[E] W. T. Eaton, *The sum of solid spheres*, Michigan Math. J. 19 (1972), pp. 193–207.
[E, G] R. D. Edwards and L. C. Glaser, *A method of shrinking decompositions of certain manifolds*, Trans. Amer. Math. Soc. 165 (1972), pp. 45–56.
[H] N. Hosay, *Erratum to "The sum of a cube and a crumpled cube is S^3"*, Notices Amer. Math. Soc. 11 (1964), p. 152.
[H, W] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, N. J., 1948.
[K] L. V. Keldys, *Topological Imbeddings in Euclidean Space*, Proc. Steklov Inst. Math. 81 (1966); English transl., Amer. Math. Soc., Providence, R. I., 1968. MR 34 # 6745; 38 # 696.
[L] L. L. Lininger, *Some results on crumpled cubes*, Trans. Amer. Math. Soc. 118 (1965), pp. 534–549. MR 31 # 2717.
[Mc1] L. F. McAuley, *Some upper semi-continuous decompositions of E^3 into E^3*, Ann. of Math. (2) 73 (1961), pp. 437–457. MR 23 # A3554.
[Mc2] — *Upper semicontinuous decompositions of E^3 into E^3 and generalizations to metric spaces*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, 21–26. MR 25 # 4502.
[M] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), pp. 416–428.
[P] T. M. Price, *Decompositions of S^3 and pseudo-isotopies*, Trans. Amer. Math. Soc. 140 (1969), pp. 295–299.
[R] M. J. Reed, *Decomposition spaces and separation properties*, Doctoral Dissertation, SUNY, Binghamton, 1971.
[S] L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), pp. 271–294.
[S, W] M. Starbird and E. Woodruff, *Decompositions of E^3 with countably many non-degenerate elements*, Geometric Topology, edited by J. C. Cantrell, Academic Press, 1979, pp. 239–252.
[Wh] G. T. Whyburn, *Analytic Topology*, Amer, Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942.
[W] E. P. Woodruff, *Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries*, Trans. Amer. Math. Soc. 232 (1977), pp. 195–204.
[W1] — *Decomposition of E^3 into cellular sets*, Geometric Topology edited by J. C. Cantrell, Academic Press, 1979, 253–257.

THE INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey 08540
and
TRENTON STATE COLLEGE
Trenton, New Jersey 08625

Accepté par la Rédaction 6. 6. 1981

ω-Trees in stationary logic

by

A. Baudisch, D. G. Seese and **H. P. Tuschik** (Berlin)

Abstract. It is proved that for all trees $\underline{A}$, $\underline{B}$ of height at most ω $\underline{A} \equiv \underline{B}(Q_1)$ implies $\underline{A} \equiv \underline{B}(aa)$. Moreover all such trees are finitely determinate and the theory of the class of all trees of height at most ω in stationary logic is decidable.

Preliminaries. The study of stationary logic $L_{\omega\omega}(aa)$ was begun by J. Barwise, M. Kaufmann and M. Makkai [1], following a suggestion of S. Shelah [8]. In their paper Barwise, Kaufmann and Makkai proved Completeness, Compactness, Downward–Löwenheim–Skolem-Theorem and Omitting Types theorems for stationary logic. The quantifier Q_1 "there exist uncountably many" is definable in stationary logic. Thus $L_{\omega\omega}(Q_1)$ is a sublogic of $L_{\omega\omega}(aa)$. We assume the reader familiar with stationary logic.

Throughout this paper L denotes an elementary language for partially ordered structures with finitely many individual constants and predicates eventually.

Structures for L are denoted by $\underline{A}$, $\underline{B}$, etc. and their universes $|\underline{A}|$, $|\underline{B}|$, etc. by the corresponding capital letters A, B, etc. For a set M let $P_{\omega_1}(M)$ denote the set of all countable subsets of M.

A set $\underset{\sim}{A} \subseteq P_{\omega_1}(A)$ is unbounded if every $B \in P_{\omega_1}(A)$ is a subset of some $C \in \underset{\sim}{A}$. $\underset{\sim}{A}$ is closed if the union of each increasing sequence $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of elements of $\underset{\sim}{A}$ is again an element of $\underset{\sim}{A}$.

Closed and unbounded (cub) subsets of $P_{\omega_1}(A)$, $P_{\omega_1}(B)$, etc. are denoted by $\underset{\sim}{A}$, $\underset{\sim}{B}$, etc.

To get $L(aa)$ we expand L by adding countably many set variables $X_1, X_2, \ldots$, the $\in$ symbol and a new quantifier aa. Formulas of $L(aa)$ are formed as usual with the new formation rule:

if φ is a formula of $L(aa)$ so is $(aaX)\varphi$ for each set variable X.

For an L-structure $\underline{A}$, $\underline{A} \models (aaX\ \varphi(X)$ holds iff there is a cub collection $\underset{\sim}{A} \subseteq P_{\omega_1}(A)$ such that for all $B \in \underset{\sim}{A}\ \underline{A} \models \varphi(B)$ hold. Let K be a class of structures for L then $\mathrm{Th}_{aa}(K)$, $\mathrm{Th}_1(K)$ denote the theory of K in the language $L(aa)$, $L(Q_1)$ respectively. In case that K has only one element $\underline{A}$ we write $\mathrm{Th}_{aa}(\underline{A})$, $\mathrm{Th}_1(\underline{A})$ instead of $\mathrm{Th}_{aa}(K)$, $\mathrm{Th}_1(K)$ respectively.

By induction on the complexity of the formulas φ of $L(aa)$ we define the quantifier rank of φ

$$\operatorname{qr}\varphi := 0 \quad \text{if } \varphi \text{ is atomic},$$
$$\operatorname{qr}\varphi\wedge\psi := \max(\{\operatorname{qr}\varphi,\ \operatorname{qr}\psi\}),$$
$$\operatorname{qr}\neg\varphi := \operatorname{qr}\varphi,$$
$$\operatorname{qr}(aaX)\varphi := \operatorname{qr}(\exists x)\varphi := \operatorname{qr}\varphi+1.$$

If φ is a formula of $L(Q_1)$, then we have to add the clause

$$\operatorname{qr}(Q_1 x)\varphi := \operatorname{qr}\varphi+1.$$

For two L structures $\underline{A}$ and $\underline{B}$ and a natural number n we write

$\underline{A}\equiv_n\underline{B}(aa)$ iff for all sentences φ of $L(aa)$ with

$$\operatorname{qr}\varphi\leqslant n:\ \underline{A}\models\varphi \text{ iff } \underline{B}\models\varphi.$$

We write

$\underline{A}\equiv_n\underline{B}(Q_1)$ iff for all sentences φ of $L(Q_1)$ with

$$\operatorname{qr}\varphi\leqslant n:\ \underline{A}\models\varphi \text{ iff } \underline{B}\models\varphi.$$

In this case $\underline{A}$ and $\underline{B}$ are said to be *(aa, n)-equivalent*, (Q_1, n)-equivalent respectively. We write $\underline{A}\equiv\underline{B}(aa)$ $(\underline{A}\equiv\underline{B}(Q_1))$ if $\underline{A}\equiv_n\underline{B}(aa)$ $(\underline{A}\equiv_n\underline{B}(Q_1)$ respectively) holds for all $n\in\omega$. Following Kaufmann [5] we define for an elementary language L, a natural number k and an L-structure $\underline{A}$:

$\underline{A}$ is finitely k-determinate if every sentence of $L(aa)$ which has the following form, and has quantifier rank at most k, is true in $\underline{A}$

$$(aa\bar{X})(\forall\bar{x})\big((aaY)\varphi(\bar{X},\bar{x},Y)\vee(aaY)\neg\varphi(\bar{X},\bar{x},Y)\big).$$

$\underline{A}$ is finitely determinate if $\underline{A}$ is finitely k-determinate for each $k\in\omega$.

Finitely determinate structures have nice model theoretic properties. A lot of results can be found in Kaufmann [5] and in Eklof and Mekler [3]. The following is taken from Kaufmann [5].

Definition. Fix two L-structures $\underline{A}$ and $\underline{B}$. $(F_k: k\leqslant n)$ is a determinate (aa, n) *back-and-forth system from $\underline{A}$ to $\underline{B}$* if it satisfies the following conditions:

(1) for all $f\in F_0$, $f\subseteq[A\times B]\cup[P_{\omega_1}(A)\times P_{\omega_1}(B)]$ and for every atomic formula φ, $\underline{A}\models\varphi(\overline{\operatorname{dom}f})$ iff $\underline{B}\models\varphi(\overline{\operatorname{rn}f})$. Here $\overline{\operatorname{dom}f}$ and $\overline{\operatorname{rn}f}$ denote corresponding enumerations of $\operatorname{dom}f$ and $\operatorname{rn}f$.

(2) $O\in F_n$.

(3) Whenever $k<l\leqslant n$ and $f\in F_l$:
  (i) $(\forall a\in A)(\exists b\in B)\,[f\cup\{(a, b)\}]\in F_k$,
  (ii) $(\forall b\in B)(\exists a\in A)\,[f\cup\{(a, b)\}]\in F_k$,
  (iii) there are cub collections $\underline{\underline{C}}\subseteq P_{\omega_1}(A)$ and $\underline{\underline{D}}\subseteq P_{\omega_1}(B)$ such that $(\forall C\in\underline{\underline{C}})(\forall D\in\underline{\underline{D}})\,[f\cup\{(C, D)\}]\in F_k$.

Theorem 1 (Kaufmann [5]). *If L is a finite language, $\underline{A}$ and $\underline{B}$ are L-structures and n is a natural number, then the following are equivalent:*
(i) *There is a determinate (aa, n) back-and-forth system from $\underline{A}$ to $\underline{B}$.*
(ii) *$\underline{A}\equiv_n\underline{B}(aa)$ and $\underline{A}$ is (and $\underline{B}$ is) finitely n-determinate.*

A partially ordered set $(A,\leqslant)$ is a tree, if for each $a\in A$, the set

$$\hat{a} := \{b:\ b\in A \text{ and } b<a\}$$

is well ordered by $<$. We define the height $h_{\underline{A}}(a)$ for each tree $\underline{A}$ and each $a\in A$ by:

$$h_{\underline{A}}(a) := \{h_{\underline{A}}(b):\ b<a\}.$$

The height $h(\underline{A})$ of the tree $\underline{A}$ we define by

$$h(\underline{A}) := \bigcup_{a\in A}\big(h_{\underline{A}}(a)\cup\{h_{\underline{A}}(a)\}\big).$$

Trees of height $\leqslant\omega$ we shall denote as ω-trees. A tree $\underline{A}$ is said to be connected if $\underline{A}\models\exists x\forall y\,(x\leqslant y)$. For each $a\in A$ we define

$$A_a := \{b:\ b\in A \text{ and } a\leqslant b\} \quad\text{and}\quad \underline{A}_a := (A_a,\ \leqslant\restriction A_a).$$

Q_1-equivalence for ω-trees. Let $\underline{A}$ and $\underline{B}$ be connected ω-trees of cardinality $\aleph_1$. We assume that n is a natural number greater 0 and that

$$f = \{(a_1, b_1), \ldots, (a_k, b_k), (A_1, B_1), \ldots, (A_l, B_l)\}$$

is a partial function from $A\cup P_{\omega_1}(A)$ in $B\cup P_{\omega_1}(B)$. Moreover we define

$$\overline{\operatorname{dom}f} := (a_1, \ldots, a_k, A_1, \ldots, A_l),$$
$$\overline{\operatorname{rn}f} := (b_1, \ldots, b_k, B_1, \ldots, B_l),$$
$$f' := \{(A_1, B_1), \ldots, (A_l, B_l)\} = f\cap(P_{\omega_1}(A)\times P_{\omega_1}(B)),$$
$$(\overline{\operatorname{dom}f'})_a := (A_1\cap A_a, \ldots, A_l\cap A_a) \quad\text{for each } a\in A$$

and

$$(\overline{\operatorname{rn}f'})_b := (B_1\cap B_b, \ldots, B_l\cap B_b) \quad\text{for each } b\in B.$$

For all $a_1, a_2\in A_a$ and all $b_1, b_2\in B_b$ we define:

$$a_1\ {}_a\!\sim_{4n}\ a_2 \text{ iff } (\underline{A}_a, (\overline{\operatorname{dom}f'})_a, a_1)\equiv_{4n}(\underline{A}_a, (\overline{\operatorname{dom}f'})_a, a_2)(Q_1),$$
$$b_1\ {}_b\!\sim_{4n}\ b_2 \text{ iff } (\underline{B}_b, (\overline{\operatorname{rn}f'})_b, b_1)\equiv_{4n}(\underline{B}_b, (\overline{\operatorname{rn}f'})_b, b_2)(Q_1).$$

It is obvious that ${}_a\!\sim_{4n}$, ${}_b\!\sim_{4n}$ are equivalence relations on A_a, B_b respectively.

Let $E_1^a, \ldots, E_{s(a)}^a$ $(E_1^b, \ldots, E_{s(b)}^b$ respectively) be all uncountable equivalence classes of ${}_a\!\sim_{4n}$ $({}_b\!\sim_{4n}$ respectively). Moreover let E^a $(E^b$ respectively) be the union of all equivalence classes of ${}_a\!\sim_{4n}$ $({}_b\!\sim_{4n}$ respectively) which are at most countable.

Now we assume that C, D are subsets of A, B respectively with the following properties

(i) card $(C) = \aleph_0$ and $\operatorname{dom}(f) \cap A \subseteq C$,
card $(D) = \aleph_0$ and $\operatorname{rn}(f) \cap B \subseteq D$,

(ii) $(\underline{A}, C) \models \forall x \forall y\ x \in C \wedge y \leqslant x \to y \in C$,
$(\underline{B}, D) \models \forall x \forall y\ x \in D \wedge y \leqslant x \to y \in D$,

(iii) $E^a \subseteq C$ and card $(C \cap E_i^a) = \aleph_0$ for all $a \in C$ and $1 \leqslant i \leqslant s(a)$,
$E^b \subseteq D$ and card $(D \cap E_i^b) = \aleph_0$ for all $b \in D$ and $1 \leqslant i \leqslant s(b)$.

LEMMA 2. *Let $\underline{A}$, $\underline{B}$, n and f be given as above and let C_1, D_1 be countable subsets of A, B respectively. Then there exist C, D with $C_1 \subseteq C \subseteq A$ and $D_1 \subseteq D \subseteq B$ such that C and D have the above properties* (i), (ii) *and* (iii).

This lemma is an easy consequence of the above definitions. The essential point is that the height of the regarded trees is smaller than ω_1.

Also the proof of the following lemma is a simple consequence of the above definitions and is left to the reader.

LEMMA 3. *Let $\underline{A}$, $\underline{B}$, n and f be given as above. For each $i \in \omega$ let C_i, D_i be countable subsets of A, B respectively with the properties* (i), (ii) *and* (iii). *Then $\bigcup_{i\in\omega} C_i$, $\bigcup_{i\in\omega} D_i$ have the properties* (i), (ii) *and* (iii), *too.*

LEMMA 4. *Let n be a natural number, $\underline{A}$ be an ω-tree and let a, b and c be elements of A. Moreover let $\overline{A'} = (A_1', \ldots, A_k')$ with $A_i' \in P_{\omega_1}(A)$ (for $1 \leqslant i \leqslant k$) and let $\bar{d} = (d_1, \ldots, d_l)$ with $d_j \in A$ (for $1 \leqslant j \leqslant l$) be given in such a way that $d_j \notin (A_a \backslash \{a\})$ (for all j: $1 \leqslant j \leqslant l$) and that*

$$(+)\quad (\underline{A}_a, A_1' \cap A_a, \ldots, A_k' \cap A_a, b) \equiv_{4n} (\underline{A}_a, A_1' \cap A_a, \ldots, A_k' \cap A_a, c)\ (Q_1).$$

Then $(\underline{A}, \overline{A'}, \bar{d}, b) \equiv_{4n} (\underline{A}, \overline{A'}, \bar{d}, c)$ (Q_1) holds.

The proof is left as an exercise for the reader (hint: use Lippner–Vinner games; player II has to play in $(A \backslash A_a)$ isomorphic and in A_a using the winning strategy for player II in the game corresponding to $(+)$).

LEMMA 5. *Let $\underline{A}$, $\underline{B}$, n $(n > 0)$ and f be as above and let C, D be subsets of A, B respectively with the properties* (i), (ii) *and* (iii). *Then $(\underline{A}, \overline{\operatorname{dom} f}) \equiv_{4n} (\underline{B}, \overline{\operatorname{rn} f})\ (Q_1)$ implies $(\underline{A}, \overline{\operatorname{dom} f}, C) \equiv_{n-1} (\underline{B}, \overline{\operatorname{rn} f}, D)$ (Q_1).*

Proof. To prove $(\underline{A}, \overline{\operatorname{dom} f}, C) \equiv_{n-1} (\underline{B}, \overline{\operatorname{rn} f}, D)$ (Q_1) we shall use Lippner–Vinner games. In the following we assume that the reader is familiar with the game.

Hence we have to prove that player II has a winning strategy in the corresponding game over $(n-1)$ rounds.

We assume that the game is played over $m < n-1$ rounds and that $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ are the already chosen elements. Moreover we assume that there are elements c_i', c_i'', c_i''', d_i', d_i'' and d_i''' (for each i with $1 \leqslant i \leqslant m$) such that the following conditions (1), …, (8) and (3′), …, (8′) are fulfilled

(1) $c_i, c_i', c_i'', c_i''' \in A$ $(1 \leqslant i \leqslant m)$ and $d_i, d_i', d_i'', d_i''' \in B$ $(1 \leqslant i \leqslant m)$.

(2) $(\underline{A}, \overline{\operatorname{dom} f}, c_1, c_1', c_1'', c_1''', \ldots, c_m, c_m', c_m'', c_m''') \equiv_{4n-4m} (\underline{B}, \overline{\operatorname{rn} f}, d_1, d_1', d_1'', d_1''', \ldots, d_m, d_m', d_m'', d_m''')$ (Q_1).

(3) If $c_i \in C$ and there is an $a \in (\operatorname{dom} f) \cap A$ with $c_i \leqslant a$, then $c_i = c_i' = c_i'' = c_i'''$.

(4) If $c_i \in C$ and there is an $j < i$ with $c_j \in C$ and $c_i \leqslant c_j$, then $c_i = c_i' = c_i'' = c_i'''$.

(5) If $c_i \in C$ and there is an $j < i$ with $c_j''' < c_j$ and $c_i \leqslant c_j'''$, then $c_i = c_i' = c_i'' = c_i'''$.

(6) If $c_i \in C$ but fulfils not the premise of (3), (4) and (5), then $c_i = c_i'' = c_i'''$ and

$$\begin{aligned} c_i' = \min\{c\colon\ c \in A \,\&\, c \leqslant c_i \,\&\, & c \text{ fulfils not the premise of (3)} \\ \&\, & c \text{ fulfils not the premise of (4)} \\ \&\, & c \text{ fulfils not the premise of (5)}\}. \end{aligned}$$

(7) If $c_i \notin C$ and there is an $j < i$ with $c_i \geqslant c_j''$ and $c_j'' \notin C$, then $c_i'' = c_j''$, $c_i''' = c_j'''$ and $c_i' = c_i$.

(8) If $c_i \notin C$ and there is no $j < i$ with $c_i \geqslant c_j''$ and $c_j'' \notin C$, then c_i''' is the immediate predecessor of c_i'',

$$c_i'' = \min\{c\colon\ c \in A \,\&\, c \leqslant c_i \,\&\, c \notin C\} \text{ and}$$
$$\begin{aligned} c_i' = \min\{c\colon\ c \in A \,\&\, c < c_i'' \,\&\, & (\text{there is no } a \in (\operatorname{dom} f) \cap A \\ & \text{with } c \leqslant a) \,\&\, (\text{there is no } j < i \text{ with } c \leqslant c_j)\}. \end{aligned}$$

The properties (3), …, (8) hold for all i with $1 \leqslant i \leqslant m$. We get the properties (3′), …, (8′) from the properties (3), …, (8) respectively by a substitution of C, c_i, c_j, c_i', …, $\operatorname{dom} f$, A by D, d_i, d_i', …, $\operatorname{rn} f$, B respectively.

Now we suppose that player I has chosen an element $c_{m+1} \in A$. Then player II has to proceed as follows. We have to regard some cases.

Case 1. $c_{m+1} \in C$ and there is an $a \in (\operatorname{dom} f) \cap A$ with $c_{m+1} \leqslant a$. At first we set $c_{m+1}' = c_{m+1}'' = c_{m+1}''' = c_{m+1}$. By (2) there exist elements d_{m+1}, d_{m+1}', d_{m+1}'', $d_{m+1}''' \in B$ with

$$(*)\quad (\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c_{m+1}''') \equiv_{4n-4(m+1)} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_{m+1}''')(Q_1).$$

Hence $d_{m+1} = d_{m+1}' = d_{m+1}'' = d_{m+1}'''$ and there is an element b from $(\operatorname{rn} f) \cap B$ with $d_{m+1} \leqslant b$. We get $d_{m+1} \in D$ by property (i) and property (i) of D.

Case 2. $c_{m+1} \in C$ and there is a $j \leqslant m$ with $c_j \in C$ and $c_{m+1} \leqslant c_j$. We set $c_{m+1}' = c_{m+1}'' = c_{m+1}''' = c_{m+1}$ and proceed as in Case 1.

Case 3. $c_{m+1} \in C$ and there is a $j \leqslant m$ with $c_j''' < c_j$ and $c_{m+1} \leqslant c_j'''$. Let j_0 be the minimal j with this property. We set $c_{m+1}' = c_{m+1}'' = c_{m+1}''' = c_{m+1}$ and proceed at first as in case 1. Hence there exist elements d_{m+1}, d_{m+1}', d_{m+1}'', $d_{m+1}''' \in B$ which fulfil $(*)$. Hence we get $d_{j_0}''' < d_{j_0}$ and $d_{m+1} \leqslant d_{j_0}''$. Using (3′), …, (7′) we get from (8′) that $d_{j_0}'' = \min\{d\colon d \in B \,\&\, d \leqslant d_{j_0} \,\&\, d \notin D\}$ and that

d_{j_0}''' is the immediate predecessor of d_{j_0}''. Hence we get $d_{m+1} \in D$, since $d_{m+1} \leqslant d_{j_0}'''$.

Case 4. $c_{m+1} \in C$ and the premises of (3), (4) and (5) are not fulfilled. Then set $c_{m+1}'' = c_{m+1}''' = c_{m+1}$ and

$$\begin{aligned} c_{m+1}' = \min\{c\colon\ & c \in A \mathbin{\&} c \leqslant c_{m+1} \mathbin{\&} c \text{ fulfils not the premise of (3)} \\ & \mathbin{\&} c \text{ fulfils not the premise of (4)} \\ & \mathbin{\&} c \text{ fulfils not the premise of (5)}\}. \end{aligned}$$

Now let $d_{m+1}^{+} \in B$ be such that

$$(**)\quad (\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c_m''', c_{m+1}') \equiv_{4n-4m-1} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}^{+}) (Q_1).$$

Using $m < n-1$ we get by (**) that

$$\begin{aligned} d_{m+1}^{+} = \min\{d\colon\ & d \in B \mathbin{\&} d \leqslant d_{m+1}^{+} \mathbin{\&} d \text{ fulfils not the premise of (3')} \\ & \mathbin{\&} d \text{ fulfils not the premise of (4')} \\ & \mathbin{\&} d \text{ fulfils not the premise of (5')}\}. \end{aligned}$$

If $d_{m+1}^{+} \in D$, then we set $d_{m+1}' = d_{m+1}^{+}$ and get (***) from (**):

$$(***)\quad (\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c_m''', c_{m+1}') \equiv_{4n-4m-1} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}') (Q_1).$$

If $d_{m+1}^{+} \notin D$, then we proceed as follows.

Obviously all elements $d \in B$ with $d < d_{m+1}^{+}$ are in D, since for these elements the premises of (3'), (4') or (5') are fulfilled. Let d^{++} be the immediate predecessor of d_{m+1}^{+}, i.e. the maximal element of B which is smaller than d_{m+1}^{+} (i.e. $d^{++} = \max\{d\colon d \in B \mathbin{\&} d < d_{m+1}^{+}\}$).

Using property (iii) of D for d^{++} we get the existence of an element d^{+++} with the following properties

$$\begin{aligned} & d^{+++} \in D, \\ & d_{m+1} \,{}_{d^{++}}{\sim_{4n}}\, d^{+++}, \\ & (\operatorname{rn} f) \cap B_{d^{+++}} = \emptyset, \\ & d_1, d_1', d_1'', d_1''', \ldots, d_m, d_m', d_m'', d_m''' \notin B_{d^{+++}}, \\ & (B_1 \cap B_{d^{+++}}) \cup \ldots \cup (B_l \cap B_{d^{+++}}) = \emptyset \end{aligned}$$

(hint: The equivalence class of d_{m+1}^{+} with respect to ${}_{d^{++}}{\sim_{4n}}$ is uncountable since $d_{m+1}^{+} \notin D$).

Between d^{++} and d^{+++} there are no elements of B (i.e. d^{++} is the immediate predecessor of d^{+++}), since $m < n-1$.

Now we set $d_{m+1}' = d^{+++}$. We know that d^{++} is the immediate predecessor of d^{+++} and of d_{m+1}^{+} and that $d_{m+1}^{+} \,{}_{d^{++}}{\sim_{4n}}\, d^{+++}$ holds. Hence we get by Lemma 4, the transitivity of $\equiv_{4n}$ (Q_1) and (**) that (***) holds.

Using (***) we get the existence of an element d_{m+1}^{-} with the property:

$$(\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c_m''', c_{m+1}', c_{m+1}) \equiv_{4n-4m-2} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}', d_{m+1}^{-}) (Q_1).$$

If $d_{m+1}^{-} \in D$, then set $d_{m+1} = d_{m+1}'' = d_{m+1}^{-}$.

If $d_{m+1}^{-} \notin D$, then we use property (iii) for d_{m+1}' and get similarly to the above an element $d_{m+1} \in D$ with

$$(\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}', d_{m+1}^{-}) \equiv_{4n-4m-2} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}', d_{m+1}) (Q_1).$$

If we set now $d_{m+1}'' = d_{m+1}''' = d_{m+1}$, then we get (*).

Case 5. $c_{m+1} \notin C$ and there exists an $j < m+1$ with $c_{m+1} \geqslant c_j''$ and $c_j'' \notin C$. At first we set $c_{m+1}'' = c_j''$, $c_{m+1}''' = c_j'''$ and $c_{m+1}' = c_{m+1}$. Using (2) we get the existence of elements $d_{m+1}, d_{m+1}', d_{m+1}''$ and $d_{m+1}''' \in B$ such that (*) is fulfilled. Hence $d_{m+1}'' = d_j''$, $d_{m+1}''' = d_j'''$, $d_{m+1}' = d_{m+1}$ and d_{m+1}''' is the immediate predecessor of d_{m+1}''. $d_j'' \notin D$ and $d_j''' \in D$ by (8'). Hence $d_{m+1} \notin D$.

Case 6. $c_{m+1} \notin C$ and there is no $j < m+1$ with $c_{m+1} \geqslant c_j''$ and $c_j'' \notin C$. Then we set

$$\begin{aligned} c_{m+1}'' &= \min\{c\colon c \in A \mathbin{\&} c \leqslant c_{m+1} \mathbin{\&} c \notin C\} \quad \text{and} \\ c_{m+1}' &= \min\{c\colon c \in A \mathbin{\&} c < c_{m+1}'' \mathbin{\&} (\text{there is no } a \in (\operatorname{dom} f) \cap A \\ &\qquad \text{with } c \leqslant a) \mathbin{\&} (\text{there is no } j < m+1 \text{ with } c \leqslant c_j)\}. \end{aligned}$$

Let c_{m+1}''' be the immediate predecessor of c_{m+1}''. Similarly as in Case 4 it is possible to show that there exists an element $d_{m+1}' \in D$ with (***). Moreover d_{m+1}' has the property that if its immediate predecessor d exists, then there exists an element $b\ (\operatorname{rn} f) \cap B$ with $d \leqslant b$ or there exists an $j < m+1$ with $d \leqslant d_j$. This we get by (***) using $m < n-1$. Using the same methods as above in Case 4 we get the existence of an element $d_{m+1}''' \in D$ with

$$(\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c_m''', c_{m+1}', c_{m+1}''') \equiv_{4n-4m-2} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}', d_{m+1}''') (Q_1).$$

Hence there exists an element d_{m+1}^{++} with

$$(\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c_m''', c_{m+1}', c_{m+1}''', c_{m+1}'') \equiv_{4n-4m-3} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d_m''', d_{m+1}', d_{m+1}''', d_{m+1}^{++}) (Q_1)$$

such that d_{m+1}''' is the immediate predecessor of d_{m+1}^{++}. If $d_{m+1}^{++} \notin D$, then set $d_{m+1}'' = d_{m+1}^{++}$. If this is not the case, then proceed as follows. By $c_{m+1}'' \notin C$, the above equivalence and property (iii) of C we get the existence of a natural number i_0 with $1 \leqslant i_0 \leqslant s(d_{m+1}''')$ such that $d_{m+1}^{++} \in E_{i_0}^{d_{m+1}'''}$. $E_{i_0}^{d_{m+1}'''}$ contains only elements which have d'''_{m+1} as immediate predecessor, since $n \neq 0$. But D is countable. Hence there is an element $d''_{m+1} \in E_{i_0}^{d'''_{m+1}} \setminus D$. Now we use Lemma 4 and get:

$$(\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c'''_m, c'_{m+1}, c''_{m+1}, c'''_{m+1}) \equiv_{4n-4m-3} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d'''_m, d'_{m+1}, d''_{m+1}, d'''_{m+1})\,(Q_1).$$

But using property (ii) of D this gives the existence of an element $d_{m+1} \in B \setminus D$ with (*).

If player I chooses a subset $A' \subseteq A$ of cardinality $\aleph_1$, then at first player II has to find a set $B' \subseteq B$ of cardinality $\aleph_1$. To do this he has to proceed as follows.

Using $\operatorname{cf}(\omega_1) = \omega_1$, property (i) and (ii) of C we get the existence of an element c'''_{m+1} and of a subset $A'' \subseteq A'$ with the following properties:

$\operatorname{card}(A'') = \aleph_1$,
all elements of A'' have the same height,
$A'' \cap C = \emptyset$,
$c'''_{m+1} \in C$,
$A'' \subseteq A_{c'''_{m+1}}$,
$(\hat{a} \setminus \hat{c}'''_{m+1}) \cap C = \{c'''_{m+1}\}$ for all $a \in A''$.

By (2) there exists an element $d'''_{m+1} \in D$ with

(o) $\quad (\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c'''_m, c'''_{m+1}) \equiv_{4n-4m-1} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d'''_m, d'''_{m+1})\,(Q_1)$.

Player II has in the game corresponding to (o) a winning strategy. We assume that player I has chosen in this game the set A''. Let B''' be those set, which player II chooses in this game using his winning strategy. B''' has a subset B'' with the following properties:

$\operatorname{card}(B'') = \aleph_1$,
$B'' \cap 0 = \emptyset$,
$B'' \subseteq B_{d'''_{m+1}}$ and $(\hat{b} \setminus \hat{d}'''_{m+1}) \cap D = \{d'''_{m+1}\}$ for all B''.

Player II chooses now in the original as B' this set B''. Player I can now choose an element $d_{m+1} \in B' = B''$. By (o) player II gets some $c_{m+1} \in A''$ with

(oo) $\quad (\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c'''_m, c'''_{m+1}, c_{m+1}) \equiv_{4n-4m-2} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d'''_m, d'''_{m+1}, d_{m+1})\,(Q_1)$.

Then we set

$$c''_{m+1} = \min\{c\colon c \in A \,\&\, c \leqslant c_{m+1} \,\&\, c'''_{m+1} < c\} \quad \text{and}$$
$$c'_{m+1} = \min\{c\colon c \in A \,\&\, c < c''_{m+1} \,\&\, (\text{there is no } a \in ((\operatorname{dom} f) \cap A) \text{ with } c \leqslant a) \,\&\, (\text{there is no } l \leqslant m \text{ with } c \leqslant c_l)\}.$$

By (oo) there exist elements d''_{m+1} and d'_{m+1} from B with

$$(\underline{A}, \overline{\operatorname{dom} f}, c_1, \ldots, c'''_{m+1}) \equiv_{4n-4m-4} (\underline{B}, \overline{\operatorname{rn} f}, d_1, \ldots, d'''_{m+1})\,(Q_1).$$

Hence we get

$$d''_{m+1} = \min\{d\colon d \in B \,\&\, d \leqslant d_{m+1} \,\&\, d'''_{m+1} < d\} \quad \text{and}$$
$$d'_{m+1} = \min\{d\colon d \in B \,\&\, d < d''_{m+1} \,\&\, (\text{there is no } b \in ((\operatorname{rn} f) \cap B) \text{ with } d \leqslant b) \,\&\, (\text{there is no } j \leqslant m \text{ with } d \leqslant d_j)\},$$

since $m < n-1$. It is obvious that $c_i, c'_i, c''_i, c'''_i\,(1 \leqslant i \leqslant m+1)$ and $d_i, d'_i, d''_i, d'''_i\,(1 \leqslant i \leqslant m+1)$ fulfil conditions (1), …, (8) and (3′), …, (8′).

In case that player I chooses an element $d_{m+1} \in B$ or a subset $B' \leqslant B$ with $\operatorname{card}(B') = \aleph_1$ we have to change then only $\underline{A}$ and $\underline{B}$ in a corresponding way. ∎

Applications.

THEOREM 6. *There is a recursive function r such that for each natural number n and all connected ω-trees $\underline{A}$ and $\underline{B}$ the following holds: $\underline{A} \equiv_{r(n)} \underline{B}(Q_1)$ implies $\underline{A} \equiv_n \underline{B}(aa)$. Moreover $\underline{A}$ and $\underline{B}$ are finitely determinate.*

Proof. We define $r(0) := 0$ and $r(n) := \sum_{i=1}^{n} 4^i$ for $n > 0$. Now we assume $\underline{A} \equiv_{r(n)} \underline{B}(Q_1)$. By Theorem 1 it is sufficient to find a determinate (aa, n) back-and-forth system from $\underline{A}$ to $\underline{B}$. We define $(F_k\colon k \leqslant n)$ as follows

$$f \in F_k \text{ iff } f\colon A \cup P_{\omega_1}(A) \to B \cup P_{\omega_1}(B),\ \operatorname{card}(\operatorname{dom} f) = n-k \text{ and } (\underline{A}, \overline{\operatorname{dom} f}) \equiv_{r(k)} (\underline{B}, \overline{\operatorname{rn} f})(Q_1).$$

Obviously properties (1), (2), (3i) and (3ii) from the definition of a determinate (aa, n) back-and-forth system are fulfilled for $(F_k\colon k \leqslant n)$. But Lemma 2, Lemma 3 and Lemma 5 imply that also property (3iii) holds for $(F_k\colon k \leqslant n)$. ∎

Remark. Using the usual technics of interpretability, which holds also for stationary logic we get that the restriction to connected ω-trees in Théorem 6 is not necessary.

COROLLARY 7. *Let $\underline{A}$ and $\underline{B}$ be two ω-trees which fulfil $\underline{A} \equiv \underline{B}(Q_1)$. Then $\underline{A} \equiv \underline{B}$ (aa) holds.*

COROLLARY 8. *Each ω-tree is finitely determinate.*

THEOREM 6. *There is a recursive function r such that for each natural number n and let $a_1, \ldots, a_k$ be elements of A. Moreover let L be an elementary language for ω-trees and let $\varphi(X_1, \ldots, X_l, x_1, \ldots, x_k, Y)$ be a formula of $L(aa)$. Then the following conditions are equivalent:*

(1) *There exists a cub system $\underline{C} \subseteq P_{\omega_1}(A)$ such that for all $C \in \underline{C}$ $\underline{A} \models \varphi(A_1, \ldots, A_l, a_1, \ldots, a_k, C)$.*

(2) *There exists a set C' with the properties* (i), (ii) *and* (iii) (*see above Lemma* 2) *for* $\underline{B} = \underline{A}$, $n = \mathrm{qr}(\varphi)$ *and* $f = \{(A_1, A_1), \ldots, (A_l, A_l), (a_1, a_1), \ldots, (a_k, a_k)\}$ *such that* $\underline{A} \models \varphi(A_1, \ldots, A_l, a_1, \ldots, a_k, C')$ *holds.*

Proof. We get the implication (1) → (2) by Lemma 2 and Lemma 3. The implication (2) → (1) follows by Lemma 5, Lemma 2 and Lemma 3. ∎

THEOREM 10. *Let T (and CT) be the class of all (connected) ω-trees. Then* $\mathrm{Th}_{aa}(T)$ *and* $\mathrm{Th}_{aa}(CT)$ *are decidable.*

Proof. Using a simple interpretability argument it is sufficient to proof the decidability of $\mathrm{Th}_{aa}(CT)$.

Let L_{II} be the monadic second order language for the class of all ω-trees. We assume that $L_{II}(Q_0, Q_1)$ is the extension of L_{II} by the quantifiers Q_0 and Q_1. Q_0 and Q_1 are used for individuales only and have the usual interpretation.

Let $(\omega - CT(\aleph_1))$ be the class of all connected ω-trees of cardinality at most $\aleph_1$. Let $\mathrm{Th}_{II,Q_0,Q_1}\big((\omega - CT(\aleph_1))\big)$ be the set of all sentences of the language $L_{II}(Q_0, Q_1)$ which are valid in $(\omega - CT(\aleph_1))$. This theory is decidable (see Seese [6]). We shall give now an interpretation of $\mathrm{Th}_{aa}\big((\omega - CT(\aleph_1))\big)$ in this theory. For each formula φ of $L(aa)$ let $\hat{\varphi}$ be the corresponding translation of in $L_{II}(Q_0, Q_1)$.

It is sufficient to show that for each formula $\varphi(X_1, \ldots, X_l, x_1, \ldots, x_k, Y)$ of the language $L(aa)$ can be effectively found a formula $\psi(X_1, \ldots, X_l, x_1, \ldots, x_k, Y)$ from the language $L_{II}(Q_0, Q_1)$ such that the following holds:

for each connected ω-tree $\underline{A}$, all sets $A_1, \ldots, A_l \in P_{\omega_1}(A)$ and all elements $a_1, \ldots, a_k \in A$ the following holds:

$$(\underline{A}, A_1, \ldots, A_l, a_1, \ldots, a_k) \models (aaY)\,\varphi(A_1, \ldots, A_l, a_1, \ldots, a_k, Y)$$

iff

$$(\underline{A}, A_1, \ldots, A_l, a_1, \ldots, a_k) \models \exists Y\big(\psi(A_1, \ldots, A_l, a_1, \ldots, a_k, Y) \wedge \wedge \hat{\varphi}(A_1, \ldots, A_l, a_1, \ldots, a_k, Y)\big).$$

This can be proved using Theorem 9. Obviously it is sufficient to find a formula $\psi(\ldots)$ with the following property:

for all sets $C' \subseteq A$ $\underline{A} \models \psi(A_1, \ldots, A_l, a_1, \ldots, a_k, C')$ holds iff C' has the properties (i), (ii) and (iii).

Hence it is sufficient to prove that properties (i), (ii) and (iii) can be expressed in $L_{II}(Q_0, Q_1)$.

For (i) and (ii) this is obvious. To see that property (iii) is expressable in $L_{II}(Q_0, Q_1)$ it is sufficient to see that the relation ${}_a\!\sim_{4n}$ is definable in this language.

To show this it is sufficient to construct effectively a formula $\chi_n(X_1, \ldots, X_l, x_1, \ldots, x_k, x, y, z)$ from $L_{II}(Q_0, Q_1)$ such that for all $A_1, \ldots, A_l \in P_{\omega_1}(A)$ and all $a_1, \ldots, a_k, a, b, c \in A$:

$$\underline{A} \models \chi_n(A_1, \ldots, A_l, a_1, \ldots, a_k, a, b, c) \quad \text{iff} \quad b\ {}_a\!\sim_{4n} c.$$

But this follows from the definition of ${}_a\!\sim_{4n}$, since the corresponding language L is a finite language (use e.g. Proposition 0.4 on page 59 of Kaufmann [5]). ∎

For a language with a binary irreflexive symmetric relation instead of $\leqslant$ all results of this paper are proved by Baudisch and Tuschik [2]. The generalizations to ω-trees are due to Seese (see Seese [6]). Furthermore Baudisch and Tuschik have the following.

THEOREM 11 (Baudisch and Tuschik [2]). *Let F be the class of all graph theoretical forests (i.e. symmetric graphs without circles, regarded as structures with one binary relation; see Harary [4]) and let L be a corresponding elementary language. Every formula $\varphi(\bar{x})$ of $L(aa)$ is equivalent relative to $\mathrm{Th}_{aa}(F)$ to a formula $\psi(\bar{x})$ of $L(Q_1)$. The correspondence is effective.*

This result is proved using a strengthening of results from Seese and Tuschik [7].

A simple corollary of Theorem 10 is also the following:

COROLLARY 12. *The theory of one equivalence relation in stationary logic is decidable and each equivalence relation is finitely determinate.*

We conclude this article with the following problems.

PROBLEM 1. Is it possible to prove an analog version of Theorem 11 also for ω-trees instead of forests?

PROBLEM 2. Is each tree of height $< \omega_1$ finitely determinate?

References

[1] J. Barwise, M. Kaufmann and M. Makkai, *Stationary logic*, Ann. Math. Logic 13 (1978), pp. 171–224.

[2] A. Baudisch and H. P. Tuschik, *$L(aa) = L(Q_1)$ for trees*, (1979), unpublished manuscript.

[3] P. C. Eklof and A. H. Mekler, *Stationary logic of finitely determinate structures*, Ann. Math. Logic 17 (1979), pp. 227–270.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Massachusetts–Menlo Park–California–London, 1969.

[5] M. Kaufmann, *Some results in stationary logic*, Ph. D. Thesis, Univ. of Wisconsin.

[6] D. G. Seese, *Stationäre Logik – Beschränkte Mengen – Dichte Systeme*, Dissertation (B), 1979.

[7] – and H. P. Tuschik, *Construction of nice trees*, Lecture Notes in Mathematics, Springer Verlag, 619 (1977), pp. 257–271.

[8] S. Shelah, *Generalized quantifiers and compact logic*, Trans. Amer. Math. Soc. 204 (1975), pp. 342–364.

Accepté par la Rédaction le 17. 8. 1981