|i) \neq u_\alpha(g|i)$. Therefore the following definition gives actually a total continuous functional $Y$ of type $(n+1)^+$:

$Y(c) :=$ least $\langle i, e \rangle$ (in $e$ steps we know $\langle c_0, \ldots, c_i \rangle \notin T^*$).

$Y$ has the property that if $\langle c_0, \ldots, c_{i-1} \rangle \in T^*$ then $Y(c|i) > i$ which shows $|Y| \geqslant$ length $(T^*) \geqslant$ length $(T)$.

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