[5] B. Knaster and S. Mazurkiewicz, *Sur un probleme concernant les transformations continues*, Fund. Math. 21 (1933), pp. 85–90.
[6] K. Kuratowski, *Theorie des continus irreductible entre deux points* II, Fund. Math. 10 (1927), pp. 225–276.
[7] — *Topology*, Vol. 2, New York–London–Warszawa 1968.
[8] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Dissertationes Math. 50 (1966).

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Completely distributive lattices

by

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Abstract. If a complete lattice with 0 and 1 satisfies the infinite distributivity laws it is called completely distributive. In this paper we give simple proofs of known characterizations of complete distributivity as well as new characterizations in terms of maps from the lattice to itself satisfying the condition $a = \bigvee\{b/a \nleq p(b)\}$ for all a in the lattice, where p: $L \to L$ is the map.

1. Introduction. Although the motivation for the results of this paper, whose purpose is to study complete distributivity of lattices, arise from Functional Analysis, we shall keep the theorems and their proofs lattice theoretic. In Functional Analysis, and more specifically in the study of invariant subspaces of operators on a normed vector space H, one examines conditions on a set L of subspaces of H to be *reflexive* in the sense that it coincides with the family of subspaces that are invariant under each operator that leaves invariant the elements of the set (see Radjavi and Rosenthal [13] for the relavant definitions). A necessary, but far from sufficient, condition for the reflexivity of L is that L is a complete lattice (under the usual lattice operations on subspaces). There are several sufficient conditions known. For instance Ringrose in [17] has shown that every complete totally ordered lattice of subspaces of a Hilbert space (*complete nest* in his terminology) is reflexive. Halmos in [7] has shown that complete atomic Boolean lattices of subspaces are also reflexive. Both these examples are examples of completely distributive lattices. Longstaff in [12] has shown that in fact complete and completely distributive lattices of subspaces of Hilbert spaces are reflexive, and so he extended the previous two cases. A necessary and sufficient condition for a complete and completely distributive lattice to be a complete atomic Boolean lattice is given in [10]. Another equivalent condition, but this time Functional Analytic, is given in [9]. It is easy to see that if the underlying Hilbert space is finite dimensional then a lattice is complete and completely distributive if and only if it is distributive. In the finite dimensional Hilbert space case R. Johnson in [8] has shown that a necessary and sufficient condition for a finite lattice to be reflexive is that it is distributive. In general Hilbert spaces neither of these two conditions implies the other. Indeed, Halmos [7] constructed a reflexive lattice which is (lattice) isomorphic to the non-modular pentagon M_5. An example in the opposite direction is due to Conway ([6]) who constructed a non-reflexive complete lattice isomorphic to a (non-atomic) Boolean algebra. Incidentally, this example also shows that distributivity alone (that is, not complete distributivity) of a complete lattice is not sufficient (unlike the finite dimensional case) for reflexivity. (Notice that by Tarski's theorem ([5], p. 119) non-atomic Boolean lattices are never completely distributive.)

Recall that a complete lattice L is called *completely distributive* if the identity

$$\bigwedge_i \bigvee_j a_{ij} = \bigvee_{f\in J^I} \bigwedge_i a_{if(i)}$$

and its dual hold, for all choises of $a_{ij}\in L$ $(i\in I, j\in J)$ where I, J are any indexing sets and J^I denotes the set of all functions $f\colon I\to J$. If the first identity holds for a lattice L, we shall say that *meets are completely distributive with respect to joins*, and if its dual hold we shall say that *joins are completely distributive with respect to meets*. (For all other standard definitions of Lattice Theory see Birkhoff [5]). In [14] Raney shows that each of the above two distributive identities implies the other and so each is equivalent to complete distributivity. He further showed in [15] that complete distributivity in a complete lattice is characterized by the condition that for each $a\in L$

$$a = \bigvee \bigcap \{M/M \text{ is a semi-ideal and } a \leqslant \bigvee M\}$$

(where $\bigcap$ denotes set theoretic intersection). We shall refer to this equivalence as *Raney's characterization*. Longstaff's approach in [12] for the reflexivity of complete and completely distributive lattices is indirect and, very briefly, runs as follows: First he shows that a complete lattice L is reflexive if for each $a\in L$ the equality $a = a_*$ holds, where

$$a_* = \bigwedge \{b_-/b \nleqslant a\}$$

and where

$$b_- = \bigwedge \{c/b \nleqslant c\} \quad (b\in L).$$

Using Raney's characterization he then shows that complete distributivity of a complete lattice is equivalent to the condition $a = a_*$ $(a\in L)$ where

$$a_\# = \bigwedge \{b/a \nleqslant b_-\}.$$

Finally, after proving the relations

$$a_* \leqslant a_{**} \leqslant a \leqslant a_{**} \leqslant a_* \quad (a\in L)$$

concludes the following:

THEOREM ([12]). *For a complete lattice L, the following are equivalent*
(i) *L is completely distributive.*
(ii) $a = a_*$ $(a\in L)$.
(iii) $a_* = a$ $(a\in L)$.

One of our aims in this paper is to present direct proof of a more general criterion for complete distributivity, and then get as corollaries known results whose proofs are scattered in the literature througout several papers.

2. Complete distributivity. In what follows we shall assume that all lattices are complete and contain a largest element 1 and a least element 0.

DEFINITION. If L is a lattice we say that a map $p\colon L\to L$ *V-defines* the lattice if for each $a\in L$ the equality $a = \bigvee\{b/a \nleqslant p(b)\}$ holds.

As we shall see, the existence of a map that V-defines a lattice is characteristic of complete distributivity. We shall also see, in the course of the proof of Theorem 1, that if a lattice is completely distributive then the map $p\colon L\to L$, $p(a) = a_-$ V-defines, where a_- is as in Longstaff's notation mentioned above. By examples we show that for a completely distributive lattice the map $p(a) = a_-$ is just one of a class of V-defining maps. Finally we mention that the equivalence of the first three statements of Theorem 1 are due to Raney ([14]), but we shall give a different and direct proof that also establishes their equivalence with the forth statement.

THEOREM 1. *If L is a lattice the following are equivalent.*
(i) *Meets are completely distributive with respect to joins.*
(ii) *Joins are completely distributive with respect to meets.*
(iii) *L is completely distributive.*
(iv) *There exists a map $p\colon L\to L$ that V-defines L.*

Proof. Omitting the obvious implications and the ones that follow by considering the dual lattice we only show (i)$\Leftrightarrow$(iv)$\Rightarrow$(ii).

(iv)$\Rightarrow$(i). Let $p\colon L\to L$ V-define L. As the reverse inclusion is always valid, to show (i) it is sufficient to show

$$\bigwedge_i \bigvee_j a_{ij} \leqslant \bigvee_{f\in J^I} \bigwedge_i a_{if(i)}.$$

Let $b\in L$ be such that

$$(1)\qquad \bigwedge_i \bigvee_j a_{ij} \nleqslant p(b).$$

Then for each i it follows that

$$\bigvee_j a_{ij} \nleqslant p(b),$$

and so for each i there corresponds at least one j such that $a_{ij} \nleqslant p(b)$ and hence, since p V-defines, $b \leqslant a_{ij}$. So we can define a function $g\colon I\to J$ in such a way that for each $i\in I$ it picks a corresponding $j\in J$ with $b \leqslant a_{ig(i)}$. Then

$$(2)\qquad b \leqslant \bigwedge_i a_{ig(i)} \leqslant \bigvee_{f\in J^I} \bigwedge_i a_{if(i)}.$$

Now by assumption the span of all b satisfying (1) equals $\bigwedge_i \bigvee_j a_{ij}$, so by taking span in (2) over all such b we obtain

$$\bigwedge_i \bigvee_j a_{ij} \leqslant \bigvee_{f\in J^I} \bigwedge_i a_{if(i)},$$

as required.

(iv) $\Rightarrow$ (ii). We only need prove that

$$\bigvee_i \bigwedge_j a_{ij} \geqslant \bigwedge_{f\in J^I} \bigvee_i a_{if(i)}.$$

Let b be such that

$$\bigwedge_{f\in J^I} \bigvee_i a_{if(i)} \nleqslant p(b).$$

Then, for all $f \in J^I$ we have

$$\bigvee_i a_{if(i)} \nleqslant p(b). \tag{3}$$

We claim that there is a $k\in I$ such that

$$b \leqslant \bigwedge_j a_{kj}. \tag{4}$$

If not, then for every $i\in I$, $b \nleqslant \bigwedge_j a_{ij}$ and so for every $i\in I$, there corresponds a $j\in J$ with $b \nleqslant a_{ij}$. Define then a function $h\colon I\to J$ which for each $i\in I$ picks a $j\in J$ with $b\nleqslant a_{ih(i)}$. As p V-defines L, $a_{ih(i)} \leqslant p(b)$. But then $\bigvee_i a_{ih(i)} \leqslant p(b)$, contradicting (3). We thus obtain from (4)

$$b \leqslant \bigvee_i \bigwedge_j a_{ij}$$

and arguing as in the last part of the previous implication, we get the desired conclusion.

(i) $\Rightarrow$ (iv). This part of the proof adapts, in a concealed way, the ideas of Raney in [15] and their modification by Longstaff in [12].

We shall show that $p\colon L\to L$, $a\mapsto a_-$ V-defines L. We only need show that $a \leqslant \bigvee\{b/a \nleqslant b_-\}$, the reverse inequality being obvious. Index by I the set $\{c/a\leqslant c_-\}$, so that $a\leqslant c_{i-}$ $(i\in I)$. For each $i\in I$ index by J_i the set $\{c/c_i \nleqslant c\}$ so that $c_i \nleqslant c_{ij}$ $(i\in I, j\in J_i)$ and $c_{i-} = \bigvee_{J_i} c_{ij}$. Allowing repetition if necessary we may assume $J_i = J$ for all $i\in I$. Clearly

$$a \leqslant \bigwedge_i c_{i-} = \bigwedge_i \bigvee_j c_{ij} = \bigvee_f \bigwedge_i c_{if(i)}.$$

For a fixed $f\in J^I$ and any $k\in I$, $c_k \nleqslant c_{kf(k)}$ and so $c_k \nleqslant \bigwedge_i c_{if(i)}$. In particular $\bigwedge_i c_{if(i)} \neq c_k$ $(k\in I)$ and so $\bigwedge_i c_{if(i)}$ does not belong to the set $\{c/a\leqslant c_-\}$. This then implies that

$$\bigwedge_i c_{if(i)} \leqslant \bigvee\{b/a \nleqslant b_-\}.$$

Hence

$$\bigvee_f \bigwedge_i c_{if(i)} \leqslant \bigvee\{b/a \nleqslant b_-\},$$

and the proof is complete.

Corollary 1 (Raney [16]). *A lattice L is completely distributive if and only if $a = \bigvee\{b/a\nleqslant b_-\}$ $(a\in L)$.*

Proof. If the stated equality holds then $a\mapsto a_-$ V-defines, so L is completely distributive. The converse is just the proof of implication (i) $\Rightarrow$ (iv) in Theorem 1.

Corollary 2 (Raney [16]). *A lattice L is completely distributive if and only if for every pair a, b in L with $a\nleqslant b$ there exist x, y in L with $a\nleqslant x$, $y\nleqslant b$ and such that for any $t\in L$ either $t\leqslant x$ or $y\leqslant t$.*

Proof. If L is completely distributive and $p\colon L\to L$ any map that V-defines L, then as $a = \bigvee\{c/a\nleqslant p(c)\}$ and $a\nleqslant b$, it follows that there is a c with $a\nleqslant p(c)$ and $c\nleqslant b$. We choose $x = p(c)$, $y = c$. If now $t\in L$ and $y\nleqslant t$ then $t\leqslant p(y) = x$, and hence x and y have all the desired properties.

Conversely, let $a\in L$ be arbitrary and let $b = \bigvee\{c/a\nleqslant c_-\}$. We shall show that $a = b$ and appeal to Corollary 1. Clearly $b\leqslant a$ so if $a\neq b$ it follows that $a\nleqslant b$. By assumption there are x, $y\in L$ with $a\nleqslant x$, $y\nleqslant b$ and $\forall t\in L$ either $t\leqslant x$ or $y\leqslant t$. As $y_- = \bigvee\{t/y\nleqslant t\}$ we have $y_-\leqslant x$ and so $a\nleqslant y_-$. But $a\nleqslant y_-$ implies $y\leqslant b$ which is a contradiction. Thus $b = a$ and $a\mapsto a_-$ V-defines L.

Corollary 1 was discovered by Raney who stated it in a different form. To prove Corollary 2 Raney used Galois connections between lattices and develops a particular type of Galois connection which he calls tight. Using Raneys original version of Corollary 1 and some manipulation, Longstaff, in a paper on Functional Analysis, stated and proved Corollary 1 in a form close to the one given. In the meantime Bandelt in [1], [2] and [3] had also stated Corollaries 1 and 2 as given here. His proofs are different from Longstaff's. The equivalence of Corollaries 1 and 2 is stated in [4].

Example (i). If L is a complete atomic Boolean lattice and a' denotes the Boolean complement of a, we define $p\colon L\to L$ by

$$p(x) = \begin{cases} 0 & \text{if } x = 0,\\ x' & \text{if } x \text{ is an atom } \neq 1,\\ 1 & \text{otherwise.}\end{cases}$$

This map V-defines L since, if $a \in L$ then a is the span of atoms containing it. So if b is an atom contained in a then $a \nleqslant b' = p(b)$. Conversely, $a \nleqslant p(b)$ implies $p(b) \neq 1$ and so b is an atom. This shows that

$$a = \bigvee \{b/b \text{ atom} \leqslant a\} = \bigvee \{b/a \nleqslant p(b)\}$$

as required. (Incidently this gives an alternative proof of one direction of Tarski's well known theorem.) Also note that as $0 \nleqslant a$ is always false, any value for $p(0)$ would do just as well.

Example (ii). If L is a complete totally ordered lattice, define $p(x) = \bigvee \{y/y < x\}$, where $<$ means strict inclusion. Clearly for every $a \in L$ we have $p(a) \leqslant a$ (where the equality is possible. In fact $p(a) = a$ if and only if a has no immediate predecessor. If a has an immediate predecessor then $p(a)$ is this predecessor). To prove that p V-defines L, we work as follows. First observe that $x \nleqslant p(y) \Rightarrow p(x) < x \Rightarrow p(y) \leqslant p(x) \Rightarrow y \leqslant x$ and hence

$$x \geqslant \bigvee \{y/x \nleqslant p(y)\}. \tag{5}$$

So if x has an immediate predecessor then x itself belongs to the set $\{y/x \nleqslant p(y)\}$, and hence the right hand side of (5) is $\geqslant x$. If on the other hand x has no immediate predecessor, then

$$x = \bigvee \{y/y < x\} \leqslant \bigvee \{y/p(y) < x\} \leqslant x$$

as required.

Remark. If p V-defines the lattice then for $y \in L$ we have $x \nleqslant p(y) \Rightarrow y \leqslant x$. Equivalently $y \nleqslant x \Rightarrow x \leqslant p(y)$ and so

$$p(y) \geqslant \bigvee \{x/y \nleqslant x\},$$

which in Longstaff's notation states $p(y) \geqslant y_-$. One can check that in the two examples above we actually have $p(y) = \bigvee \{x/y \nleqslant x\}$ $(y \in L)$ but we show by an example that this is not always the case. The example that follows is just one of a class of different types of examples.

Example (iii). Let L be the interval $[0, 1]$ of real numbers between 0 and 1 with its usual ordering, and let $p\colon L \to L$ be the map

$$p(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It is easy to see that $x_- = x = p(x)$ if x is rational and $x_- = x < p(x)$ if x is irrational. Also if $y \in [0, 1]$ is any fixed real, the relation $p(x) < y$ implies $p(x) \neq 1$, and so x is rational. Hence

$$\begin{aligned} \bigvee \{x/y \nleqslant p(x)\} &= \bigvee \{x|\ p(x) < y\} = \bigvee \{x|\ x \in a \text{ and } p(x) < y\} \\ &= \bigvee \{x|\ x \in a \text{ and } x < y\} = y. \end{aligned}$$

However the following holds, where we put $p(0) = 0$ to exclude empty statements.

Theorem 2. *If L is a complete atomic Boolean lattice then there exists a unique $p\colon L \to L$ that V-defines L. In this case $p(0) = 0$, $p(a) = a'$ if a is an atom, and $p(a) = 1$ otherwise (which says $p(a) = a_-$ for $a \in L$).*

Proof. The existence of p has been shown concretely in example (i) (mere existence follows from Theorem 1) and is of the form given. Let now $a \neq 1$ be an atom of L. A fortiori a is not the span of elements strictly smaller than it, since these are equal to zero. Hence the relation $a = \bigvee \{b/a \nleqslant p(b)\}$ implies that one of the b with $a \nleqslant p(b)$ (and so with $b \leqslant a$) must equal a, and so $a \nleqslant p(a)$ which in turn implies $p(a) \leqslant a'$. But a' is the span of the rest of the atoms of L (and each such atom x satisfies $a \nleqslant x$). So

$$p(a) \leqslant a' \leqslant \bigvee \{x/a \nleqslant x\} \leqslant p(a).$$

If on the other hand a is not an atom, then the set $\{b/a \nleqslant b\}$ contains all atoms. Therefore $\bigvee \{b/a \nleqslant b\} = 1$ and

$$1 = p(a) \geqslant \bigvee \{b/a \nleqslant b\} = 1.$$

Combining the two cases we see that the only $p\colon L \to L$ that V-defines L is the one given.

3. The dual of V-defining maps. We now work with the dual concept to that of V-defining maps. We shall not elaborate on some of the proofs of the theorems, for they follow by arguments similar to the ones given above.

Definition. A map $p\colon L \to L$ is said to *Λ-define* the lattice if $a = \bigwedge \{p(b)/b \nleqslant a\}$ $(a \in L)$.

It is easy to see that if p either V-defines or Λ-defines the lattice then

$$b \nleqslant a \Rightarrow a \leqslant p(b). \tag{6}$$

Another simple observation to make is that if the map p Λ-defines then it also has the property

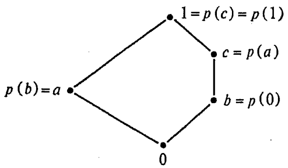
$$a = \bigwedge \{p(b)/a \leqslant p(b)\}, \tag{7}$$

since

$$a = \bigwedge \{p(b)/b \nleqslant a\} \geqslant \bigwedge \{p(b)/a \leqslant p(b)\} \geqslant a.$$

If a map has properties (6) and (7) it does not necessarily Λ-define L. We shall show by an easy example that a lattice may have these properties under some map $p\colon L \to L$ but the lattice is not distributive, that is, not only p does not Λ-define L but no map does. However properties (6) and (7) are usefull in practice to check distributivity in a lattice.

EXAMPLE (iv). In the non-modular five element pentagon $0 < a < 1$, $0 < b < c < 1$ we put $p(1) = 1$, $p(a) = c$, $p(b) = a$, $p(c) = 1$, $p(0) = b$. An easy verification shows



that p has the desired properties.

Inspite of this example we have the following theorem, which can also be proved from Corollary 2, but we prefer the direct proof.

THEOREM 3. *If* $p\colon L \to L$ *satisfies the conditions*

(i) $b \nleqslant a \Rightarrow a \leqslant p(b)$,

(ii) $a = \bigwedge \{p(b)/a \leqslant p(b)\}$ $(a \in L)$

and if in addition $a \nleqslant p(a)$ *for each* a *in* L *with* $p(a) \neq 1$, *then* L *is completely distributive.*

Proof. By Theorem 1 we only need show that

$$\bigwedge_i \bigvee_j a_{ij} \leqslant \bigvee_{f \in J^I} \bigwedge_i a_{if(i)}.$$

We can assume that the right hand side is not equal to 1 and so there is a b such that $p(b) \neq 1$ and

$$\bigvee_{f \in J^I} \bigwedge_i a_{if(i)} \leqslant p(b). \tag{8}$$

This implies that for every $f\colon I \to J$

$$\bigwedge_i a_{if(i)} \leqslant p(b)$$

and so, choosing f to be the constant function $f(i) = j$,

$$\bigwedge_i a_{ij} \leqslant p(b) \qquad (i \in J).$$

We claim that this implies that there is a $k \in I$ with

$$\bigvee_j a_{kj} \leqslant p(b). \tag{9}$$

Suppose on the contrary, $\forall i \in I$, $\bigvee_i a_{ij} \nleqslant p(b)$. So, as before, there is a function $g\colon J \to I$ such that $a_{ig(i)} \nleqslant p(b) \Rightarrow b \leqslant a_{ig(i)}$ (all i) and hence

$$b \leqslant \bigwedge_i a_{ig(i)} \leqslant \bigvee_{f \in J^I} \bigwedge_i a_{if(i)} \leqslant p(b),$$

contradicting the assumption $b \nleqslant p(b)$ for $p(b) \neq 1$. The proof now continues as before by using (ii).

The example preceding Theorem 3 shows that from assumptions (i) and (ii) alone we cannot expect distributivity. We shall show however that for a wide class of subsets of L meets are completely distributivite with respect to joins and dually.

Let L be indexed by an indexing set K and denote by L_k the set $\{a/a \geqslant a_k\}$. Now, each L_k can be indexed by a set J_k, so that $L_k = \{a_{kj}/j \in J_k\}$. Let I be an arbitrary subset of K, and denote the cartesian product of the $J_l\,(l \in I)$ by ΠJ_l. We have the following:

THEOREM 4. *Suppose* $p\colon L \to L$ *satisfies conditions* (i) *and* (ii) *of Theorem* 3. *Then*

$$\bigvee_i \bigwedge_{i \in J_i} p(a_{ij}) = \bigwedge_{f \in \Pi J_l} \bigvee_i p(a_{if(i)}).$$

Proof. We can assume that all the indexing sets J_i are the same by just replacing the J_i's with their union and by letting $a_{ij} = 1$ if $i \in J - J_i$. Thus we simply have to prove that

$$\bigvee_i \bigwedge_j p(a_{ij}) = \bigwedge_{f \in J^I} \bigvee_i p(a_{if(i)}).$$

Let $b_i = \bigwedge_j p(a_{ij})$ and let b be such that

$$b \geqslant \bigvee_i \bigwedge_j p(a_{ij}).$$

For each i, $b \geqslant b_i$ and so $b \in L_i$. This shows that for each $i \in I$ there is a $j \in J$ with $b = a_{ij}$, and so there is a function $g\colon I \to J$ with $b = a_{ig(i)}$. Hence also

$$p(b) = \bigvee_i p(a_{ig(i)}) \geqslant \bigwedge_{f \in J^I} \bigvee_i p(a_{if(i)})$$

and we can continue as usually.

4. Semi-simplicity. Recall that a lattice is called *semi-simple* if the intersection of its maximal ideals is $\{0\}$. It is well known (see for example [5]) that every Boolean lattice is semi-simple. In the next theorem we replace the condition $a \nleqslant p(a)$ $(p(a) \neq 1)$ of Theorem 3 by a stronger one and we get a surprisingly stronger result. This result strengthens that in [10] which in turn strengthens Tarski's result mentioned above which states that a completely distributive complete lattice is a Boolean lattice if and only if it is atomic. First we need a lemma.

LEMMA 1. *Let* $p\colon L \to L$ *satisfy conditions* (i) *and* (ii) *of Theorem* 3, *and suppose that* $a \wedge p(a) = 0$, *for every* $a \in L$ *with* $p(a) \neq 1$. *Then* L *is a complete atomic Boolean lattice.*

Proof. The proof follows closely the one given in [10] so we omit it. (See also [11] for correction of some printing mistakes of [10].)

THEOREM 5. *If L is completely distributive then L is a complete atomic Boolean lattice if and only if L is semi-simple.*

Proof. As in [10]. One only has to observe that if $a \in L$ satisfies $p(a) \neq 1$ then $a \wedge p(a)$ belongs to each maximal ideal.

Immediate corollaries of the above come from the considerations: It is well known that the set theoretic complement of an ideal of a lattice is a filter (dual ideal) if and only if the ideal is prime. In a Boolean lattice prime ideals are maximal and so the set theoretic complement of a maximal ideal is an ultrafilter and conversely. From Theorem 5 the following (known) equivalences can be shown for a completely distributive lattice L.

(i) L is a Boolean lattice,

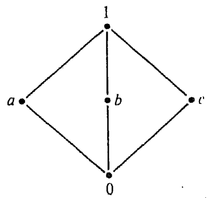
(ii) the set theoretic complement of every maximal ideal is an ultrafilter,

(iii) the dual of (ii).

Indeed, if (ii) holds and $a \in L$ is non-zero, there is by Zorn's Lemma an ultrafilter containing it (consider the filter $\{x/a \leqslant x\}$). The set theoretic complement of this ultrafilter is a maximal ideal not containing a. By varying a we conclude that L is semi-simple and Theorem 5 applies. To prove (ii) $\Rightarrow$ (i) it is sufficient to observe that the dual of a completely distributive lattice is also completely distributive (this follows from the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 1).

Another immediate corollary to Theorem 5 is that if the intersection of the ultrafilters of a completely distributive lattice is $\{1\}$ then again L is a complete atomic Boolean lattice. We have mantioned these corollories because we want to show by counterexamples that the assumptions of Theorem 5 cannot be weakened.

EXAMPLE (v). Complete atomic semi-simple lattices are not necessarily completely distributive, even if we assume the lattice to be modular. That this fails can be seen by considering the five element diamond (double triangle by some authors) $0 < a < 1$, $0 < b < 1$, $0 < c < 1$



Here the maximal ideals are $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$, so L is semi-simple and clearly atomic.

EXAMPLE (vi). If we weaken the complete distributivity assumption of Theorem 5 we still do not get the same conclusion. We construct an example of a complete lattice L with the following properties:

(a) L is distributive. In fact the infinite distributive law

$$x \wedge \left(\bigvee_i x_i\right) = \bigvee_i (x \wedge x_i)$$

holds,

(b) L is semi-simple.

Yet, we show that the lattice is neither Boolean nor atomic. In fact we show that even the intersection of its ultrafilters is not $\{1\}$:

Let $(X, \mathscr{T})$ be a topological space and let L be the (complete) lattice of all open sets in $\mathscr{T}$ (with infinite) operations:

$$\bigvee T_a = \bigcup T_a, \quad \bigwedge T_a = \left(\bigcap T_a\right)^0.$$

Then

(i) If $(X, \mathscr{T})$ is a compact Hausdorff space then L is semi-simple.

(ii) If singletons are closed but not open in $(X, \mathscr{T})$ then the intersection of ultrafilters is not $\{1\}$.

In particular, L has the desired properties.

Proofs. (i) Denote the set $\{T \in L/x \notin T\}$, where $x \in X$, by M_x. If J is a (proper) ideal of L, we show that there is an $x \in X$ such that $J \subseteq M_x$. Indeed, if not, then for each $x \in X$ there is a $T_x \in J$ such that $x \in T_x$. But then $X = \bigcup_x T_x$ and by compactness $X = \bigcup_{i=1}^{n} T_{x_i}$ for some $n \in N$. As each $T_x \in J$, it follows that X also belongs to J_i, a contradiction to the fact that J is proper. So after all $J \subseteq M_x$ for some x. Clearly M_x itself is a (proper) ideal, so if J is a maximal ideal we have $J = M_x$ for some x. Next we show that each M_y $(y \in X)$ is a maximal ideal. If not, then there is a maximal ideal M properly containing it, and by the above $M = M_z$ for some $z \in X$, so $M_y \subset M_z$. By the Hausdorff assumption there is a $T \in \mathscr{T}$ such that $z \in T$ but $y \notin T$, so this T is in M_y but not in M_z, giving a contradiction.

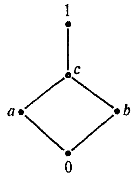
To show that L is semi-simple, let $S \in L$ be a non-empty set in T and let $w \in S$. Then clearly $S \notin M_w$ and M_w is a maximal ideal.

(ii) We show that for each singleton $\{x\}$ the set $X - \{x\}$ of L belongs to all ultrafilters of L. Let U be an ultrafilter of L and suppose that for some $x \in X$, $X - \{x\} \notin U$. If $T \in U$ then by definition of filters $T \nsubseteq X - \{x\}$ and hence $x \in T$. Therefore

$$U \subseteq \{T \in L/x \in T\}.$$

The set on the right is a (proper) filter of L and by maximality we have equality. Consider now the (not necessarily proper) filter V generated by U and $X-\{x\}$. As $X-\{x\}\notin U$, the filter V is strictly larger than U and so coincides with L. This implies that there is a $T\in U$ with $T\cap(X-\{x\})=\emptyset$ and so $T=\{x\}$. This contradicts the fact that T is open and $\{x\}$ is not. It follows that, after all, $X-\{x\}$ belongs to the intersection of all ultrafilters and completes the example.

5. Questions. It has been observed that among V-defining maps, $a\mapsto a_-$ is a most usefull one. This particular map has properties that arbitrary V-defining maps do not necessarily enjoy. For instance it is easy to see that $a\mapsto a_-$ is a $\bigvee$-homomorphism, $(\bigvee a_i)_-=\bigvee a_{i-}$, (however not always a $\bigwedge$-homomorphism) but the following example shows that not every V-defining map is: For the lattice $0<a<c<1$, $0<b<c<1$.



Put $p(1)=1$, $p(a)=b$, $p(b)=a$, $p(c)=1$, $p(0)=0$. Then $p(a\vee b)=1\neq c=p(a)\vee p(b)$, although p V-defines. Combining Theorems 3 and 4 it is tempting to ask whether a lattice L satisfying

$$(*)\qquad a=\bigwedge\{b_-/a\leqslant b_-\}\quad(a\in L)$$

is completely distributive. Below we give an example, due to H.-J. Bandelt (private communication), that this is not the case. However the following weaker distributive law holds:

$$\bigwedge\{c_-/a\leqslant c_-\}\vee b=\bigwedge\{b\vee c_-/a\leqslant c_-\}\qquad(a,b\in L).$$

Indeed, as it is sufficient to prove that the left hand side is $\leqslant$ the right hand side, let $d\in L$ be such that $\bigwedge\{c_-/a\leqslant c_-\}\vee b\leqslant d_-$. Then

$$d_-\geqslant\bigwedge\{c_-/a\leqslant c_-\}=a\quad\text{and}\quad d_-\geqslant b.$$

So d_- is one of the c with $c_-\geqslant a$ and so

$$d_-=d_-\vee b\geqslant\bigwedge\{b\vee c_-/a\leqslant c_-\}.$$

The rest follows as usually.

The example of H.-J. Bandelt of a lattice satisfying $(*)$ but not being completely distributive is the following

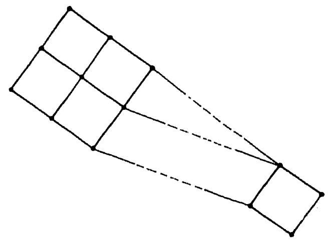


Fig. 1

We leave the details to the reader.

References

[1] H.-J. Bandelt, *On complete distributivity and maximal d-intervals in complete lattices*, Coll. Math. Soc. J. Bolyai, Szeged (1974), pp. 30–43, MR 55 # 12589.
[2] — *Zur Konkreten Charakterisierung von Kongruenzverbänden*, Arch. Math. 26 (1975), pp. 8–13, MR 52 # 2980.
[3] — *On congruence lattices of 2-valued algebras*, Coll. Math. Soc. J. Bolyai, Szeged (1975), pp. 27–31.
[4] — *Regularity and complete distributivity*, Semigroup Forum 19 (1980), pp. 123–126.
[5] G. Birkhoff, *Lattice Theory*, AMS Colloquium Publications, 3rd edition, N. Y. 1979.
[6] J. B. Conway, *A complete Boolean algebra of subspaces which is not reflexive*, Bull. Amer. Math. Soc. 79 (1973), pp. 720–722, MR 47 # 9312.
[7] P. R. Halmos, *Reflexive lattices of subspaces*, J. London Math. Soc. 4 (1971), pp. 257–263, MR 44 # 5808.
[8] R. E. Johnson, *Distinguished rings of linear transformations*, Trans. Amer. Math. Soc. 111 (1964), pp. 400–412, MR 28 # 5088.
[9] M. S. Lambrou, *Complete atomic Boolean lattices*, J. London Math. Soc. 15 (2) (1977), pp. 387–390.
[10] — *Semisimple completely distributive lattices are Boolean algebras*, Proc. Amer. Math. Soc. 68 (1978), pp. 217–219, MR 57 # 3030.
[11] — *Erratum to "Semisimple completely distributive lattices are Boolean algebras"*, Proc. Amer. Math. Soc. 68 (1978), pp. 217–219, MR 80d:06010.
[12] W. E. Longstaff, *Strongly reflexive lattices*, J. London Math. Soc. (2) 11 (1975), pp. 491–498, MR 52 # 15036.
[13] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer Ergebnisse Band 77, New York 1973.
[14] G. N. Raney, *Completely distributive complete lattices*, Proc. Amer. Math. Soc. 3 (1952), pp. 677–680, MR 14, 612.
[15] — *A subdirect-union representation for completely distributive complete lattices*, Proc. Amer. Math. Soc. 4 (1953), pp. 518–522, MR 15, 389.

[16] G. N. Raney, *Tight Galois connections and complete distributivity*, Trans Amer. Math. Soc. 97 (1960), pp. 418–426, MR 22 # 10928.
[17] J. R. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. 15 (1965), pp. 61–83, MR 30 # 1405.

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Remarks on intrinsic isometries

by

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Abstract. A map $f\colon A \to A'$ of metric spaces is said to be an intrinsic isometry if it preserves the length of every arc. It is shown in this note that the Euclidean n-space E^n is intrinsically isometric to a subset A of E^{n+1} with arbitrarily small diameter $\delta(A)$. We also consider the intrinsic metric of a product of metric spaces.

1. Introduction. The notion of the intrinsic metric for metric spaces and related notions were introduced by K. Borsuk [1]. Let us say that a space A (with metric ϱ) is *geometrically acceptable* (notation: $A \in \mathrm{GA}$) if

(1.1) for every two points $x, y \in A$ there exists an arc $L \subset A$ with finite length such that $x, y \in L$

and

(1.2) for every point $x \in A$ and for every $\varepsilon > 0$ there is a neighborhood U of x in A such that for every point $y \in U$ there exists in A an arc L containing the two points x, y and such that the length $|L| < \varepsilon$.

Then setting

(1.3) $\varrho_A(x, y)$ = lower bound of the length of all arcs $L \subset A$ containing the two points x, y,

one gets a metric ϱ_A in A called the *intrinsic metric in A*. The topology in $A \in \mathrm{GA}$ induced by the metric ϱ_A is the same as the topology induced by the metric ϱ.

A function f mapping a GA-space A onto another GA-space A' is said to be an *intrinsic isometry* provided

$$\varrho_A(x, y) = \varrho_{A'}(f(x), f(y)) \quad \text{for every } x, y \in A. \tag{1.4}$$

A map f is an intrinsic isometry if and only if it preserves the length of every arc. Every intrinsic isometry is a homeomorphism.

K. Borsuk has proved [1] that for every $\varepsilon > 0$ there exists an intrinsic isometry mapping the Euclidean n-space E^n onto a subset $A \subset E^{2n}$ such that the diameter of A (by the usual metric in E^{2n}) is less than ε. We will prove the following